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# On the Power Subgroups of the Extended Modular Group $\overline{\Gamma}$

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**Abstract**

In this paper we describe the group structure of power subgroups $\overline{\Gamma}^m$ of the extended modular group $\overline{\Gamma}$ and the quotients to them. Then we give some relations between the power subgroups $\overline{\Gamma}^m$, the commutator subgroups $\overline{\Gamma}'$ and $\overline{\Gamma}''$ and also the information of interest about free normal subgroups of the extended modular group $\overline{\Gamma}$.

**Key Words:** Extended Modular Group, Power Subgroup, Commutator Subgroup, Free Subgroup

## 1. Introduction

The modular group $\Gamma$ is the discrete subgroup of $PSL(2,\mathbb{Z})$ generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \qquad \text{and} \qquad U(z) = z + 1.$$

Let $S = T \cdot U$, that is,

$$S(z) = -\frac{1}{z+1}.$$

Then modular group $\Gamma$ has a presentation

$$\Gamma = < T, S \mid T^2 = S^3 = I > \cong C_2 * C_3.$$

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By adding the reflection $R(z) = 1/\ \overline{z}$ to the generators of the modular group $\Gamma$, the extended modular group $\overline{\Gamma}$ has been defined in [1]. The extended modular group $\overline{\Gamma}$ has a presentation

$$\overline{\Gamma} =< T, S, R \mid T^2 = S^3 = R^2 = I,\ RT = TR, RS = S^{-1}R >$$

or

$$\overline{\Gamma} =< T, S, R \mid T^2 = S^3 = R^2 = (RT)^2 = (RS)^2 = I > \cong D_2 *_{\mathbb{Z}_2} D_3. \qquad (1)$$

The modular group $\Gamma$ is a subgroup of index 2 in $\overline{\Gamma}$.

Let us define $\overline{\Gamma}^m$ to the subgroup generated by the $m^{\text{th}}$ powers of all elements of $\overline{\Gamma}$, for some positive integer $m$. $\overline{\Gamma}^m$ is called the $m^{\text{th}}$- *power subgroup* of $\overline{\Gamma}$. As fully invariant subgroups, they are normal in $\overline{\Gamma}$.

From the definition one can easily deduce that

$$\overline{\Gamma}^{mk} < \overline{\Gamma}^m$$

and that

$$\overline{\Gamma}^{mk} < (\overline{\Gamma}^m)^k.$$

Also, it is easy to deduce that

$$\overline{\Gamma}^m . \overline{\Gamma}^k = \overline{\Gamma}^{(m,k)},$$

where $(m, k)$ denotes the greatest common divisor of $m$ and $k$.

The power subgroups of the modular group $\Gamma$ was studied by [4]. In [4], M. Newman showed that

$$\begin{aligned} &\Gamma^2 =< S > * < TST >, \\ &\Gamma^3 =< T > * < STS^2 > * < S^2TS >, \\ &\Gamma' = \Gamma^2 \cap \Gamma^3\ ,\ \Gamma' =< TSTS^2 > * < TS^2TS > \text{ and } \Gamma'' \subset \Gamma^6 \subset \Gamma'. \end{aligned} \qquad (2)$$

Also, M. Newman proved that the groups $\Gamma^{6m}$ are free groups and the index $\left|\Gamma : \Gamma^{6m}\right| = \infty$ for $m \geq 72$ and $\left|\Gamma : \Gamma^{6m}\right|$ when $2 \leq m \leq 71$ is unknown. $\Gamma^6$ is a free group of rank 37.

The commutator subgroup of $\overline{\Gamma}$ is denoted by $\overline{\Gamma}'$ and defined by

$$< [g, h] \mid\ g, h \in \overline{\Gamma} >,$$

where $[g,h]= ghg^{-1}h^{-1}$. $\overline{\Gamma}'$ is a normal subgroup of $\overline{\Gamma}$, and therefore we can form the quotient group $\overline{\Gamma}/\overline{\Gamma}'$.

The commutator subgroup $\overline{\Gamma}'$ of the extended modular group $\overline{\Gamma}$ was investigated in [1], and it was shown that

$$\begin{aligned} &\left|\overline{\Gamma} : \overline{\Gamma}'\right| = 4, \\ &\overline{\Gamma}' =< S > * < TST >, \\ &\left|\overline{\Gamma} : \overline{\Gamma}''\right| = 36, \end{aligned} \tag{3}$$

so that $\overline{\Gamma}''$ is a free group with basis $[S, TST]$, $\left[S, TS^2T\right]$, $\left[S^2, TST\right]$, $\left[S^2, TS^2T\right]$.

The purpose of this paper is to determine the structure of the power subgroups $\overline{\Gamma}^m$ of the extended modular group $\overline{\Gamma}$ and to give some relations between them, the commutator subgroups $\overline{\Gamma}'$ and $\overline{\Gamma}''$ and also to investigate free normal subgroups of the extended modular group $\overline{\Gamma}$. In our discussion we use Reidemeister-Schreier method, (for more detail about this method, see [2]).

## 2. The Power Subgroups of the Extended Modular Group

We consider the presentation of the extended modular group $\overline{\Gamma}$ given in (1):

$$\overline{\Gamma} =< T, S, R \mid T^2 = S^3 = R^2 = (RT)^2 = (RS)^2 = I > .$$

We find a presentation for the quotient $\overline{\Gamma}/\ \overline{\Gamma}^m$ by adding the relation $X^m = I$ to the presentation of $\overline{\Gamma}$. The order of $\overline{\Gamma}/\ \overline{\Gamma}^m$ gives us the index. We have

$$\begin{aligned} \overline{\Gamma}/\ \overline{\Gamma}^m \cong< T, S, R \mid\ & T^2 = S^3 = R^2 = (TR)^2 = (RS)^2 = I, \\ & T^m = S^m = R^m = (TR)^m = (RS)^m = I > . \end{aligned} \tag{4}$$

Thus we use Reidemeister-Schreier process to find the presentation of the power subgroups $\overline{\Gamma}^m$. First we have the following theorem.

**Theorem 2.1** *i) The normal subgroup $\overline{\Gamma}^2$ is isomorphic to the free product of two finite* *cyclic groups of order* 3*. Also*

$$\left|\overline{\Gamma}:\overline{\Gamma}^2\right| = 4,$$
$$\overline{\Gamma}^2 =< S > * < TST >,$$
$$\overline{\Gamma} = \overline{\Gamma}^2 \cup T\,\overline{\Gamma}^2 \cup R\,\overline{\Gamma}^2 \cup TR\,\overline{\Gamma}^2.$$

*The elements of* $\overline{\Gamma}^2$ *are characterised by the property that the sum of the exponents of* $T$ *is even.*

*ii) The normal subgroup* $\overline{\Gamma}^3$ *is isomorphic to the extended modular group* $\overline{\Gamma}$*, i.e.*

$$\overline{\Gamma}^3 = \overline{\Gamma}.$$

**Proof.** i) By (4), we have

$$\overline{\Gamma}/\,\overline{\Gamma}^2 \cong < T,S,R \mid T^2 = S^3 = R^2 = (TR)^2 = (RS)^2 = I,$$
$$T^2 = S^2 = R^2 = (TR)^2 = (RS)^2 = I > .$$

Since

$$S^3 = S^2 = I,$$

we obtain $S = T^2 = R^2 = I$. Therefore

$$\overline{\Gamma}/\,\overline{\Gamma}^2 \cong < T,R \mid T^2 = R^2 = (TR)^2 = I > \cong D_2$$

and

$$\left|\overline{\Gamma}:\overline{\Gamma}^2\right| = 4.$$

Now we choose $\{I,T,R,TR\}$ as a Schreier transversal for $\overline{\Gamma}^2$. According to the Reidemeister-Schreier method, we can form all possible products :

$$\begin{array}{lll}
I.T.(T)^{-1} = I, & I.S.(I)^{-1} = S, & I.R.(R)^{-1} = I, \\
T.T.(I)^{-1} = I, & T.S.(T)^{-1} = TST, & T.R.(TR)^{-1} = I, \\
R.T.(TR)^{-1} = RTRT, & R.S.(R)^{-1} = RSR, & R.R.(I)^{-1} = I, \\
TR.T.(R)^{-1} = TRTR, & TR.S.(TR)^{-1} = TRSRT, & TR.R.(T)^{-1} = I.
\end{array}$$

Since $RTRT = I$, $TRTR = I$, $RSR = S^{-1}, TRSRT = TS^{-1}T = (TST)^{-1}$, the generators are $S$ and $TST$. Thus we have

$$\overline{\Gamma}^2 =< S, TST \mid S^3 = (TST)^3 = I > \cong C_3 * C_3,$$

and

$$\overline{\Gamma}^2 = \overline{\Gamma}^2 \cup T\,\overline{\Gamma}^2 \cup R\,\overline{\Gamma}^2 \cup TR\,\overline{\Gamma}^2.$$

ii) By (4), we have

$$\begin{aligned}\overline{\Gamma}/\,\overline{\Gamma}^3 \cong &< T, S, R \mid T^2 = S^3 = R^2 = (TR)^2 = (RS)^2 = I,\\ &T^3 = S^3 = R^3 = (TR)^3 = (RS)^3 = I > .\end{aligned}$$

Therefore we find $S = T = R = I$ from the relations

$$R^2 = R^3 = I,\ S^3 = (SR)^2 = I,\ T^2 = T^3 = I.$$

Thus we have

$$\left|\overline{\Gamma} : \overline{\Gamma}^3\right| = 1;$$

that is,

$$\overline{\Gamma}^3 = \overline{\Gamma}.$$

□

The following results are easy to see:

**Theorem 2.2** *i)* $\overline{\Gamma}^2 = \Gamma^2 = \overline{\Gamma}' = \overline{\Gamma}^2 \cap \overline{\Gamma}^3$

*ii)* $(\overline{\Gamma}')^3 \subset \overline{\Gamma}''$.

Now we have

**Theorem 2.3** *Let $m$ be a positive integer. The normal subgroups $\overline{\Gamma}^m$ satisfy the following:*

*i)* $\overline{\Gamma}^m = \overline{\Gamma}$ *if* $2 \nmid m$,

*ii)* $\overline{\Gamma}^m = \overline{\Gamma}^2$ *if* $2 \mid m$ *but* $6 \nmid m$.

**Proof.** i) If $2 \nmid m$ then by (4), we find $S = T = R = I$ from the relations

$$R^2 = R^m = I,\ S^3 = S^m = (SR)^2 = (SR)^m = I = I,\ T^2 = T^m = I.$$

Thus $\overline{\Gamma}/\ \overline{\Gamma}^m$ is trivial and hence $\overline{\Gamma}^m = \overline{\Gamma}$.

ii) If $2 \mid m$ but $6 \nmid m$ then $(m, 3) = 1$. By (4), we obtain $S = T^2 = R^2 = I$ from the relations

$$R^2 = R^m = I,\ S^3 = S^m = I,\ T^2 = T^m = I$$

as $2 \mid m$ but $6 \nmid m$. These show that

$$\overline{\Gamma}/\ \overline{\Gamma}^m \cong < T, R \mid T^2 = R^2 = (TR)^2 = I > \cong D_2$$

and

$$\left|\overline{\Gamma} : \overline{\Gamma}^m\right| = 4.$$

Since $\overline{\Gamma}^2$ is the only normal subgroup of index 4 we have $\overline{\Gamma}^m = \overline{\Gamma}^2$. □

Therefore the only case left is that when $m$ is divisible by 6. In this case, the above techniques do not say much about $\overline{\Gamma}^m$. To do this we use the second commutator subgroup $\overline{\Gamma}''$ of $\overline{\Gamma}$.

**Theorem 2.4** *Let $m$ be a positive integer. The groups $\overline{\Gamma}^{6m}$ are the subgroups of the second commutator subgroup $\overline{\Gamma}''$.*

**Proof.** i) Since $\overline{\Gamma}^6 \subset \left(\overline{\Gamma}^2\right)^3 \subset \overline{\Gamma}^2$ and $\overline{\Gamma}' = \overline{\Gamma}^2$implies that $\overline{\Gamma}^6 \subset \left(\overline{\Gamma}'\right)^3 \subset \overline{\Gamma}'$and $\overline{\Gamma}^{6m} \subset \overline{\Gamma}^6 \subset \overline{\Gamma}''$. Since $\overline{\Gamma}'$ does not contain any reflection, $\overline{\Gamma}^{6m}$ does not contain any reflection. Also we know that $\Gamma^{6m} \subset \overline{\Gamma}^{6m}$. Thus we get

$$\overline{\Gamma}^{6m} = \Gamma^{6m} \subset \overline{\Gamma}''.$$

□

Then because $\overline{\Gamma}''$ is a free group and $\overline{\Gamma}^{6m} \subset \overline{\Gamma}''$, we have by Schreier's theorem the following theorem

**Theorem 2.5** *The groups $\overline{\Gamma}^{6m}$ are free groups.*

Therefore

$$\begin{aligned} \left|\overline{\Gamma} : \overline{\Gamma}^{6m}\right| &= \left|\overline{\Gamma} : \Gamma^{6m}\right| \\ &= \left|\overline{\Gamma} : \Gamma\right| . \left|\Gamma : \Gamma^{6m}\right| \\ &= 2\left|\Gamma : \Gamma^{6m}\right| \end{aligned}$$

since $\left|\overline{\Gamma} : \Gamma\right| = 2$. In [4], the index $\left|\Gamma : \Gamma^{6}\right|$ was computed as 216. Therefore

$$\left|\overline{\Gamma} : \overline{\Gamma}^{6}\right| = 432.$$

Also, the index $\left|\overline{\Gamma} : \overline{\Gamma}^{6m}\right|$ is unknown since $\left|\Gamma : \Gamma^{6m}\right|$ , $2 \leq m \leq 71$, is unknown.

**Corollary 2.6** $\overline{\Gamma}^{6}$ *is a free group of rank* 37.

## 3. Free Normal Subgroups of the Extended Modular Group

As $\overline{\Gamma}$ is isomorphic to the free product of dihedral groups $D_2$ and $D_3$ with amalgamation $\mathbb{Z}_2$, it has two kinds of normal subgroups : Free ones and free products of some infinite cyclic groups, some cyclic groups of order 2 and order 3, some dihedral groups $D_2$ and $D_3$ with some dihedral groups $D_2$ and $D_3$ with amalgamation $\mathbb{Z}_2$. Therefore the study of free normal subgroups and their group theoretical structures will be important to us. Here we discuss them for extended modular group $\overline{\Gamma}$. This has been done for modular group by Newman in [3]. His results can be generalized to the extended modular group.

Before giving the main theorem we need the following lemmas.

**Lemma 3.1** *Let $N$ be a non-trivial normal subgroup of finite index in $\overline{\Gamma}$. Then $N$ is free if and only if it contains no elements of finite order.*

**Proof.** By (1), $\overline{\Gamma}$ is isomorphic to a free product of $D_2 = C_2 \times C_2$ and $D_3 = C_2 \times C_3$ each amalgamated over $\mathbb{Z}_2$. A subgroup of finite index in $\overline{\Gamma}$ is isomorphic to a free product of the groups $F$, $C_r$, and $D_{m_1} *_{\mathbb{Z}_2} D_{m_2}$, where $r$ and each $m_i$ divide 2 or 3. Thus if $N$ is a subgroup of finite index in $\overline{\Gamma}$, it follows that

$$N = F * \prod\nolimits_{*} C_r * \prod\nolimits_{*} (D_{m_1} *_{\mathbb{Z}_2} D_{m_2}), \tag{5}$$

where $F$ is either free or $\{I\}$ and each $C_r$ is conjugate to $\{T\}$ or to $\{S\}$ or to $\{R\}$ and each $D_{m_i}$ is conjugate to $\{T, R\}$ or to $\{S, R\}$. As $N$ contains no elements of finite order the free product $\prod_* C_r * \prod_* (D_{m_1} *_{\mathbb{Z}_2} D_{m_2})$ is vacuous; and also as $N$ is non-trivial, $N$ must be free.

Conversely, if $N$ is free, then by definition, it contains no elements of finite order. □

**Lemma 3.2** *The only normal subgroups of finite index in* $\overline{\Gamma}$ *containing elements of finite order are*

$$\overline{\Gamma}, \Gamma, \Gamma^2 \text{ and } \Gamma^3.$$

**Proof.** Let $N$ be a normal subgroup of finite index in $\overline{\Gamma}$ containing an element of finite order. Then $N$ contains an element of order 2 or an element of order 3 or two elements of order 2 or two elements of order 2 and 3 or three elements so that two elements of order 2 and an element of order 3. An element of order 2 in $\overline{\Gamma}$ is conjugate to $T$ or to $R$ and an element of order 3 in $\overline{\Gamma}$ is conjugate to a power of $S$. Therefore if a normal subgroup $N$ contains an element of finite order, then it contains $T$ or $R$ or $S$. Therefore there are seven cases:

(i) $N$ contains $T$, $R$ and $S$. Then $N = \overline{\Gamma}$.

(ii) $N$ contains $T$ but not $R$ and $S$. Then $N \neq \overline{\Gamma}$, $\Gamma$ and $\Gamma^3 \subset N$, as $N$ is normal. Since$\left|\overline{\Gamma} : \Gamma^3\right| = 6$ we have $N = \Gamma^3$.

(iii) $N$ contains $T$, $R$ but not $S$. Then $N \neq \overline{\Gamma}$ and $\Gamma^3 \subset N$, the fact that $N$ is normal and by (ii). Since $\left|\overline{\Gamma} : \Gamma^3\right| = 6$, we have $N = \overline{\Gamma}$ or $\Gamma$ or $\Gamma^3$. But this is not possible since $S \in \overline{\Gamma}$, $S \in \Gamma$ and $R \notin \Gamma^3$.

(iv) $N$ contains $T$ and $S$, but not $R$. Then $N \neq \overline{\Gamma}$ and $\Gamma \subset N$, by (1) and the fact that $N$ is normal. Since $\left|\overline{\Gamma} : \Gamma\right| = 2$ it follows that $N = \Gamma$.

(v) $N$ contains $S$ but not $T$ and $R$. Then $N \neq \overline{\Gamma}$ and $\Gamma^2 \subset N$, by (2) and the fact that $N$ is normal. Since $\left|\overline{\Gamma} : \Gamma^2\right| = 4$, it follows that $N = \Gamma^2$.

(vi) $N$ contains $S$, $R$ but not $T$. Then $N \neq \overline{\Gamma}$ and $\Gamma^2 \subset N$, as $N$ is normal and by (v). Since $\left|\overline{\Gamma} : \Gamma^2\right| = 4$, we have $N = \overline{\Gamma}$ or $\Gamma$ or $\Gamma^2$. But this is not possible since $T \in \overline{\Gamma}$, $T \in \Gamma$ and $R \notin \Gamma^2$.

(vii) $N$ contains $R$ but not $T$ and $S$. This is not possible by (iii) and by (vi). □

**Theorem 3.3** *Let $N$ be a non-trivial normal subgroup of finite index in $\overline{\Gamma}$ different from $\overline{\Gamma}$, $\Gamma$, $\Gamma^2$, $\Gamma^3$. Then $N$ is a free group.*

**Proof.** It can be easily seen as an immediate consequence of the lemmas. □

**Theorem 3.4** *Let $N$ be a normal subgroup of finite index in $\overline{\Gamma}$ different from $\overline{\Gamma}$, $\Gamma$, $\Gamma^2$, $\Gamma^3$ such that $\left|\overline{\Gamma} : N\right| = \mu < \infty$. Then $\mu$ is divisible by* 12.

**Proof.** The quotient group contains subgroups of orders 2, 4 and 6, so its order is divisible by 12. □

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