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A Quasi–Linear Manifolds and Quasi–Linear Mapping Between Them

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Abstract

In this article a special class of Banach manifolds (called QL-manifolds) and mapping between them (QL-mappings) are introduced and some examples are given.

0. Introduction

We further develop in this article the theory of QL- mappings, which was started by A. I. Shnirelman ([5]), continued by M.A.Ephendiev ([3]) and also by myself ([1]). As was proved in [1], the classes FQL and $FSQL$-mappings coincide; however the latter class is more adapted to expansion on affine bundles, which are used in definition of the QL-manifold. As an example, we introduce a QL-manifold structure on the Banach manifold $H_s(S^1, S^2)$. This example shows that QL-manifold structures can be introduced on various classes of mappings. As an example of a QL-mapping, we can take F_f : $H_s(S^1, S^2) \to H_s(S^1, S^2)$, where $f : S^2 \to S^2$ is diffeomorphism. We provide definitions of FQL and $FSQL$-mappings in the appendix.

1. Definitions

Let X be a real infinite-dimensional Banach manifold, and $\{X_j\}$, $X_{j-1} \subset X_j$, $j = 1, 2, ...$ is a system of open sets, exhausting X, i.e. $X = \cup X_j$. Let us suppose $\xi_j = \left(Y_j, \psi_j, B_{n_j}\right)$ is an affine bundle, where Y_j is a total space, B_{n_j} is a basis which is a finite-dimensional manifold with boundary, and $\psi_j : Y_y \to B_{n_j}$ is the continuous epimorphism. Let Ω_j be a bounded domain in Y_j, $\varphi_j : X_j \to \Omega_j$ be a homeomorphism. (φ_j, X_j) will be called a chart on X. After carrying out the conditions given above we say that on X_j a linear ($L-$) structure is introduced. If a $L-$structure is defined on X_{j+1}, then obviously, it has been defined on X_j, too (as an induced structure). If $\varphi_{j^/} : X_{j^/} \to \Omega_{j^/}$, $\varphi_{j^{//}} : X_{j^{//}} \to \Omega_{j^{//}}$, $j^/, j^{//} \geq j$, are two $L-$structures on X_j, then the mappings of transition $\varphi_{j^{//}} \circ \varphi_{j^/}^{-1} : \Omega_{j^/} \to \Omega_{j^{//}}$ and $\varphi_{j^/} \circ \varphi_{j^{//}}^{-1} : \Omega_{j^{//}} \to \Omega_{j^/}$ will arise. Let us consider them in charts of affine bundles $\xi_{j^/} = \left(Y_{j^/}, \psi_{j^/}, B_{n_{j^/}}\right)$ and $\xi_{j^{//}} = \left(Y_{j^{//}}, \psi_{j^{//}}, B_{n_{j^{//}}}\right)$. Let us suppose that they are FQL-mappings (see [5]). In that case, we say that two $L-$structures on X_j are equivalent.

Definition 1 *A class of equivalent $L-$structures on X_j is called a FQL- structure on X_j.*

Obviously, the FQL- structure on X_{j+1} an induces FQL- structure on X_j, as well. The FQL- structure on X_j is coordinated with the FQL- structure on X_{j+1}, if it coincides with the induced structure.

Definition 2 *A collection of FQL- structures on X_j, j = 1,2,3,..., coordinated between each other is called a FQL- structure on X.*

The Banach manifold X with the FQL- structure is called a FQL-manifold.

Now let us define a FSQL-mapping between FQL- manifolds.

Let X, Y be FQL-manifolds, $X = \cup X_i, X_i \subset X_{i+1} \quad \forall i$, $Y = \cup Y_j, Y_j \subset Y_{j+1} \quad \forall j$, (φ_i, X_i), (ψ_j, Y_j) be $L-$charts on X, Y, $\varphi_i(X_i) = \Omega_i$, and $\psi_j(Y_j) = \Theta_j$ be bounded domains of affine bundles ξ_i, η_j, respectively.

Definition 3 *A continuous mapping $f : X \to Y$ between FQL-manifolds X and Y is called a FSQL-mapping, if*

a) $\forall i \quad \exists j(i)$, $f(X_i) \subset Y_{j(i)}$; and

b) $\psi_j \circ f \circ \varphi_i^{-1} : \Omega_i \to \Theta_j$ is FSQL-mapping (see [1]).

2. Example of *FQL*-Manifold

Let S^1 be circle, x be coordinate on S^1, $0 \leq x < 2\pi$; S^2 be 2-dimensional sphere, embedded in R^3, $i : S^2 \to R^3$ be embedding mapping. Let a set X consist of mappings $u : S^1 \to S^2$ of class H_s, i.e. $\partial_x^k (i \circ u) \in L^2\left(S^1, R^3\right)$, $0 \leq k \leq s$,

$$\|u\|_s^2 = \sum_0^s \int_0^{2\pi} \left\| \partial_x^k (i \circ u)(x) \right\|_{R^3}^2 dx \quad , \tag{1}$$

where s is some natural number. Obviously, one can introduce in X the structure of infinite-dimensional smooth manifold (see [4]). Its model space is the real Hilbert space $H_s\left(S^1, R^2\right)$.

Now let us introduce a FQL- structure on X. Suppose that X is naturally embedded in $H_s(S^1, R^3)$ with norm (1), $X_j = \{u \in X \mid \|u\|_s < j\}$, j be some natural number. For the solution of this problem we will: construct an affine bundle (Y_j, P_j, B_j) with finite-dimensional base B_j; pick out in Y_j a bounded domain D_j; construct homeomorphisms $\Phi_j : D_j \to X_j$ (linear charts), $j = 1, 2, 3, ...$; and prove that homeomorphisms $\Phi_i^{-1} \circ \Phi_j : D_j \to D_i$ are FQL-mappings.

It is easy to prove the following lemma.

Lemma 4 $\exists \delta(j, s) > 0$, $\forall u \in X_j \quad \exists y(u) \in S^2$, $\|y - u(x)\|_{R^3} > \delta \quad \forall x \in S^1$.

Let N be some natural number, and $x_1, ..., x_N$ be N equidistant points on S^1. Let us put in a correspondence to each mapping $u \in X_j$ in point $p_N(u) = (u(x_1), ..., u(x_N)) \in \left[S^2\right]^N$.

Let $B_N = \{\bar{y} = (y_1, ..., y_N) \in \left[S^2\right]^N \mid \exists u \in X_j,\ u(x_1) = y_1, u(x_2) = y_2, ..., u(x_N) = y_N\}$.

Obviously, B_N is a domain in $\left[S^2\right]^N$; therefore it will be a manifold of dimension $2N$.

Lemma 5 *At fixed j and sufficiently large N for each point $\bar{y} \in B_N$, there exists a mapping $U_{\bar{y}} \in H_s(S^1, S^2)$, satisfying conditions $U_{\bar{y}}(x_i) = y_i$, $i = \overline{1, N}$.*

Proof. Let $\bar{U}_{\bar{y}} : S^1 \to R^3$ be a mapping such that $\bar{U}_{\bar{y}}(x_i) = y_i$, $i = \overline{1, N}$ and $\left\|\bar{U}_{\bar{y}}\right\|_s$ has a minimum among all these mappings. That such a mapping $\bar{U}_{\bar{y}}(x)$ exists, is unique and continuously depends on $\bar{y}$, follows from the convexity of functional $u \mapsto \|u\|_s^2$ (see [6]). In this case, for $\|\bar{U}_{\bar{y}}\|_s < j$, as according to the construction, there exists such mapping $u \in X_j$, such that $p_N(u) = \bar{y}$, and for all this $u(x)$, $\left\|\bar{U}_{\bar{y}}\right\|_s \le \|u\|_s$.

As known, S^2 has some tubular neighborhood in R^3. Let us denote its radius by $\varepsilon > 0$. In this neighborhood for each point y exists the nearest point $\psi(y) \in S^2$ to it;

moreover the mapping $y \mapsto \psi(y)$ is smooth, surjective and nondegenerative in it. As $\|u\|_{C^1} \leq K \cdot \|u\|_s$ at $s \geq 3$, then from $\|u\|_s < j$ it follows that $\|u\|_{C^1} \leq K \cdot j$; that is $\forall u \in H_s\left(S^1, R^3\right)$, $\|u\|_s < j$, $\|u'(x)\|_{R^3} < K \cdot j$. Then

$\forall x_1, x_2 \in S^1$, $\forall u \in H_s\left(S^1, R^3\right)$, $\|u\|_s < j$, and $\|u(x_1) - u(x_2)\|_{R^3} < K \cdot j \cdot |x_1 - x_2|$.

Let us suppose that $|x_1 - x_2| < \varepsilon/(K \cdot j)$. Then $K \cdot j \cdot |x_1 - x_2| < \varepsilon$. Therefore,

$\forall u \in H_s\left(S^1, R^3\right)$, $\|u\|_s < j$, and $\|u(x_1) - u(x_2)\|_{R^3} < \varepsilon$ at $|x_1 - x_2| < \varepsilon/(K \cdot j)$.

Let N be such that the distance between neighbor points $x_1, ..., x_N \in S^1$ is less than $\varepsilon/(K \cdot j)$. Then

$$\forall u \in H_s\left(S^1, R^3\right), \quad \|u\|_s < j, \quad \forall i = \overline{1, N} \quad \|u(x_i) - u(x_{i+1})\| < \varepsilon.$$

Let $x \in S^1$. Obviously, $\exists i, |x - x_i| < \varepsilon/(K \cdot j)$. Therefore

$$\forall u \in H_s\left(S^1, R^3\right), \quad \|u\|_s < j, \quad \|u(x) - u(x_i)\|_{R^3} < \varepsilon \quad .$$

From all this follows that the curve $u(x)$ belongs to the ε- tubular neighborhood of S^2 in R^3, if $\|u\|_s < j$ and $u(x_i) \in S^2$, $i = \overline{1, N}$. Therefore it can be smoothly projected on S^2. As $\left\|\bar{U}_{\bar{y}}\right\| < j$, then all of this is right for $\bar{U}_{\bar{y}}$. Let us denote $U_{\bar{y}}(x) = \psi \circ \bar{U}_{\bar{y}}(x)$. According to the construction, this mapping also belongs to $p_N^{-1}(\bar{y})$, that is $U_{\bar{y}}(x_i) = y_i, i = \overline{1, N}$.

By this the proof of the lemma 5 is finished. □

From smoothness of ψ follows

$$\|U_{\bar{y}}\|_s \leq C \cdot \left\|\bar{U}_{\bar{y}}\right\|_s < C \cdot j.$$

So, generally, $U_{\bar{y}} \notin X_j$, but also $U_{\bar{y}} \in X_{C \cdot j}$.

Now let $\exp_y : T_y S^2 \to S^2$ be an exponential mapping. As it is known, $\exp_y(\vec{g})$ is diffeomorphism from some $\delta_1(y)$-neighborhood of zero in $T_y S^2$on some $\varepsilon_1(y)$-neighborhood of point $y \in S^2$. We can suppose that $\varepsilon_1(y)$ and $\delta_1(y)$ are independent on $y \in S^1$, because $\exp_y(\vec{g})$ is smooth and S^2 is compact.

Let us prove that the ε_1-neighborhood of the curve $u(x)$, $u \in X_{C \cdot j}$, includes all the curves from $p_N^{-1}(p_N(u)) \cap X_{C \cdot j}$ for a large enough N. Analogous to what was proved earlier, it can be shown that

$$\exists K_1 \geq K, \quad \forall u \in X_{C\cdot j}, \quad \forall x_1, x_2 \in S^1 \quad \|u(x_1) - u(x_2)\|_{R^3} < K_1 \cdot |x_1 - x_2|.$$

Then $\forall u_1 \in p_{_N}^{-1}(p_{_N}(u)) \cap X_{C\cdot j} \quad \|u_1(x) - u(x)\|_{R^3} \leq \|u_1(x) - u_1(x_i)\|_{R^3} + \|u_1(x_i) - u(x_i)\|_{R^3} + \|u(x_i) - u(x)\|_{R^3} < 2K_1|x_{i+1} - x_i|$.

Let N be such a natural number that $\forall i \quad |x_i - x_{i+1}| < \varepsilon_1/(2 \cdot K_1)$. Then

$$\forall u \in X_{C\cdot j}, \quad \forall u_1 \in p_{_N}^{-1}(p_{_N}(u)) \cap X_{C\cdot j} \quad \|u_1(x) - u(x)\|_{R^3} < \varepsilon,$$

as was confirmed above. Because of the arbitrary $u \in X_{C\cdot j}$ this statement is also right for the element $U_{\bar{y}}$.

Let $\bar{y}_0 \in B_N$. Let us construct in the neighborhood of curve $U_{\bar{y}_0}(x)$ two vector fields tangent to S^2, orthogonal to each other and having the unit length. Let us denote them by $\vec{g}_1(y)$ and $\vec{g}_2(y) : (\vec{g}_1(y), \vec{g}_2(y)) \equiv \delta_{1,2}$, where $\delta_{1,2}$ is the Kronecker symbol. At first, such fields can be constructed on R^2, then transferred on S^2, by lemma 4 and stereographic projection. According to the construction, such vector fields will be defined on each curve $U_{\bar{y}}(x)$, where $\bar{y} \in \theta_{\bar{y}_0}$, $\theta_{\bar{y}_0}$ is δ-neighborhood of point $\bar{y}_0$ in B_N. B_N can be covered by finite number of such δ-neighborhoods $\theta_{\bar{y}_1}$, ..., $\theta_{\bar{y}_l}$, where $\bar{y}_1$,..., $\bar{y}_l$ are some points from B_N, as B_N is relatively compact and finite-dimensional. Let $F^N = \left\{\vec{v} \in S^1 \to R^2 | \vec{v} \in H_s, v(x_1) = \cdots = v(x_N) = 0\right\}$, which a linear subspace of $H_s\left(S^1, R^2\right)$, with finite-co-dimension $2N$. Let $Y\left(\theta_{\bar{y}_p}\right) = \theta_{\bar{y}_p} \times F^N \quad p = \overline{1, l}$, $\{\vec{e}_1, \vec{e}_2\}$ be on orthonormed base in R^2. Obviously, each function $\vec{v} \in F^N$ has the following form in this base: $\vec{v}(x) = v_1(x) \cdot \vec{e}_1 + v_2(x) \cdot \vec{e}_2$, where $v_k(x)$, $k = 1, 2$, is scalar function, $v_k \in H_s\left(S^1, R^1\right)$, $v_k(x_i) = 0$, $k = 1, 2$, $i = \overline{1, N}$. Let us consider the mapping

$$\Phi_p : \theta_{\bar{y}_p} \times F^N \to p_{_N}^{-1}\left(\theta_{\bar{y}_p}\right), \quad \Phi_p(\bar{y}, \vec{v}) = \exp_{U_{\bar{y}(x)}} \vec{g}(x), \quad p = \overline{1, l},$$

where $\vec{v}(x) = v_1(x) \cdot \vec{e}_1 + v_2(x) \cdot \vec{e}_2$, $\vec{g}(x) = v_1(x) \cdot \vec{g}_1(U_{\bar{y}}(x)) + v_2(x) \cdot \vec{g}_2(U_{\bar{y}}(x))$. Obviously,

1) at $\bar{y}' \neq \bar{y}''$, $\bar{y}', \bar{y}'' \in \theta_{\bar{y}_p}$, $\Phi_p(\bar{y}', \vec{v}) \neq \Phi_p(\bar{y}'', \vec{w}) \quad \forall \vec{v}, \vec{w} \in F^N$, as (according to the construction) $\Phi_p(\bar{y}', \vec{v}) \in p_{_N}^{-1}(\bar{y}')$, and $\Phi_p(\bar{y}'', \vec{w}) \in p_{_N}^{-1}(\bar{y}'')$,

2) at $\|\vec{v}\|_C < \delta_1$, $\|\vec{w}\|_C < \delta_1$, $\vec{v} \neq \vec{w}$, $\Phi_p(\bar{y}, \vec{v}) \neq \Phi_p(\bar{y}, \vec{w}) \quad \forall \bar{y} \in \theta_{\bar{y}_p}, \forall p = \overline{1, l}$, as $\exp_y \vec{g}$ is diffeomorphism in δ_1 - neighborhood of $0_y \in T_y S^2$.

From these reasons, it follows, that the mapping Φ_p, $p = \overline{1,l}$, is a diffeomorphism between $\theta_{\bar{y}_p} \times \left\{ \vec{v} \in F^N \left| \|\vec{v}\|_C < \delta_1 \right. \right\}$ and neighborhood $\left\{ u(x) \left| \|U_{\bar{y}}(x) - u(x)\|_C < \varepsilon_1 \right. \right\}$, where $\bar{y} \in \theta_{\bar{y}_p}$, $p_{_N}(u(x_i)) = p_{_N}(U_{\bar{y}}(x_i))$,$i = \overline{1,N}$. According to the construction, this neighborhood contains the set $p_{_N}^{-1}\left(\theta_{\bar{y}_p}\right) \cap X_j$.

Obviously, $D_p = \Phi_p^{-1}\left(p_{_N}^{-1}\left(\theta_{\bar{y}_p}\right) \cap X_j\right)$ is a bounded domain from $Y\left(\theta_{\bar{y}_p}\right)$. Let us paste together domains D_p, $D_{p'}$, $p, p' = \overline{1,l}$, by diffeomorphisms $\Phi_p^{-1} \circ \Phi_{p'}$. As a result we get some set D_j. Now let us construct an affine bundle, in which D_j will be a bounded domain. Let $(\vec{g}_{1,p}(y), \vec{g}_{2,p}(y))$, $(\vec{g}_{1,p'}(y), \vec{g}_{2,p'}(y))$ be vector fields, defined in neighborhoods of the curves $U_{\bar{y}_p}(x)$ and $U_{\bar{y}_{p'}}(x)$, $\bar{y} \in \theta_{\bar{y}_p} \cap \theta_{\bar{y}_{p'}}$, respectively.

Let $\lambda_{p,p',\bar{y}}(x)$ be an orthogonal matrix, transferring the first base to the second in point $y = U_{\bar{y}}(x)$. Let us put in correspondence to the element $(\bar{y}, \vec{v}) \in \theta_{\bar{y}_p} \times F^N$ the element $(\bar{y}, \vec{w}) \in \theta_{\bar{y}'} \times F^N$, where

$$\vec{w}(x) = \lambda_{p,p',\bar{y}}(x) \cdot \vec{v}(x). \tag{2}$$

This mapping is a linear isomorphism, depending smoothly on $\bar{y} \in \theta_{\bar{y}_p} \cap \theta_{\bar{y}_{p'}}$. Pasting together all simple bundles $\theta_{\bar{y}_p} \times F^N$, $p = \overline{1,l}$, by these diffeomorphisms, we get an affine bundle. Let us denote it by (Y_j, P_j, B_j). It can be shown, that $\Phi_p^{-1} \circ \Phi_{p'} : (\bar{y}, \vec{v}) \mapsto (\bar{y}, \vec{w})$, where $\vec{w}(x) = \lambda_{p,p',\bar{y}}(x) \cdot \vec{v}(x)$.

Hence it follows that D_j is the bounded domain in Y_j.

Now let us paste together diffeomorphisms $\Phi_1, ..., \Phi_l$ by transition functions. As a consequence we get one diffeomorphism from D_j on X_j. Let us denote it by Φ_j. With it is completed construction of the linear chart $\left(\Phi_j^{-1}, X_j\right)$ on X_j.

Now let us show that the linear structures on X_j and X_i are coordinated at different j and i, that is the mapping of transition $\Phi_j^{-1} \circ \Phi_i$ is a *FSQL*- mapping between domains of affine bundles.

Let $(x_1, ..., x_N)$, $(x_1', ..., x_L')$ be points on S^1, used as a definition of $L-$structure on X_j and X_i, $\bar{y} = (y_1, ..., y_N)$, $\bar{y}' = (y_1', ..., y_L')$ be points from B_j, B_i, respectively, $U_{\bar{y}}(x)$, $U_{\bar{y}'}(x)$ be corresponding mappings constructed by the method mentioned above. Let

$F^N = \{\vec{v} \in H_s(S^1, R^2) | v(x_1) = \cdots = v(x_N) = 0\}, F^L = \{\vec{v} \in H_s(S^1, R^2) | v(x'_1) = \cdots$

$= v(x'_L) = 0\}$

be vector subspaces $H_s\left(S^1, R^2\right)$ (co-dimensions $2N$ and $2L$), which are isomorphic to layers from (Y_j, P_j, B_j), (Y_i, P_i, B_i), respectively. Without loss of generality, it can be supposed that $x_m \neq x'_n$, $m = \overline{1, N}$,$n = \overline{1, L}$. Obviously,

$$F^{N+L} = \left\{\vec{v} \in H^s\left(S^1, R^2\right) \middle| v\left(x_m\right) = v\left(x'_n\right) = 0, m = \overline{1, N}, n = \overline{1, L}\right\}, \quad F^N = F^{N+L} + F_L,$$

where F_L is orthogonal complement to F^{N+L} in F^N. And

$\theta_{\bar{y}_p} \times F^N = \left(\theta_{\bar{y}_p} \times F_L\right) \times F^{N+L} = \underset{\bar{y}}{\cup}\underset{\alpha}{\cup} F^{N+L}_{\bar{y},\alpha}, \quad \bar{y} \in \theta_{\bar{y}_p}, \quad \alpha \in F_L$, where $F^{N+L}_{\bar{y},\alpha} = (\bar{y}, \alpha) \times F^{N+L}$, $p = \overline{1, l}$. Moreover, $\theta_{\bar{y}_p} \times F_L = \underset{\bar{y}}{\cup} F_{L,\bar{y}}$, where $F_{L,\bar{y}} = \bar{y} \times F_L$, $\bar{y} \in \theta_{\bar{y}_p}$,$p = \overline{1, l}$. Pasting together simple bundles $\left(\theta_{\bar{y}_y} \times F_L\right) \times F^{N+L}$, $p = \overline{1, l}$, by diffeomorphisms (2), we get an affine bundle, which is subbundle of (Y_j, P_j, B_j), in this case each layer of the last bundle "divided" into parallel planes by layers of subbundle. Let us denote this subbundle by $(Y_j, P_{j,i}, B_{j,i})$. Let us paste together simple bundles $\theta_{\bar{y}_p} \times F_L$,$p = \overline{1, l}$, by these diffeomorphisms. As a consequence we get a finite-dimensional (namely, $2 \cdot (N + L)$-dimension) affine bundle. Without restriction of generality, it can be supposed that $B_{j,i}$ is a total space of last. Let $(\bar{y}, z) \in \theta_{\bar{y}_p} \times F_L$. Let us consider the function

$$u\left(x\right) = \exp_{U_{\bar{y}(x)}} \left(\sum_{k=1}^{2} \left(z_k\left(x\right) + v_k\left(x\right)\right) \vec{g}_k\left(U_{\bar{y}}\left(x\right)\right) \right),$$

where $v_k\left(x_m\right) = v_k\left(x'_n\right) = 0$, that is $\vec{v} = (v_1, v_2) \in F^{N+L}$. For all such $u\left(x\right)$, $u\left(x_m\right) = y_m$, $u\left(x'_n\right) = y'_n$, $m = \overline{1, N}$, $n = \overline{1, L}$. Therefore

$$\exp^{-1}_{U_{\bar{y}'(x)}} u\left(x\right) = \left(\bar{y}', w\left(x\right)\right), \quad \bar{y}' = \left(y'_1, ..., y'_L\right),$$

for all these $u\left(x\right)$. Otherwise, $\Phi_i^{-1} \circ \Phi_j$ will transfer the layer $P^{-1}_{j,i}\left(\bar{y}, \vec{z}\right)$ over point $(\bar{y}, \vec{z})$ into layer $P_i^{-1}\left(\bar{y}'\right)$ over point $\bar{y}'$, where $\bar{y}' = \left(u\left(x'_1\right), ..., u\left(x'_L\right)\right)$. Therefore it transfers

$P_{j,i}^{-1}(\theta_{\bar{y},\vec{z}})$ into $P_i^{-1}\left(\theta_{\bar{y}'_q}\right)$, where $\bar{y'} \in \theta_{\bar{y}'_q}$, $\theta_{\bar{y}'_q}$ is some chart from a fixed atlas on B_i, and $\theta_{\bar{y},\vec{z}}$ is some neighborhood of $(\bar{y},\vec{z})$ in $B_{j,i}$. This function of transition has the following form:

$$(\bar{y},\vec{z},\vec{v}) \mapsto \left(\bar{y'},(w_1,w_2)\right) = \left(\bar{y'},\vec{w}\right),$$

where $u(x) = \Phi_j(\bar{y},\vec{z}+\vec{v})$, $\bar{y'} = (u(x'_1),...,u(x'_L))$
and $w_k(x) = \left(\vec{g}_k\left(U_{\bar{y'}}(x)\right), h(x)\right)$, ($h(x) = \exp^{-1}_{U_{\bar{y'}}(x)} u(x)$, $h(x) \in T_{U_{\bar{y'}}(x)}S^2$), $k = 1,2$, is scalar multiples of vectors, tangent to S^2 at point $U_{\bar{y'}}(x)$. From mentioned formulas follow that the function of transition $\Phi_i^{-1} \circ \Phi j$ between linear charts on X_j and X_i is given by operators of composition with smooth functions in charts of the corresponding bundles. According to [5] such an operator defines a QL-mapping. $\Phi_i^{-1} \circ \Phi j$ will be FQL and therefore a $FSQL$-mapping in charts of affine bundles, as all used functions have different from zero gradients at all points. So, all conditions of the definition of $FSQL$-mapping are satisfied ([1]). That is why $\Phi_i^{-1} \circ \Phi_j$ will be a $FSQL$-mapping between domains of affine bundles.

From all of this follows that the structure introduced in X is Fredholm Quasi-Linear.

3. Example of *FSQL*-Mapping

Let $f : S^2 \to S^2$ be diffeomorphism, X, X' be FQL-manifolds, $X = X' = H_s\left(S^1, S^2\right)$.

Thus we have the mapping $F_f : X \to X'$, $F_f : u \mapsto f(u)$, which, incidentally is a diffeomorphism (inverse mapping is $F_{f^{-1}}$). Let us show that F_f is $FSQL$-mapping between FQL-manifolds. Let us denote that F_f is a bounded mapping, as $\|f \circ u\|_s \leq C \cdot \|u\|_s$. On the other hand, let $X_1,...,X_j,...$, $X_1 \subset X_2 \subset \cdot\cdot \subset X_j \subset \cdots$, $\underset{j}{\cup} X_j = X$ and $X'_1,...,X'_i,...$, $X'_1 \subset X'_2 \subset \cdots \subset X'_i \subset \cdots$, $\underset{j}{\cup} X'_j = X'$ be domains, taken as in the definition of QL-manifold. From the boundedness F_f it follows that $\forall j \quad \exists i$, $F_f(X_j) \subset X'_i$. Let (Y_j, P_j, B_j), (Y'_i, P'_i, B'_i) be affine bundles, according to X_j, X'_i, and defined as in the example of FQL-manifold. Let $(x_1,...,x_N)$, $(x'_1,...,x'_L)$ be points on S^1, used as a definition of L-structure on X_j and X'_i, and $\bar{y} = (y_1,...,y_N)$, $\bar{y}' = (y'_1,...,y'_L)$ be points from B_j and B'_i, respectively. As in the first example, let us take “dividing”[1]

[1] An (affine) bundle (Y_1, p_1, B_1) is called a “dividing” of a (affine) bundle (Y_2, p_2, B_2), if $Y_1 = Y_2$ and $\forall \alpha \in B_1 \exists \beta \in B_2$, $p_1^{-1}(\alpha) \subset p_2^{-1}(\beta)$.

$(Y_j, P_{j,i}, B_{j,i})$ and some layer on $(\bar{y}, \vec{z})^2$, $(\bar{y}, \vec{z}) \in \theta_{\bar{y}_p} \times F_L$. As we have noticed in the first example, the function

$$u(x) = \exp_{U_{\bar{y}}(x)} \left(\sum_1^2 (z_k(x) + v_k(x)) \cdot \vec{g}_k \left(U_{\bar{y}}(x) \right) \right),$$

where $v_k(x_m) = v_k(x'_n) = 0$, $m = \overline{1, N}$, $n = \overline{1, L}$, translates the points x_m, x'_n to points $y_m = u(x_m)$, $y'_n = u(x'_n)$. Then the mapping $\bar{u}(x) = f(u(x))$ will translate the points x_m, x'_n to points $t_m, t'_n \in S^2$, where $t_m = \bar{u}(x_m)$, $t'_n = \bar{u}(x'_n)$, $m = \overline{1, N}$, $n = \overline{1, L}$. Hence the layer on point $(\bar{y}, \vec{z})$ will be mapped (by operator F_f) in layer on point $\bar{t}'$, $\bar{t}' = (t'_1, ..., t'_L)$. That is why for some neighborhood $\theta_{\bar{y},\bar{z}}$ of point $(\bar{y}, \vec{z})$ the set $P_{j,i}^{-1}\left(\theta_{\bar{y},\bar{z}}\right)$ will be translated in $(P'_i)^{-1}\left(\theta_{\bar{t}'_q}\right)$, where $\bar{t}' \in \theta_{\bar{t}'_q}$ and $\theta_{\bar{t}'_q}$ is a chart from fixed atlas on B'_i. In charts of aforesaid bundles, F_f appears as follows: $(\bar{y}, \vec{z}, \vec{v}) \mapsto (\bar{t}', w_1, w_2) = (\bar{t}', \vec{w})$, $u(x) = \exp_{U_{\bar{y}}(x)} \left(\sum_1^2 (z_k(x) + v_k(x)) \cdot \vec{g}_k(U_{\bar{y}}(x)) \right)$, $\bar{t}' = (f(u(x'_1)), ..., f(u(x'_L)))$; $\vec{h}'(x) = \exp^{-1}_{U_{\bar{t}'}(x)} f(u(x))$ $(\vec{h}'(x) \in T_{U_{\bar{t}'}(x)} S^2)$, $w_k(x) = \left(\vec{g}'_k\left(U_{\bar{t}'}(x)\right), \vec{h}'(x) \right)$, $k = 1, 2$, is scalar multiple of vectors, tangent to S^2 at point $U_{\bar{t}'}(x)$. The above formulas show that F_f is defined by operators of composition with smooth functions in charts of bundles of X_j and X'_i. According to [5], such an operator defines a *QL*-mapping between local charts of affine bundles. As f is a diffeomorphism and all used functions have different from zero gradients at all points, then according to [5], F_f will be *FQL* and hence, *FSQL*-mapping (see [1]) in charts of affine bundles. Therefore F_f will be *FQL*-mapping between linear charts of *FQL*-manifolds X and X', hence *FSQL*- mapping between X and X'.

4. Appendixes[3]

A*) FQL*-mapping. Let X, Y be real Banach spaces, Ω be a bound domain in X, X_n be a n-dimensional space. Let $X_\alpha^n = \pi^{-1}(\alpha)$, $\alpha \in X_n$.

Definition 6 *A continuous mapping $f^n : \Omega \to Y$ is called a Fredholm Linear (FL), if*

[2]For all $\gamma \in B$, the set $p^{-1}(\gamma)$ is called a layer of (affine) bundle (Y, p, B) on point $\gamma \in B$.

[3]Appendix (A) is taken from article [5] and appendix (B) from article [1].

a) some linear mapping $\pi_n : X \to X_n$ *is fixed;*
b) on each plane $X^n_\alpha = \pi^{-1}(\alpha)$, $\alpha \in X_n$, *passing through* Ω, $f^n_\alpha \equiv f^n|_{X^n_\alpha}$ *is an affine invertible mapping from* X^n_α *on to its image* $Y^n_\alpha = f(X^n_\alpha)$, *that is, closed in* Y *and has co-dimension* n;
c)f^n_α *depends continuously on* α.

Definition 7 *A continuous mapping* $f : X \to Y$ *is said to be Fredholm Quasi-Linear (FQL), if there exists a sequence FL-mgappings* $\{f^{n_k}\}$, *uniformly approximating* f *on each bounded domain* $\Omega \subset X$, *such that*

$$\|f^{n_k}_\alpha\| < C(\Omega), \left\|(f^{n_k}_\alpha)^{-1}\right\| < C(\Omega),$$

with $k > k_0(\Omega)$, *if* $\alpha \in \pi_{n_k}(\Omega)$ *and* $C(\Omega)$ *does not depend on*k, *and if* $k > k_0(\Omega)$.

B) FSQL- mapping. *Let* H_1 *and* H_2 *be real Hilbert spaces,* $\|\|_1$, $\|\|_2$ *be the corresponding norms in them. Let* $\{X^n_\alpha\}$, $\alpha \in M_n$, *be a family of pairs of disjoint closed planes in* H_1 *of codimension* n, *continuously depending on* α, M_n *is manifold of dimension* n. *Suppose that* $\left\{Y^n_\beta\right\}$, $\beta \in N_n$, *is an analogous family in* H_2. *Let* $\tilde{M}_n = \bigcup_\alpha X^n_\alpha$, $\tilde{N}_n = \bigcup_\beta X^n_\beta$. *Let us determine the projections* $\pi_n : \tilde{M}_n \to M_n$, $p_n : \tilde{N}_n \to N_n$ *in the following way* $\pi_n : x \mapsto \alpha$, *if* $x \in X^n_\alpha$; $p_n : y \mapsto \beta$, *if* $y \in Y^n_\beta$. *It is obvious that the triples* $\xi = (\pi_n, \tilde{M}_n, M_n)$ *and* $\eta = (p_n, \tilde{N}_n, N_n)$ *are affine bundles.*

Definition 8 *A continuous mapping* $f : \tilde{M}_n \to \tilde{N}_n$ *is called Fredholm Special Linear (FSL), if* $\forall\alpha \in M_n$, $f^n_\alpha \equiv f|_{X^n_\alpha}$ *is an affine invertible mapping from* X^n_α *on some* Y^n_β, $f^n_\alpha \in Aff(X^n_\alpha, Y^n_\beta)$ *and* f^n_α *depends continuously on* α.

The restriction of FSL-mapping on any domain Ω, $\bar{\Omega} \subset \tilde{M}_n$, *is also called FSL-mapping.*

It is obvious that FSL-mapping induces bimorphism between affine bundles ξ *and* η.

Let Ω, $\bar{\Omega} \subset \tilde{M}_n$, *be a bounded domain in* H_1, $f : \Omega \to H_2$ *be an FSL-mapping and*

$$|||f|||_\Omega = \sup_{x^n_\alpha \cap \Omega \neq \phi} \inf\{C \mid \|f^n_\alpha(x)\|_2 \le C(1 + \|x\|_1), \|x\|_1 \le C(1 + \|f^n_\alpha(x)\|_2), x \in X^n_\alpha \}.$$

Definition 9 *A continuous mapping $f : \Omega \to H_2$ is called Fredholm Special Quasi-Linear (FSQL), if there exists a sequence of FSL-mappings $f^{n_i} : \Omega \to H_2$, $i = 1, 2, \ldots$, uniformly approximating f on Ω and*

$$|||f|||_\Omega \leq C(\Omega), \quad \forall i > i(\Omega);$$

moreover, $C(\Omega)$ does not depend on i for $i > i(\Omega)$.

Definition 10 *A continuous mapping $f : H_1 \to H_2$ is called FSQL-mapping, if in any bounded domain $\Omega \subset H_1$ it is the FSQL-mapping.*

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