Turk J Math
28 (2004) , 177 – 184.

Modules Supplemented Relative to A Torsion Theory

Tamer Koşan, Abdullah Harmancı

Abstract

This article introduces the concept of a τ-supplemented module as follows: Given a hereditary torsion theory in Mod R with associated torsion functor τ, we say that a module M is τ-supplemented when for every submodule N of M there exists a direct summand K of M such that $K \leq N$ and N/K is τ-torsion module. We present here some fundamental properties of this class of modules and study the decompositions of τ-supplemented modules under certain conditions on modules. The question of which direct sum of τ-supplemented R-modules are τ-supplemented is treated here.

Key Words: Torsion Theory, Supplemented Module.

1. Introduction

Let τ be a class of right modules over a ring. Motivated by the notion τ-complemented modules studied in [8] we introduce and study τ-supplemented modules. In what follows R will denote any ring with an identity and all modules will be unital right R-modules. τ will denote the torsion functor associated with an arbitrary torsion theory on the category Mod R of all right R-modules. A module M is lifting is a (or $(D1)$-module) if, for any given $A \leq M$, there exists a direct summand K of M such that $M = K \oplus L$ and $K \leq A$ with $A \bigcap L$ is small in L. This article introduces the concept of τ-supplemented modules as follows. Let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory. Then τ is uniquely determined by its associated class $\mathcal{T}$ of τ-torsion modules $\mathcal{T} = \{M \in \text{Mod } R \mid \tau(M) = M\}$ where for a

1991 *Mathematics Subject Classification:* 16S90

module M, $\tau(M) = \sum\{N \mid N \leq M, N \in \mathcal{T}\}$ and $\mathcal{F}$ is referred as a τ-torsion free class and $\mathcal{F} = \{M \in Mod-R \mid \tau(M) = 0\}$. A module in $\mathcal{T}$(or $\mathcal{F}$) is called a τ-torsion module (τ-torsionfree module). Every torsion class $\mathcal{T}$ determines in every module M a unique maximal $\mathcal{T}$-submodule $\tau(M)$, the *τ-torsion submodule* of M, and $\tau(M/\tau(M)) = 0$, i.e., $M/\tau(M)$ is $\mathcal{F}$-module and τ-torsionfree. In what follows τ will represent a hereditary torsion theory, that is, if $\tau = (\mathcal{T}, \mathcal{F})$ then the class $\mathcal{T}$ is closed under taking submodules, direct sums, images and extensions by short exact sequences, equivalently the class $\mathcal{F}$ is closed under submodules, direct products, injective hulls and isomorphic copies. We refer the reader to [3] and [9] as torsion theoretic sources sufficient for our purposes and [1] and [10] for the other notations in this paper.

Given a hereditary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ in Mod R we say that a module M is *τ-supplemented* if every submodule A of M contains a direct summand B of M such that A/B is τ-torsion. We say that a submodule A of M satisfies the *τ-supplemented condition* if A contains a direct summand B such that A/B is τ-torsion. M is τ-supplemented if and only if every submodule of M satisfies τ-supplemented condition. For the torsion class Mod R, we denote the corresponding torsion functor by χ, and if the torsion class is the class of zero modules we denote the corresponding torsion functor by ξ. In this notation $\xi = (0, \text{Mod } R)$ and $\chi = (\text{Mod } R, 0)$ where 0 denotes the class of zero modules. The torsion functor for the dual Goldie torsion theory will be denoted by τ_*. Then the dual Goldie torsion theory $\tau_* = (\mathcal{T}_*, \mathcal{F}_*)$ is generated by the class of small R-modules. A module M is τ_*-torsion if and only if $M = Z^*(M)$, where $Z^*(M) = \{n \in M : nR$ is small$\}$ (see [4] and [6]).

Examples 1.1. *Let R be any ring. Then*

(i) Every R-module is χ-supplemented.

(ii) An R-module M is ξ-supplemented if and only if M is semisimple.

(iii) Every lifting R-module is τ_*-supplemented.

Proof. (i) and (iii) Clear. (ii) By [1, Theorem 9.6]. □

Example 1.2. *Let R be any domain which is not right primitive. Then every R-module is τ_*-supplemented.*

Proof. By [7, Corollary 2.5]. □

Example 1.3. *Let I be an idempotent ideal of an arbitrary ring R. Let τ_I denote the hereditary torsion theory defined by I with torsion class $\mathcal{T}_I = \{N \in Mod\ R \mid NI = 0\}$.*

Then an R-module M is τ_I-supplemented if and only if NI is a direct summand of M for each submodule N of M.

Proof. The sufficiency is clear. Conversely, let N be any submodule of M. There exists a direct summand K of M such that K is contained in NI and NI/K is τ_I-torsion. Then $NI = (NI)I$ contained in K contained in NI, and hence $NI = K$. □

Corollary 1.4. *Let I be an idempotent ideal of a ring R such that the R-module R is τ_I-supplemented. Then $I = eR$ for some idempotent element e of R.*

Proof. By Example 1.3. □

2. Properties of τ-Supplemented Modules

Lemma 2.1. *Let M be a module. Then*

(i) M is τ-supplemented module if and only if every submodule A of M can be written as $A = B \bigoplus C$ with B is direct summand of M and C is τ-torsion submodule of M.

(ii) Every submodule of a τ-supplemented module is τ-supplemented.

Proof. Clear from definitions. □

We do not know if there is a torsion theory τ and a τ-supplemented module M such that some homomorphic image of M is not τ-supplemented nor do we know, in general, when a finite direct sum of τ-supplemented modules is τ-supplemented.

Proposition 2.2 *Let $M = M' \oplus M''$ be a direct sum of a τ-supplemented module M' and a τ-torsion module M''. Then M is τ-supplemented.*

Proof. Let N be a submodule of the module M. Then $N \cap M'$ is a submodule of M'. There exists a direct summand K of M'(hence also of M) such that $(N \cap M')/K$ is τ-torsion. But $N/(N \cap M')$ is isomorphic to $(N + M')/M'$, so is τ-torsion. Thus N/K is τ-torsion. It follows that M is τ-supplemented. □

Corollary 2.3. *Let $M = M' \oplus M''$ be a direct sum of a semisimple module M' and a τ-torsion module M''. Then M is τ-supplemented.*

Proof. By Proposition 2.2. □

Corollary 2.4. *Let I be an idempotent ideal of a ring R such that $I = Re$ for some* *idempotent element e of R. Then an R-module M is τ_I-supplemented if and only if $M = M' \oplus M''$ is a direct sum of a semisimple submodule M' and a τ_I-torsion submodule M''.*

Proof. The sufficiency is clear by Corollary 2.3. Conversely, suppose that M is τ_I-supplemented. Note that eR is contained in $I = Re$. Therefore Me is a submodule of M. Let K be any submodule of Me. Then $K = Ke = KRe = KI$, so that K is a direct summand of M and hence also of Me, by Example 1.3. Thus Me is semisimple. Moreover Me is a direct summand of M, say $M = Me \oplus N$ for some submodule N of M. Because N is isomorphic to M/Me, we have $NI = Ne = 0$. □

Lemma 2.5. *Let M be a module. Assume that M is a τ-supplemented module. Then any τ-torsion free submodule is direct summand.*

Proof. Let M be a τ-supplemented module and L a τ-torsion free submodule of M. There exist submodules K and K' of M such that $M = K \oplus K'$, K is contained in L and L/K is τ-torsion. Clearly $L = K \oplus (L \cap K')$. But $L \cap K'$ is contained in $L \cap \tau(M)$ so that $L \cap K' = 0$ and $L = K$. □

Corollary 2.6. *Let M be a τ-torsionfree module. Then the following statements are equivalent.*
(i) M is a τ-supplemented module.
(ii) M is a semisimple module.

Proof. Let M be a τ-torsion free module. Every submodule of M is τ-torsion free. By Lemma 2.5 the proof is clear. □

Lemma 2.7. *Any τ-supplemented module M is a direct sum $M' \oplus M''$ of a semisimple submodule M' and a τ-supplemented module M'' such that $\tau(M'')$ is an essential submodule of M''.*

Proof. Let K be complement of $\tau(M)$ in M. By Lemma 2.5, K is semisimple and $M = K \oplus K'$ for some submodule K' of M. Note that $\tau(M) \oplus K$ is an essential submodule of M and $\tau(M) = \tau(K) \oplus \tau(K') = \tau(K')$ so that $\tau(M) = (\tau(M) \oplus K) \cap K'$ is an essential submodule of K'. □

A torsion theory τ is called *stable* if the class of τ-torsion right R-modules is closed under essential extensions; equivalently, it is closed under injective hulls. For example, Goldie torsion theory is stable [9, page 153 Proposition 7.3].

Theorem 2.8. *Let τ be a stable torsion theory. Then the following statements are equivalent for a module M.*
(i) M is τ-supplemented.
(ii) Every τ-torsionfree submodule is a direct summand of M.
(iii) $M = M' \oplus M''$ is a direct sum of a semisimple submodule M' and a τ-torsion submodule M''.

Proof. $(i) \Rightarrow (ii)$ By Lemma 2.5.

$(ii) \Rightarrow (iii)$ Let K be a complement of $\tau(M)$ in M. By hypothesis, $M = K \oplus K'$ for some submodule K' of M and K is semisimple. Because $\tau(M) = \tau(K')$ is essential submodule of K' and τ is stable, we have K' is τ-torsion.

$(iii) \Rightarrow (i)$ By Corollary 2.3. □

Corollary 2.9. *Let τ be a stable hereditary torsion theory. Then any finite direct sum of τ-supplemented modules is τ-supplemented.*

Proof. By Theorem 2.8. □

We shall show in Section 3 that Theorem 2.8 fails for non-stable torsion theories.

Lemma 2.10. *An indecomposable module is τ-supplemented if and only if every proper submodule of M is τ-torsion.*

Proof. Clear. □

We shall say a module M is *almost τ-torsion* if every proper submodule of M is τ-torsion. Note that τ-torsion modules are almost τ-torsion and almost τ-torsion modules are τ-supplemented. Let M be an almost τ-torsion module which is not τ-torsion. Let $T = \tau(M)$. Then T does not equal M. Let m be an element of M not in T. By hypothesis, $M = mR$ and M is local module with unique maximal submodule T.

Theorem 2.11. *Let M be a τ-supplemented module which satisfies dcc or acc on direct summands. Then M is a finite direct sum of almost τ-torsion submodules.*

Proof. By hypothesis, $M = M_1 \oplus M_2 \oplus ... \oplus M_n$ is a finite direct sum of indecomposable submodules M_i $(1 \leq i \leq n)$. By Lemma 2.1 and 2.10, M_i is almost τ-torsion for each $1 \leq i \leq n$. □

Corollary 2.12. *Let R be a right Noetherian ring and let M be a τ-supplemented R-module. Then $M/\tau(M)$ is a semisimple module.*

Proof. Let $T = \tau(M)$. Let m belong to M. By Lemma 2.1, mR is τ-supplemented and hence, by Theorem 2.11, $mR/(mR \cap T)$ is semisimple. Thus $(mR + T)/T$ is semisimple for each m in M. Hence M/T is semisimple. □

3. Examples

Theorem 2.8 fails for non-stable torsion theories.

Example 3.1. Let R denote the ring of all upper triangular 2×2 matrices with entries in the ring $\mathbb{Z}$ of integers and let I denote the ideal of R which is generated as a right ideal by the idempotent where the $(1,1)$ entry 1 and all other entries 0. Let τ denote the hereditary torsion theory such that a module M is torsion provided $MI = 0$. Then the torsion submodule $\tau(R)$ of the right R-module R consists of all matrices in R with $(1,1)$ entry is 0. Clearly $\tau(R)$ is an essential submodule of R. But the right ideal N generated by the element a with $(1,1)$ entry 2 and all other entries are 0 does not contain a direct summand K of R such that N/K is τ-torsion. Thus every τ-torsion-free submodule of R is a direct summand of R but R is not τ-supplemented.

Example 3.2. Let F be any field and let S be any F-algebra. Let R denote the subring of the ring of all 2 by 2 matrices over S consisting of all 2 by 2 matrices with second column having entries from S, with $(1,1)$ entry from F and with $(2,1)$ entry 0. Let I denote the ideal generated as a right ideal by the idempotent e in R with $(1,1)$ entry 1 and all other entries 0. Let τ denote the torsion theory where a module M is τ-torsion if $MI = 0$. It can be shown that any submodule of the right R-module R is τ-torsion or contains e. It follows that R is a τ-supplemented R-module but $\tau(R)$ is not a direct summand of R.

There are modules M and torsion theories τ such that M is τ-supplemented but not lifting.

Example 3.3. Let M denote the $\mathbb{Z}$-module $\mathbb{Z}/8\mathbb{Z} \bigoplus \mathbb{Z}/2\mathbb{Z}$. Let $V = (\overline{4}, \overline{0})\mathbb{Z}$, $U = (\overline{2}, \overline{1})\mathbb{Z}$, $V_1 = (\overline{4}, \overline{1})\mathbb{Z}$, $U_1 = (\overline{1}, \overline{1})\mathbb{Z}$, $U_2 = (\overline{2}, \overline{0})\mathbb{Z}$, $N = (\overline{1}, \overline{0})\mathbb{Z}$, $K = (\overline{0}, \overline{1})\mathbb{Z}$. They are all proper submodules of M. N, K and U_1 are direct summands and $V \cong V_1$ and $U \cong U_2$. Since $M = U + U_1$ and $U \bigcap U_1 = V$ and V is small in U, U is a supplement of U_1. But U is not a direct summand. By [5, page 58, Proposition 4.8] M is not a lifting module. Let $\tau = \xi(U)$ denote the smallest hereditary torsion theory relative to which U is torsion. The direct summands N, K, U_1 satisfy the τ-supplemented condition, and as U, V, U_2 and V_1 are τ-torsion, they also satisfy the τ-supplemented condition. It follows that M is τ-supplemented.

There are modules M and torsion theories τ such that M is not τ-supplemented, but lifting.

Example 3.4. Let F be a field and R the upper triangular matrix ring $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Let M denote the right R-module R, e_{ij} the matrix units in R. Let $I = e_{12}R + e_{22}R$. Then I is an idempotent ideal and so defines a hereditary torsion theory τ_I with torsion class $\mathcal{T}_I = \{N \in \text{Mod } R \mid NI = 0\}$. By [5, page 71, Theorem 4.41] M is a lifting module. Let $K = e_{12}R$. Then K is not a direct summand since K is essential in the direct summand $e_{11}R$. K is not τ_I-torsion since $KI = e_{12}R$. K is simple module. Hence K can not contain any submodule A such that A is direct summand and K/A is τ_I-torsion. Thus M is not τ_I-supplemented.

Acknowledgement

We would like to express our gratefulness to the referee for his/her valuable suggestions and contributions.

References

[1] F.W.Anderson and K.R.Fuller, *Rings and Categories of Modules*, Springer-Verlag, New York. 1992.

[2] G.F.Birkenmeir and R.Wiegandt, *Pseudocomplements in the Lattice of Torsion Classes*, Comm. Alg. 26(1) (1998), 197-220

[3] J.Golan, *Torsion Theories*, Pitman Mon. and Surveys in Pure and Appl.Math. 29, 1986.

[4] C.Lomp, *Dual Continous Modules*, Master's thesis, Glasgow Univ., 1996.

[5] S.H.Mohammed and B.J.Müller, *Continous and Discrete Modules*, London Math.Soc.,LN 147, Cambridge Univ.Press, 1990.

[6] A.C.Özcan and A.Harmanci, *Characterizations of Some Rings by Functor Z*(.)*, Turkish J.Math. 21 (1997) 325-331.

[7] A.C.Özcan and P.F.Smith, *The Z* functor for rings whose primitive images are artinian*, Comm. in Algebra, 30 (2002) 4915-4930.

[8] P.F.Smith, A.M.Viola-Prioli and J.Viola-Prioli, *Modules complemented with respect to a torsion theory*, Comm.Alg. 25 (1997), 1307-1326.

[9] B.Strenström, *Rings of Quotients*, Springer-Verlag, Berlin, 1975.

[10] R.Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Reading, 1991.

Tamer KOŞAN, Abdullah HARMANCI
Received 11.03.2003
Hacettepe University,
Department of Mathematics,
Beytepe, Ankara-TURKEY.
e-mail: tkosan@hacettepe.edu.tr,
e-mail: harmanci@hacettepe.edu.tr