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On θ-Euclidean L-Fuzzy Ideals of Rings

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Abstract

In this paper we define a θ-Euclidean level subset and a θ-Euclidean level ideal. We also give some properties of a θ-Euclidean level subset.

Key Words: Ideal; Fuzzy ideal; Level subset; θ-Euclidean level ideal; θ-Euclidean L-fuzzy ideal.

1. Introduction

In this paper we define a new set denoted by μ_{θ_y}. We call it a θ-Euclidean level subset. Then we will show that for $0 \neq y \in R$, the set μ_{θ_y} is an ideal of R if $\mu\colon R \to L$ is an L-fuzzy ideal of R. We also give a theorem on θ-Euclidean L-fuzzy ideal of R.

2. Preliminaries

Throughout this paper, R denotes a commutative ring with identity. L denotes a lattice with the least element 0 and the greatest element 1. Unless stated otherwise L is complete and completely distributive in the sense that it satisfies the following law:

$$\bigvee\{a_i \mid i \in I\} \wedge \bigvee\{b_j \mid j \in J\} = \bigvee\{a_i \wedge b_j \mid i \in I, j \in J\} \qquad [3]$$

for all $a_i, b_j \in L$.

Definition 2.1 [3]. *An L-fuzzy ideal is a function $J\colon R \to L$ satisfying the following axioms:*

(i) $J(x+y) \geq J(x) \wedge J(y)$,

(ii) $J(-x) = J(x)$,

(iii) $J(xy) \geq J(x) \vee J(y)$.

Since we are considering L-fuzzy ideals over a fixed lattice L, we shall call them fuzzy ideals only.

Definition 2.2 [5]. *Let* μ *be any fuzzy subset of a set* S *and let* $t \in [0,1]$. *The set* $\mu_t = \{ x \in S \mid \mu(x) \geq t \}$ *is called a level subset of* μ.

Proposition 2.3 [2, 3].

(i) A function $J\colon R \to L$ *is a fuzzy ideal iff*

$J(x-y) \geq J(x) \wedge J(y)$ *and* $J(xy) \geq J(x) \vee J(y)$.

(ii) If $J\colon R \to L$ *is a fuzzy ideal, then*

(a) $J(0) \geq J(x) \geq J(1)$, *for all* $x \in R$;

(b) $J(x-y) = J(0)$ *implies* $J(x) = J(y)$ *for all* $x, y \in R$;

(c) the level cuts $J_\alpha = \{x \in R \mid J(x) \geq \alpha\}$ *are ideals of* R, *for* $\alpha \leq J(0)$. *Conversely, if each* J_α *is an ideal, then* J *is a fuzzy ideal.*

3. θ-Euclidean L-fuzzy ideal

Definition 3.1 [1]. *Let* $\theta\colon R \to L$ *be a non-constant fuzzy subset of* R. *A function* $\mu\colon R \to L$ *is called a* θ*-Euclidean L-fuzzy ideal if* μ *satisfies the following axioms:*

(i) $\mu(x+y) \geq \mu(x) \wedge \mu(y)$ *for all* x, y *in* R,

(ii) $\mu(-x) = \mu(x)$;

(iii) $\mu(xy) \geq \mu(x) \vee \mu(y)$;

(iv) For any $x, y \in R$, *with* $y \neq 0$, *there exist elements* $q, r \in R$ *such that* $x = yq + r$ *where either* $r = 0$ *or else* $\mu(r) \vee \theta(r) \geq \mu(y) \vee \theta(y)$.

Now we will define a new set called a θ*-Euclidean Level subset and examine this set.*

Definition 3.2 *Let* $\mu\colon R \to L$ *and* $\theta\colon R \to L$ *be fuzzy sets. For* $0 \neq y \in R$, *the set* $\mu_{\theta_y} = \{x \in R \mid$ *there exist elements* $q, r \in R$ *such that* $x = yq + r$ *where either* $r = 0$ *or else* $\mu(r) \geq \mu(y) \vee \theta(y)\}$ *is called a* θ*-Euclidean level subset of* μ.

Theorem 3.3 *Let $\mu\colon R \to L$ be an L-fuzzy ideal of R .Then for $0 \neq y \in R$, μ_{θ_y} is an ideal of R. Also $\mu_{\theta_y} \neq \{0\}$.*

Proof. Let x_1 and x_2 be elements of μ_{θ_y}. We have to show that $x_1 + x_2 \in \mu_{\theta_y}$.

Then there exist elements $q_1, r_1 \in R$ such that $x_1 = yq_1 + r_1$ where either $r_1 = 0$ or else $\mu(r_1) \geq \mu(y) \vee \theta(y)$. And then there exist elements $q_2, r_2 \in R$ such that $x_2 = yq_2 + r_2$ where either $r_2 = 0$ or else $\mu(r_2) \geq \mu(y) \vee \theta(y)$. Hence $x_1 + x_2 = y(q_1 + q_2) + r_1 + r_2$ and $q_1 + q_2$, $r_1 + r_2 \in R$. Also we know that $\mu(r_1 + r_2) \geq \mu(r_1) \wedge \mu(r_2)$ since μ is an L-fuzzy ideal of R. If $r_i \neq 0$ for $i = 1$ or $i = 2$, then $\mu(r_i) \geq \mu(y) \vee \theta(y)$. Therefore we can say that there exist elements $q_1 + q_2, r_1 + r_2 \in R$ such that $x_1 + x_2 = y(q_1 + q_2) + r_1 + r_2$ where either $r_1 + r_2 = 0$ or else $\mu(r_1 + r_2) \geq \mu(y) \vee \theta(y)$. This means that $x_1 + x_2$ is an element of μ_{θ_y}.

Now, we need to show that $ax \in \mu_{\theta_y}$ for all $x \in \mu_{\theta_y}$ and $a \in R$. Let $a \in R$ and $x \in \mu_{\theta_y}$. Then there exist elements $q, r \in R$ such that $x = yq + r$ where either $r = 0$ or else $\mu(r) \geq \mu(y) \vee \theta(y)$. Suppose that $r \neq 0$. Then $\mu(r) \geq \mu(y) \vee \theta(y)$. Since μ is an L-fuzzy ideal of R, we can write

$$\mu(ar) \geq \mu(a) \vee \mu(r) \geq \mu(r) \geq \mu(y) \vee \theta(y).$$

Hence there exist elements $aq, ar \in R$ such that $ax = yaq + ar$ where either $ar = 0$ or else $\mu(ar) \geq \mu(y) \vee \theta(y)$. Therefore we get $ax \in \mu_{\theta_y}$. This means that μ_{θ_y} is an ideal of R. We can write $y = y.1 + 0$ where $q = 1$, $r = 0 \in R$. Then $0 \neq y \in \mu_{\theta_y}$. Thus $\mu_{\theta_y} \neq \{0\}$. □

Definition 3.4 *Let μ be any L-fuzzy ideal of a ring R. The ideals μ_{θ_y} are called θ-Euclidean level ideals of μ.*

Corollary 3.5 *Let $\mu\colon R \to L$ be an L-fuzzy ideal of R and $\theta\colon R \to L$ be a fuzzy set. If $\mu(1) \geq \mu(y) \vee \theta(y)$ for $0 \neq y \in R$, then $\mu_{\theta_y} = R$.*

Proof. μ_{θ_y} is an ideal of R from Theorem 3.3. So $\mu_{\theta_y} \subseteq R$. To obtain $R \subseteq \mu_{\theta_y}$, we have to show that $1 \in \mu_{\theta_y}$. Since R is a ring with identity, we can write $1 = y.0 + 1$ where $r = 1$ and $q = 0$.

There exist elements $0 = q, 1 = r \in R$ such that $1 = y.0 + 1$ where $\mu(r) = \mu(1) \geq \mu(y) \vee \theta(y)$. So $1 \in \mu_{\theta_y}$. Since $1 \in \mu_{\theta_y}$ and μ_{θ_y} is an ideal of R, we obtain that $R \subseteq \mu_{\theta_y}$. Thus $R = \mu_{\theta_y}$. □

Definition 3.6 *Let $\mu\colon R \to L$ and $\theta\colon R \to L$ be fuzzy sets. For $0 \neq y \in R$ we define $\mu^*_{\theta_y}$ as follows:*

$\mu^*_{\theta_y} = \{x \in R \mid$ *there exist elements $q, r \in R$ such that $x = yq + r$ where*

$$\theta(r) \geq \mu(y) \vee \theta(y)\}.$$

Theorem 3.7 *Suppose that L is a chain and that $\mu\colon R \to L$ and $\theta\colon R \to L$ are fuzzy sets. Then $\mu\colon R \to L$ is a θ-Euclidean L-fuzzy ideal of R if and only if μ_α is an ideal of R for $\alpha \leq \mu(0)$ and $R = \mu_{\theta_y} \cup \mu^*_{\theta_y}$ for all $0 \neq y \in R$.*

Proof. Suppose first that $\mu\colon R \to L$ is a θ-Euclidean L-fuzzy ideal of R. Then $\mu\colon R \to L$ is an L-fuzzy ideal of R. Because of Proposition 2.3, μ_α is an ideal of R. Now it must be shown that $R = \mu_{\theta_y} \cup \mu^*_{\theta_y}$ for all $0 \neq y \in R$. Let $0 \neq y \in R$ and $x \in \mu_{\theta_y} \cup \mu^*_{\theta_y}$. Then $x \in R$. So $\mu_{\theta_y} \cup \mu^*_{\theta_y} \subseteq R$.

Let $a \in R$. Since $\mu\colon R \to L$ is a θ-Euclidean L-fuzzy ideal of R, for $0 \neq y, a \in R$ there exist elements q , $r \in R$ such that $a = yq + r$ where either $r = 0$ or else $\mu(r) \vee \theta(r) \geq \mu(y) \vee \theta(y)$. If $\mu(r) \vee \theta(r) = \mu(r)$, then we get $\mu(r) \geq \mu(y) \vee \theta(y)$. So $a \in \mu_{\theta_y}$.

If $\mu(r) \vee \theta(r) = \theta(r)$, then $a \in \mu^*_{\theta_y}$. So $a \in \mu_{\theta_y} \cup \mu^*_{\theta_y}$. Thus $R \subseteq \mu_{\theta_y} \cup \mu^*_{\theta_y}$. We obtain that $R = \mu_{\theta_y} \cup \mu^*_{\theta_y}$.

Conversely, suppose that μ_α is an ideal of R for $\alpha \leq \mu(0)$ and $R = \mu_{\theta_y} \cup \mu^*_{\theta_y}$ for all $0 \neq y \in R$. Because of Proposition 2.3, $\mu\colon R \to L$ is an L-fuzzy ideal.

Let $0 \neq y, x \in R$. Since $\mu_{\theta_y} \cup \mu^*_{\theta_y} = R$, $x \in \mu_{\theta_y} \cup \mu^*_{\theta_y}$. If $x \in \mu_{\theta_y}$, then there exist elements q , $r \in R$ such that $x = yq + r$ where either $r = 0$ or else $\mu(r) \geq \mu(y) \vee \theta(y)$. We can write $\mu(r) \vee \theta(r) \geq \mu(r) \geq \mu(y) \vee \theta(y)$. If $x \in \mu^*_{\theta_y}$, then there exist elements $q_1, r_1 \in R$ such that $x = yq_1 + r_1$ where $\theta(r_1) \geq \mu(y) \vee \theta(y)$. Also $\mu(r_1) \vee \theta(r_1) \geq \theta(r_1) \geq \mu(y) \vee \theta(y)$. Finally $\mu\colon R \to L$ is a θ-Euclidean L-fuzzy ideal of R. □

Properties of μ_{θ_y}

(i) Let $\mu\colon R \to L$ and $\theta\colon R \to L$ be fuzzy sets. For $0 \neq y \in R$,

(a) $\mu_\alpha \subseteq \mu_{\theta_y}$ for all $\alpha \in L$ such that $\alpha \geq \mu(y) \vee \theta(y)$;

(b) $(y) \subseteq \mu_{\theta_y}$;

(c) $\mu_{\theta_1} = R$ (for $y = 1$).

(ii) Let L be a chain, $\mu\colon R \to L$ be an L-fuzzy ideal, $\theta\colon R \to L$ be a fuzzy set and $0 \neq y \in R$. If x , $y \in \mu_{\theta_y} \cap \mu_{\theta_x}$ and $\theta(x) = \theta(y)$, then $\mu_{\theta_x} = \mu_{\theta_y}$.

Proof.

(a) Let $x \in \mu_\alpha$. Then $x \in R$ and $\mu(x) \geq \alpha$. We can write $x = y.0 + x$ where $q = 0, r = x \in R$. Also $\mu(r) = \mu(x) \geq \alpha \geq \mu(y) \vee \theta(y)$. Hence $x \in \mu_{\theta_y}$. So $\mu_\alpha \subseteq \mu_{\theta_y}$.

(b) We know that $y \in \mu_{\theta_y}$. So $(y) \subseteq \mu_{\theta_y}$.

(c) Let $a \in R$. We can write $a = 1.a + 0$ where $r = 0, q = a \in R$. Thus $a \in \mu_{\theta_1}$ and $R \subseteq \mu_{\theta_1}$. Also $\mu_{\theta_1} \subseteq R$. So $R = \mu_{\theta_1}$.

(ii) Since $x \in \mu_{\theta_y}$, there exist elements q , $r \in R$ such that $x = yq + r$ where either $r = 0$ or else $\mu(r) \geq \mu(y) \vee \theta(y)$.

Let $t \in \mu_{\theta_x}$. Then there exist elements q_1 , $r_1 \in R$ such that $t = xq_1 + r_1$ where either $r_1 = 0$ or else $\mu(r_1) \geq \mu(x) \vee \theta(x)$. We can write

$$t = (yq + r)q_1 + r_1 = yqq_1 + rq_1 + r_1$$

and qq_1 , $rq_1 + r_1 \in R$.

If $r = r_1 = 0$, then $rq_1 + r_1 = 0$.

If $r = 0$ and $r_1 \neq 0$, then $\mu(rq_1 + r_1) = \mu(r_1) \geq \mu(x) \vee \theta(x) = \mu(yq + r) \vee \theta(y)$
$= \mu(yq) \vee \theta(y) \geq \mu(y) \vee \theta(y)$.

If $r \neq 0$ and $r_1 = 0$, then $\mu(rq_1 + r_1) = \mu(rq_1) \geq \mu(r) \geq \mu(y) \vee \theta(y)$.

Suppose that $r \neq 0$ and $r_1 \neq 0$. Then $\mu(rq_1 + r_1) \geq \mu(rq_1) \wedge \mu(r_1) \geq \mu(r) \wedge \mu(r_1)$.

If $r \neq 0$, $r_1 \neq 0$ and $\mu(r) \wedge \mu(r_1) = \mu(r)$, then $\mu(rq_1 + r_1) \geq \mu(r) \geq \mu(y) \vee \theta(y)$.

If $r \neq 0$, $r_1 \neq 0$ and $\mu(r) \wedge \mu(r_1) = \mu(r_1)$, then

$\mu(rq_1 + r_1) \geq \mu(r_1) \geq \mu(x) \vee \theta(x) = \mu(yq + r) \vee \theta(y)$
$\geq \{\mu(y) \wedge \mu(r)\} \vee \theta(y) = \mu(y) \vee \theta(y)$.

Finally we can say that there exist elements qq_1 , $rq_1 + r_1 \in R$ such that $t = y(qq_1) + (rq_1 + r_1)$ where either $rq_1 + r_1 = 0$ or else

$$\mu(rq_1 + r_1) \geq \mu(y) \vee \theta(y).$$

This means that $\mu_{\theta_x} \subseteq \mu_{\theta_y}$.

We obtain $\mu_{\theta_y} \subseteq \mu_{\theta_x}$ in a similiar way. So $\mu_{\theta_y} = \mu_{\theta_x}$. □

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