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Radical Submodules and Uniform Dimension of Modules

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Abstract

We investigate the relations between a radical submodule N of a module M being a finite intersection of prime submodules of M and the factor module M/N having finite uniform dimension. It is proved that if N is a radical submodule of a module M over a ring R such that M/N has finite uniform dimension, then N is a finite intersection of prime submodules. The converse is false in general but is true if the ring R is fully left bounded left Goldie and the module M is finitely generated. It is further proved that, in general, if a submodule N of a module M is a finite intersection of prime submodules, then the module M/N can have an infinite number of minimal prime submodules.

1. Introduction

Throughout this note all rings are associative with identity and all modules are unital left modules. Let R be a ring and let M be an R-module. A submodule K of M is called *prime* if $K \neq M$ and whenever $r \in R$ and L is a submodule of M such that $rL \subseteq K$ then $rM \subseteq K$ or $L \subseteq K$. In this case, the ideal $P = \{r \in R : rM \subseteq K\}$ is a prime ideal of R and we call K a P-*prime* submodule of M. For more information about prime submodules of M see, for example, [3]–[8] and [10]. A submodule N of a module M is called a *radical* submodule if N is an intersection of prime submodules of M. Note that radical submodules are proper submodules of M.

Given a submodule N of a module M, a decomposition $N = K_1 \cap \cdots \cap K_n$ in terms of submodules $K_i (1 \leq i \leq n)$ of M, where n is a positive integer, is called *irredundant* if $N \neq K_1 \cap \cdots \cap K_{i-1} \cap K_{i+1} \cap \cdots \cap K_n$ for all $1 \leq i \leq n$. In [11], a submodule N of a module M is said to have a *prime decomposition* if N is the intersection of a finite collection of prime submodules of M. Let N be a submodule of an R–module M such that N has a prime decomposition. Then N will be said to have a *normal prime decomposition* if there exists a positive integer n, distinct prime ideals $P_i (1 \leq i \leq n)$ of R and P_i-prime submodules K_i $(1 \leq i \leq n)$ of M such that $N = K_1 \cap \cdots \cap K_n$ is an irredundant decomposition.

Lemma 1.1 *(See [11, Corollary 2, Theorem 3 and Lemma 14].) Let R be any ring and let N be a submodule of an R-module M such that N has a prime decomposition. Then N has a normal prime decomposition. Moreover, if $N = K_1 \cap \cdots \cap K_n$ and $N = L_1 \cap \cdots \cap L_k$ are normal prime decompositions of N where K_i is P_i-prime for some prime ideal $P_i (1 \leq i \leq n)$ and L_j is Q_j-prime for some prime ideal $Q_j (1 \leq j \leq k)$, then $n = k$ and $\{P_i : 1 \leq i \leq n\} = \{Q_j : 1 \leq j \leq k\}$.*

In Lemma 1.1, the prime ideals P_i $(1 \leq i \leq n)$ are called the *associated prime ideals* of N. Given submodules G, H of an R-module M we set $(G : H) = \{r \in R : rH \subseteq G\}$. Note that $(G : H)$ is an ideal of R. Moreover, $(G : H) = R$ if and only if $H \subseteq G$.

Lemma 1.2 *(See [11, Theorem 6].) Let R be any ring and let N be a submodule of an R-module M such that N has a prime decomposition. Then a prime ideal P of R is an associated prime ideal of N if and only if $P = (N : L)$ for some submodule L of M.*

A module M has *finite uniform dimension* if M does not contain a direct sum of an infinite number of non-zero submodules. Also, a non-zero module M is *uniform* if $X \cap Y \neq 0$ for all non-zero submodules X and Y of M.

Lemma 1.3 *(See [9, 2.2.7, 2.2.8, 2.2.9].) A non-zero R-module M has finite uniform dimension if and only if there exist a positive integer n and independent uniform submodules U_i $(1 \leq i \leq n)$ of M such that $U_1 \oplus \cdots \oplus U_n$ is an essential submodule of M. Moreover, if $V_i (1 \leq i \leq k)$ are independent uniform submodules of M such that $V_1 \oplus \cdots \oplus V_k$ is essential in M then $n = k$.*

In Lemma 1.3, the positive integer n is called the *uniform* (or *Goldie*) *dimension* of M and is denoted by $u(M)$. Let N be a submodule of a module M. By Zorn's Lemma the collection of submodules L of M such that $L \cap N = 0$ has a maximal member and any such is called a *complement of* N (*in* M). A submodule K of M is called a *complement* (*in* M) if there exists a submodule N of M such that K is a complement of N.

Lemma 1.4 *(See [2, 1.10 and 5.10].) Let L, N be submodules of a module M with $L \cap N = 0$. Then there exists a complement K of N such that $L \subseteq K$. Moreover, if M has finite uniform dimension then $u(M) = u(N) + u(K) = u(M/K) + u(K)$.*

We shall require the following result later. Its proof is included for completeness.

Lemma 1.5 *Given a positive integer n, a module M has uniform dimension n if and only if there exist submodules L_i $(1 \leq i \leq n)$ such that*

(a) M/L_i is a uniform module for all $1 \leq i \leq n$,

(b) $0 = L_1 \cap \cdots \cap L_n$, and

(c) $0 \neq L_1 \cap \cdots L_{i-1} \cap L_{i+1} \cap \cdots \cap L_n$ for all $1 \leq i \leq n$.

Note that in Lemma 1.5, (b) and (c) can be restated thus: $0 = L_1 \cap \cdots \cap L_n$ is an irredundant decomposition.

Proof. Suppose first that M has uniform dimension n. By Lemma 1.3, there exist independent uniform submodules $U_i(1 \leq i \leq n)$ of M such that $U_1 \oplus \cdots \oplus U_n$ is an essential submodule of M. For each $1 \leq i \leq n$, let K_i be a complement of U_i in M such that $U_1 \oplus \cdots \oplus U_{i-1} \oplus U_{i+1} \oplus \cdots \oplus U_n \subseteq K_i$ (Lemma 1.4). By Lemma 1.4, M/K_i is a uniform module for each $1 \leq i \leq n$. Suppose that $K_1 \cap \cdots \cap K_n \neq 0$. Then $(K_1 \cap \cdots \cap K_n) \cap (U_1 \oplus \cdots \oplus U_n) \neq 0$. Let $0 \neq x = U_1 + \cdots + U_n$ where $x \in K_1 \cap \cdots \cap K_n$ and $u_i \in U_i$ $(1 \leq i \leq n)$. Then $u_1 = x - u_2 - \cdots - u_n \in K_1 \cap U_1 = 0$, so that $u_1 = 0$. Similarly, $u_i = 0(2 \leq i \leq n)$, and hence $x = 0$, a contradiction. Therefore $0 = K_1 \cap \cdots \cap K_n$. Moreover, for each $1 \leq i \leq n$, $0 \neq U_i \subseteq K_1 \cap \cdots \cap K_{i-1} \cap K_{i+1} \cap \cdots K_n$.

Conversely, suppose that M contains submodules L_i $(1 \leq i \leq n)$ satisfying (a), (b) and (c). Define a mapping $\phi : M \to (M/L_1) \oplus \cdots \oplus (M/L_n)$ by $\phi(m) = (m + L_1, \cdots, m + L_n)$ for all $m \in M$. By (b), ϕ is a monomorphism. Let $1 \leq i \leq n$. By (c) there exists $0 \neq m_i \in L_1 \cap \cdots \cap L_{i-1} \cap L_{i+1} \cap \cdots \cap L_n$ and hence $m_i \notin L_i$ by (b). It follows that $0 \neq (0, \cdots, 0, m_i + L_i, 0, \cdots, 0) = \phi(m_i) \in \phi(M)$. Hence $\phi(M) \cap (0 \oplus \cdots \oplus 0 \oplus (M/L_i) \oplus 0 \oplus \cdots \oplus 0) \neq 0$ for all $1 \leq i \leq n$. Hence $\phi(M)$ is an essential submodule of

$(M/L_1) \oplus \cdots \oplus (M/L_n)$ and hence $u(M) = u(\phi(M)) = n$ by Lemma 1.3 and (a). □

Before proceeding we make two comments about Lemma 1.5. Firstly, note that a non-zero module M has finite uniform dimension if and only if the zero submodule is the intersection of a finite collection of irreducible submodules. Recall that a submodule N of M is called *irreducible* if the factor module M/N is uniform. The second comment is that condition (a) in Lemma 1.5 is crucial because if K and L are non-zero submodules of a module M such that $K \cap L = 0$ and M/K and M/L both have finite uniform dimension then $u(M) \leq u(M/K) + u(M/L)$ but it is not necessarily the case that $u(M) = u(M/K) + u(M/L)$. A simple example can be given to illustrate this fact. Let $\mathbb{Z}$ denote the ring of rational integers and $\mathbb{Q}$ the field of rational numbers. Let M denote the $\mathbb{Z}$-module $\mathbb{Q} \oplus \mathbb{Q}$ so that $u(M) = 2$. Let $K = \{(q, q) : q \in \mathbb{Q}\}$. Then $M = K \oplus (\mathbb{Q} \oplus 0)$ so that $u(M/K) = 1$. Let n be any positive integer and let π be any collection of n distinct primes in $\mathbb{Z}$. Let X denote the submodule $\sum_{p \notin \pi} \sum_{k=1}^{\infty} \mathbb{Z}(1/p^k)$ of $\mathbb{Q}$. Note that X consists of all s/t in $\mathbb{Q}$ such that $s, t \in \mathbb{Z}, t \neq 0$ and t is not divisible by any prime p in π. Note that $\mathbb{Q}/X \cong (\mathbb{Q}/\mathbb{Z})/(X/\mathbb{Z})$ so that $u(\mathbb{Q}/X) = n$. Let L denote the submodule $0 \oplus X$ of M. Then K and L, are non-zero submodules of M such that $K \cap L = 0$, $u(M/K) = 1, u(M/L) = n + 1$ and $u(M) = 2$, so that $u(M) \neq u(M/K) + u(M/L)$.

We complete this section with two results about prime submodules.

Lemma 1.6 *(See [8, Proposition 1.4(ii)].) Let N be a P-prime submodule of an R-module M, for some prime ideal P of R, and let K be a proper submodule of M containing N such that K/N is a complement in M/N. Then K is a P-prime submodule of M.*

In what follows we shall be particularly interested in irreducible prime submodules of a module M, i.e. prime submodules K of M such that M/K is a uniform module. For example, in the $\mathbb{Z}$-module $\mathbb{Q}$, the zero submodule of $\mathbb{Q}$ is an irreducible prime submodule. Lemma 1.6 has the following consequence.

Corollary 1.7 *Let N be a P-prime submodule of an R-module M, for some prime ideal P of R, and let L be a non-zero submodule of M such that $N \cap L = 0$. Let K be a complement of L in M such that $N \subseteq K$. Then K is a P-prime submodule of M. Moreover, if L is a uniform module then K is an irreducible P-prime submodule of M.*

Proof. Note that $K \cap L = 0$ and $L \neq 0$ together imply $K \neq M$. It is easy to check that K/N is a complement of $(L+N)/N$ in M/N. By Lemma 1.6, K is a P-prime submodule of M. Now suppose that L is uniform. By Lemma 1.4, $u(M/K) = u(L) = 1$, i.e. K is an irreducible prime submodule of M. □

2. Modules with finite uniform dimension

In this section we shall prove that any radical submodule N of a module M such that the factor module M/N has finite uniform dimension has a prime decomposition and we shall investigate the associated prime ideals of N.

Let U be a uniform R-module. Let $P = \{r \in R : rV = 0$ for some non-zero submodule V of $U\}$. Then P is an ideal of R. Following [1] we shall call P the assassinator of U. It can easily be checked that if $PW = 0$ for some non-zero submodule W of U then P is a prime ideal of R.

Lemma 2.1 *Let U be a uniform submodule of an R-module M and let P be the assassinator of U. Suppose that $PM \cap U = 0$. Then there exists an irreducible P-prime submodule K of M such that $K \cap U = 0$.*

Proof. Note that $PU = 0$ so that P is a prime ideal of R. Let K be a complement of U in M such that $PM \subseteq K$ (Lemma 1.4). Let $r \in R$ and let L be a submodule of M containing K such that $rL \subseteq K$. Then $r(L \cap U) \subseteq K \cap U = 0$. Either $L \cap U = 0$ in which case $L = K$ or $L \cap U \neq 0$ in which case $r \in P$ because P is the assassinator of U. It follows that K is a P-prime submodule of M. By Lemma 1.4, M/K is a uniform module and hence K is an irreducible P-prime submodule of M. □

Lemma 2.2 *Let M be an R-module such that the zero submodule of M is a radical submodule. Let U be a uniform submodule of M with assassinator P. Then $PM \cap U = 0$.*

Proof. Let A be a finitely generated left ideal of R such that $A \subseteq P$. There exists a non-zero submodule V of U such that $AV = 0$. There exist prime submodules K_λ $(\lambda \in \Lambda)$ of M such that $0 = \cap_{\lambda \in \Lambda} K_\lambda$. Let $\lambda \in \Lambda$. If $V \nsubseteq K_\lambda$ then $AV = 0 \subseteq K_\lambda$ gives $AM \subseteq K_\lambda$. Hence $AM \cap V \subseteq K_\lambda$. Thus $AM \cap V \subseteq \cap_{\lambda \in \Lambda} K_\lambda = 0$. Next $(AM \cap U) \cap V = AM \cap V = 0$, so that $AM \cap U = 0$ because U is uniform. Clearly it follows that $PM \cap U = 0$. □

Theorem 2.3 *Let R be any ring and let M be a non-zero R-module such that the zero submodule of M is a radical submodule. Then the following statements are equivalent.*

(i) The zero submodule of M is a finite intersection of irreducible prime submodules of M.

(ii) M has finite uniform dimension.

Moreover, in this case if $0 = K_1 \cap \cdots \cap K_n$ is any irredundant decomposition, where K_i is an irreducible prime submodule of M for each $1 \leq i \leq n$, then $n = u(M)$.

Proof. (i) $\Rightarrow$ (ii) and last part By Lemma 1.5.

(ii) $\Rightarrow$ (i) Suppose that M has finite uniform dimension. Let U_1 be any uniform submodule of M and let P_1 be the assassinator of U_1. By Lemma 2.2, $P_1 M \cap U_1 = 0$ and by Lemma 2.1 there exists an irreducible P_1-prime submodule K_1 of M such that $K_1 \cap U_1 = 0$. If $u(M) = 1$ then $K_1 = 0$ and the result is proved.

Suppose that $u(M) \geq 2$. Let U_2 be any uniform submodule of M such that $U_1 \cap U_2 = 0$. If $K_1 \cap (U_1 \oplus U_2) = 0$ then set $K_2 = M$. Suppose that $K_1 \cap (U_1 \oplus U_2) \neq 0$. Note that $K_1 \cap (U_1 \oplus U_2)$ embeds in U_2 (because $K_1 \cap U_1 = 0$) and hence $K_1 \cap (U_1 \oplus U_2)$ is a uniform submodule of M. Let P_2 be the assassinator of $K_1 \cap (U_1 \oplus U_2)$. As above, by Lemmas 2.2 and 2.1 there exists an irreducible P_2-prime submodule K_2 of M such that $K_2 \cap \{K_1 \cap (U_1 \oplus U_2)\} = 0$ and hence $(K_1 \cap K_2) \cap (U_1 \oplus U_2) = 0$. If $u(M) = 2$ then $U_1 \oplus U_2$ is essential in M and hence $K_1 \cap K_2 = 0$ so that again the result is true because $K_2 = M$ or K_2 is an irreducible prime submodule.

Suppose that $u(M) \geq 3$. Let U_3 be any uniform submodule of M such that $(U_1 \oplus U_2) \cap U_3 = 0$. By the above argument there exists a submodule K_3 of M such that $(K_1 \cap K_2 \cap K_3) \cap (U_1 \oplus U_2 \oplus U_3) = 0$ and either $K_3 = M$ or K_3 is an irreducible prime submodule of M. Repeat this process to obtain a sequence $U_i (i \geq 1)$ of independent uniform submodules and a sequence $K_i (i \geq 1)$ of submodules such that K_1 is an irreducible prime submodule and for each $i \geq 2$ the submodule $K_i = M$ or K_i is irreducible prime satisfying

$$(K_1 \cap \cdots \cap K_s) \cap (U_1 \oplus \cdots \oplus U_s) = 0$$

for each positive integer s. Let $n = u(M) \geq 1$. Then $U_1 \oplus \cdots \oplus U_n$ is an essential submodule of M and hence $K_1 \cap \cdots \cap K_n = 0$. □

Corollary 2.4 *Let N be a radical submodule of an R-module M. Then N is a finite intersection of irreducible prime submodules of M if and only if M/N has finite uniform dimension. In this case, N has a prime decomposition.*

Proof. By Theorem 2.3. □

In certain circumstances, every radical submodule of a module M is an intersection of irreducible prime submodules. In order to prove this we begin with the following lemma.

Lemma 2.5 *Let P be a prime ideal of a ring R and let M be an R-module such that 0 is a P-prime submodule of M and every non-zero submodule contains a uniform submodule of M. Then the zero submodule is an intersection of irreducible P-prime submodules of M.*

Proof. By Zorn's Lemma M contains a maximal independent collection of uniform submodules $U_\lambda (\lambda \in \Lambda)$ and by hypothesis $\oplus_{\lambda \in \Lambda} U_\lambda$ is an essential submodule of M. Let $\mu \in \Lambda$ and let $L_\mu = \oplus_{\lambda \neq \mu} U_\lambda$. Note that L_μ is a submodule of M such that $L_\mu \cap U_\mu = 0$. By Lemma 1.4 there exists a complement K_μ of U_μ in M such that $L_\mu \subseteq K_\mu$. Now Lemma 1.6 gives that K_μ is P-prime. It is easy to check that $(\cap_{\lambda \in \Lambda} K_\lambda) \cap (\oplus_{\lambda \in \Lambda} U_\lambda) = 0$ and hence $\cap_{\lambda \in \Lambda} K_\lambda = 0$ where K_λ is a P-prime submodule of M for each $\lambda \in \Lambda$. □

We shall say that a (non-zero) R-module M has *many uniforms* if for every prime submodule K of M and for each element $m \in M \backslash K$, the submodule $(Rm+K)/K$ contains a uniform submodule.

Theorem 2.6 *Let M be an R-module with many uniforms. Then, for any prime ideal P of R, every P-prime submodule of M is an intersection of irreducible P-prime submodules of M. Moreover, every radical submodule of M is an intersection of irreducible prime submodules of M.*

Proof. Let P be a prime ideal of R and let K be a P-prime submodule of M. Applying Lemma 2.5 to the module M/K we see that $0 = \cap_{\lambda \in \Lambda} K_\lambda / K$ where K_λ is a submodule containing K such that K_λ / K is an irreducible P-prime submodule of M/K for each $\lambda \in \Lambda$. Clearly $K = \cap_{\lambda \in \Lambda} K_\lambda$ where K_λ is an irreducible P-prime submodule of M for each $\lambda \in \Lambda$. The last part is clear. $\square$

Note that if R is a left Noetherian ring then every non-zero left R-module has many uniforms. More generally, if a ring R has left Krull dimension then every non-zero left R-module has many uniforms by [9, 6.2.4 and 6.2.6]. A ring R is called *left semi-artinian* if every non-zero cyclic left R-module contains a simple submodule. For example, right perfect rings are left semi-artinian. Clearly if R is a left semi-artinian ring then every non-zero left R-module has many uniforms. (For more information on left semi-artinian rings see [2, pp26-28].) In the next section we shall show that if R is any commutative ring, or more generally any ring satisfying a polynomial identity, then every non-zero R-module has many uniforms.

Next we give a characterization of the associated prime ideals of a radical submodule N in case M/N has finite uniform dimension (compare Lemma 1.2).

Theorem 2.7 *Let N be a radical submodule of an R-module M such that M/N has finite uniform dimension. Then P is an associated prime ideal of N if and only if P is the assassinator of a uniform submodule of the module M/N.*

Proof. Suppose first that L is a submodule of M containing N such that L/N is a uniform module. Let P be the assassinator of L/N. By Lemma 2.2, $P = (N : L)$ and by Lemma 1.2, P is an associated prime ideal of N.

Conversely, suppose that P is an associated prime ideal of N. Let $N = K_1 \cap \cdots \cap K_n$ be a normal prime decomposition of N where K_i is a P_i-prime submodule of M for some prime ideal P_i for each $1 \leq i \leq n$ and n is a positive integer. Without loss of generality, we can suppose that $P = P_1$ (Lemma 1.1). If $n = 1$ then $N = K_1$ and so N is a P-prime submodule of M. Let H be a submodule of M properly containing N such that H/N is a uniform module. Clearly P is the assassinator of H/N.

Now suppose that $n \geq 2$. Since $K_2 \cap \cdots \cap K_n \neq N$ it follows that there exists a submodule G of $K_2 \cap \cdots \cap K_n$ properly containing N such that G/N is a uniform module. Note that $PG \subseteq K_1 \cap \cdots \cap K_n = N$. On the other hand, let $r \in R$ and let J be a submodule of G such that $rJ \subseteq N$. Then $rJ \subseteq K_1$. Either $J \subseteq K_1$–in which case $J \subseteq K_1 \cap \cdots \cap K_n = N$–or $r \in P$. It follows that P is the assassinator of the uniform submodule G/N of M/N. $\square$

Corollary 2.8 *Let N be a radical submodule of an R-module M such that M/N has finite uniform dimension. Then a prime ideal P of R is the assassinator of a uniform submodule of the module M/N if and only if $P = (N : L)$ for some submodule L of M.*

Proof. By Lemma 1.2 and Theorem 2.5. □

3. Modules over fully bounded rings

We now consider when it is the case that every submodule N of a module M with N having a prime decomposition has the property that the factor module M/N has finite uniform dimension. Note that if F is a field and V an infinite dimensional vector space over F then the zero subspace of V is a prime submodule, but the F-module V does not have finite uniform dimension. Because of this example we shall consider finitely generated modules. But even for finitely generated modules there are problems. In [1, Example 1.22] an example is given of a right Noetherian domain such that the left R-module R does not have finite uniform dimension. Thus we shall also restrict the choice of the ring R.

A prime ring R is *left bounded* if every essential left ideal contains a non-zero two-sided ideal. A general ring R is a *fully left bounded left Goldie ring* (*left FBG-ring* for short) if, for each prime ideal P of R, the prime ring R/P is a left bounded left Goldie ring. Clearly commutative rings are (left) FBG-rings, as are rings with polynomial identity by [9, 13.6.6].

Let R be a prime left Goldie ring. An element c of R is *regular* if $cr \neq 0$ and $rc \neq 0$ for every non-zero element r of R. An R-module M is called *torsion-free* if $cm \neq 0$ for every regular element c of R and non-zero element m of M. On the other hand, M is a *torsion* module if for each $m \in M$ there exists a regular element c of R such that $cm = 0$.

Lemma 3.1 *(See [8, Lemma 2.6].) Let P be a prime ideal of a ring R such that R/P is a left bounded left Goldie ring and let K be a submodule of an R-module M. Then K is a P-prime submodule of M if and only if $P = (K : M)$ and the (R/P)-module M/K is torsion-free.*

Let P be a prime ideal of a ring R. By a *maximal P-prime* submodule of an R-module M we mean a P-prime submodule K of M such that K is not properly contained in any P-prime submodule of M. By a *maximal prime* submodule of M we shall mean a submodule which is a maximal Q-prime submodule of M for some prime ideal Q of R. In [7], given a prime ideal P of R, a submodule L of a module M is called P-*maximal* if L is maximal in the collection of submodules H of M such that $P = (H : M)$.

Lemma 3.2 *Let P be a prime ideal of a ring R. Consider the following statements about a submodule K of an R-module M.*

(i) K is P-maximal;

(ii) K is maximal P-prime;

(iii) K is irreducible P-prime.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Moreover, if R/P is a left bounded left Goldie ring then (iii) $\Rightarrow$ (ii). If in addition M is finitely generated, then (ii) $\Rightarrow$ (i).

Proof. (i) $\Rightarrow$ (ii) Let K be a P-maximal submodule of M. Note that $P = (K : M)$. Let $r \in R$ such that $rL \subseteq K$ for some submodule L of M properly containing K. Let $A = (L : M)$. Then $P \subset A$ because K is P-maximal. Now $rAM \subseteq rL \subseteq K$, so that $rA \subseteq P$ and hence $r \in P$. It follows that K is P-prime. Clearly K is a maximal P-prime submodule of M.

(ii) $\Rightarrow$ (iii) Let K be a maximal P-prime submodule of M. Let L be any submodule of M properly containing K. Let H be a submodule of M containing K such that H/K is a complement of L/K in M/K. Since $L/K \neq 0$ it follows that $H/K \neq M/K$. By Lemma 1.6, H is a P-prime submodule of M. Then $H = K$. It follows that L/K is an essential submodule of M/K. Therefore M/K is a uniform module and K is an irreducible P-prime submodule of M.

Now suppose that R/P is a left bounded left Goldie ring. Let K be an irreducible P-prime submodule of M. Let G be any submodule of M properly containing K. Let $m \in M$. Since G/K is an essential submodule of the (R/P)-module M/K it follows that $\overline{E}(m + G) = 0$ for some essential left ideal $\overline{E}$ of the ring R/P. By [9, 2.3.5.] there exists a regular element $\overline{c}$ of R/P such that $\overline{c}(m + G) = 0$. It follows that M/G is a torsion (R/P)-module for every submodule G properly containing K. By Lemma 3.1, N is a maximal P-prime submodule of M.

Finally, suppose that M is a finitely generated module (and R/P is left bounded left Goldie). Let K be an irreducible P-prime submodule of M and let G be any submodule of M properly containing N. As before, M/G is a torsion (R/P)-module. By hypothesis, there exists an ideal A of R properly containing P such that $AM \subseteq G$. Thus $P \subset (G : M)$. It follows that K is P-maximal. □

Let M be a finitely generated R-module. Then $g(M)$ will denote the least number of elements in a smallest generating set of M.

Lemma 3.3 *Let R be a prime left Goldie ring and let M be a finitely generated torsion-free R-module. Then M has finite uniform dimension and $u(M) \leq g(M)u(R)$.*

Proof. Suppose that $M \neq 0$ and $g(M) = k$, for some positive integer k. There exists an epimorphism $\phi : R^{(k)} \to M$. Let $K = \ker \phi$. Then $R^{(k)}/K$ is torsion-free so that K is a complement submodule of $R^{(k)}$ by [2, 1.10]. By Lemma 1.4,

$$ku(R) = u(R^{(k)}) = u(K) + u(R^{(k)}/K) \geq u(R^{(k)}/K) = u(M).$$

□

Corollary 3.4 *Let P be a prime ideal of a ring R such that the ring R/P is left bounded left Goldie and let K be a P-prime submodule of a finitely generated R-module M. Then the R-module M/K has finite uniform dimension and $u(M/K) \leq g(M/K)u(R/P)$.*

Proof. By Lemmas 3.1 and 3.3 □

Theorem 3.5 *Let R be a left FBG-ring. Then the following statements are equivalent for a submodule N of a finitely generated R-module M.*

(i) N is a radical submodule of M and M/N has finite uniform dimension.

(ii) N is a finite intersection of maximal prime submodules of M.

(iii) N has a prime decomposition.

Proof. (i) $\Rightarrow$ (ii) By Corollary 2.4 and Lemma 3.2.

(ii) $\Rightarrow$ (iii) Clear.

(iii) $\Rightarrow$ (i) Suppose that N has a prime decomposition. Then N is a radical submodule of M. Let $N = K_1 \cap \cdots \cap K_n$ be a prime decomposition where K_i is a P_i-prime submodule of M for some prime ideal P_i of R for each $1 \leq i \leq n$. For each $1 \leq i \leq n$, the prime ring R/P_i is left bounded left Goldie. By Corollary 3.4, the R-module M/K_i has finite uniform dimension. Since M/N embeds in $(M/K_1) \oplus \cdots \oplus (M/K_n)$ it follows that M/N has finite uniform dimension. □

Theorem 3.6 *Let R be a left FBG-ring and let M be a non-zero R-module. Then, for any prime ideal P of R, every P-prime submodule of M is an intersection of maximal P-prime submodules of M. Moreover, every radical submodule of M is an intersection of maximal prime submodules of M.*

Proof. We shall prove that M has many uniforms. Let Q be a prime ideal of R and let K be a Q-prime submodule of M. Let $m \in M \backslash K$. Note that the ring R/Q is a left bounded left Goldie ring and the (R/Q)-module M/K is torsion-free (see Lemma 3.1). Hence $(Rm + K)/K$ is a torsion-free cyclic (R/Q)-module. There exists a non-essential left ideal $\overline{L}$ of $\overline{R} = R/Q$ such that $(Rm + K)/K \cong \overline{R}/\overline{L}$. Next there exists a uniform left ideal $\overline{U}$ of $\overline{R}$ such that $\overline{L} \cap \overline{U} = 0$, and hence $\overline{U}$ embeds in $(Rm + K)/K$. It follows that M has many uniforms. By Theorem 2.6 and Lemma 3.2, every P-prime submodule is an intersection of maximal P-prime submodules of M, for each prime ideal P of R. The last part is clear.

Next we shall examine the fully left bounded condition further. We begin with the following result. □

Lemma 3.7 *Let R be a prime ring such that every ideal is finitely generated as a left ideal and let M be a finitely generated R-module such that the zero submodule $0 = K_1 \cap \cdots \cap K_n$ where n is a positive integer and K_i is a maximal 0-prime submodule of M for each $1 \leq i \leq n$. Let L be a submodule of M such that $L \cap K_1 \cap \cdots \cap K_{i-1} \cap K_{i+1} \cap \cdots \cap K_n \nsubseteq K_i$ for each $1 \leq i \leq n$. Then there exists a non-zero ideal A of R such that $AM \subseteq L$.*

Proof. The result is proved by induction on n. Suppose that $n = 1$. Then 0 is a maximal 0-prime submodule of M and L is a non-zero submodule of M. Let

$H = \{m \in M : Bm \subseteq L$ for some non-zero ideal B of $R\}$. It is easy to check that H is a submodule of M. Let $x \in M$ such that $Cx \subseteq H$ for some non-zero ideal C of R. There exist a positive integer k and elements $c_i \in C(1 \leq i \leq k)$ such that $C = Rc_1 + \cdots + Rc_k$. For each $1 \leq i \leq k$ there exists a non-zero ideal D_i of R such that $D_i c_i x \subseteq L$. Let $D = D_1 \cdots D_k C$. Then D is a non-zero ideal of R such that $Dx = D_1 \cdots D_k Cx = \sum_{i=1}^{k} D_1 \cdots D_k c_k x \subseteq L$, and hence $x \in H$. It follows that if $H \neq M$ then H is a 0-prime submodule of M. Because 0 is a maximal 0-prime submodule of M, we deduce that $H = M$. Now M is finitely generated and it easily follows that $AM \subseteq L$ for some non-zero ideal A of R.

Now suppose that $n \geq 2$. Let $K = K_1 \cap \cdots \cap K_{n-1}$. Note that $\{[(L \cap K_n) + K]/K\} \cap [(K_1/K) \cap \cdots \cap (K_{i-1}/K) \cap (K_{i+1}/K) \cap \cdots \cap (K_{n-1}/K)] \nsubseteq K_i/K$ for all $1 \leq i \leq n-1$. By induction on n there exists a non-zero ideal A_1 of R such that $A_1(M/K) \subseteq [(L \cap K_n) + K]/K$, i.e. $A_1 M \subseteq (L \cap K_n) + K$. On the other hand, $L \cap K \nsubseteq K_n$ so that, by the case $n = 1$, there exists a non-zero ideal A_2 of R such that $A_2(M/K_n) \subseteq [(L \cap K) + K_n]/K_n$, i.e. $A_2 M \subseteq (L \cap K) + K_n$. Let $A = A_1 A_2$. Then A is a non-zero ideal of R and

$$AM \subseteq [(L \cap K_n) + K] \cap [(L \cap K) + K_n] \subseteq (L \cap K) + (L \cap K_n) \subseteq L,$$

because $K \cap K_n = 0$. □

Corollary 3.8 *Let R be a prime ring such that every ideal is finitely generated as a left ideal and let M be a finitely generated left R-module such that the zero submodule is the intersection of a finite collection of maximal 0-prime submodules. Let L be an essential submodule of M. Then there exists a non-zero ideal A of R such that $AM \subseteq L$.*

Proof. There exist a positive integer n and maximal 0-prime submodules $K_i(1 \leq i \leq n)$ such that $0 = K_1 \cap \cdots \cap K_n$ and $0 \neq K_1 \cap \cdots \cap K_{i-1} \cap K_{i+1} \cap \cdots \cap K_n$ for all $1 \leq i \leq n$. Clearly $L \cap K_1 \cap \cdots \cap K_{i-1} \cap K_{i+1} \cap \cdots \cap K_n \nsubseteq K_i$ for all $1 \leq i \leq n$. The result follows by Lemma 3.6. □

Theorem 3.9 *The following statements are equivalent for a left Noetherian ring R.*

(i) R is fully left bounded.

(ii) Every radical submodule of every finitely generated R-module is a finite intersection of maximal prime submodules of M.

(iii) Every radical submodule of the R-module R is a finite intersection of maximal prime submodules of the R-module R.

(iv) Every prime ideal P of R is a finite intersection of maximal P-prime submodules of the R-module R.

Proof. (i) $\Rightarrow$ (ii) By Theorem 3.5.

(ii) $\Rightarrow$ (iii) Clear.

(iii) $\Rightarrow$ (iv) Let P be any prime ideal of R. By (iii) there exist a positive integer n, prime ideals $P_i (1 \leq i \leq n)$ and maximal P_i-prime submodules $K_i (1 \leq i \leq n)$ of R such that $P = K_1 \cap \cdots \cap K_n$ and $P \neq K_1 \cap \cdots \cap K_{i-1} \cap K_{i+1} \cap \cdots \cap K_n$ for all $1 \leq i \leq n$. For each $1 \leq i \leq n$, $PR \subseteq K_i$ so that $P \subseteq (K_i : R) = P_i$. Suppose that $P \neq P_i$ for some $1 \leq i \leq n$. Then $P_i(K_1 \cap \cdots \cap K_{i-1} \cap K_{i+1} \cap \cdots \cap K_n) \subseteq P$, so that $K_1 \cap \cdots \cap K_{i-1} \cap K_{i+1} \cap \cdots \cap K_n = P$, a contradiction. Thus $P = P_i (1 \leq i \leq n)$. This proves (iv).

(iv) $\Rightarrow$ (i) Let Q be any prime ideal of R. Let M denote the R-module R/Q. Then the (R/Q)-module M satisfies the hypotheses of Corollary 3.8. Let E be any left ideal of R containing Q such that E/Q is an essential left ideal of R/Q. By Corollary 3.8 there exists an ideal A of R properly containing Q such that $(A/Q)(R/Q) \subseteq E/Q$, i.e. $A \subseteq E$. Hence R/Q is left bounded. □

Finally, note that if R is an arbitrary ring and N is a radical submodule of an R-module M such that the module M/N has only a finite number of minimal prime submodules then N has a prime decomposition (see [8, p.1059]). The converse is false. Consider the following result.

Theorem 3.10 *Let P and Q be prime ideals of a ring R such that $P \nsubseteq Q$ and $Q \nsubseteq P$ and let N be the submodule $P \oplus Q$ of the R-module $R \oplus R$. Then $N = K \cap L$ where K is the P-prime submodule $P \oplus R$ and L is the Q-prime submodule $R \oplus Q$ of M. Moreover,* *the minimal prime submodules of M/N are $K/N, L/N$ and BM/N where $P+Q \subseteq B$ and $B/(P+Q)$ is a minimal prime ideal of the ring $R/(P+Q)$.*

Proof. The first part is clear. Let G be a submodule of M containing N such that G/N is a minimal prime submodule of M/N. Note that G is a prime submodule of M. Now $P(R \oplus 0) \subseteq G$ gives $R \oplus 0 \subseteq G$ or $PM \subseteq G$. If $R \oplus 0 \subseteq G$ then $R \oplus Q \subseteq G$ and $(R \oplus Q)/N$ is a prime submodule of M/N so that $G/N = (R \oplus Q)/N$. Suppose that $PM \subseteq G$. Next $Q(0 \oplus R) \subseteq G$ gives that $G/N = (P \oplus R)/N$ or $QM \subseteq G$. Suppose that $QM \subseteq G$. Then $(P+Q)M \subseteq G$. Because $P+Q$ is contained in the prime ideal $(G : M)$ there exists a prime ideal B of R such that $P+Q \subseteq B \subseteq (G : M)$ and $B/(P+Q)$ is a minimal prime ideal of the ring $R/(P+Q)$. Note that BM/N is a prime submodule of M/N such that $BM/N \subseteq G/N$. Then $G/N = BM/N$. □

Let S be a commutative domain such that there exists a proper ideal A of S such that the ring S/A has an infinite number of minimal prime ideals. Let R denote the polynomial ring $S[X]$ where X is the set of indeterminates $\{x_a : a \in A\}$. Let $P = \sum_{a \in A} Rx_a$ and let $Q = \sum_{a \in A} R(x_a - a)$. Then P and Q are prime ideals of R because $R/P \cong R/Q \cong S$. Moreover, $P+Q = P+A$ and $R/(P+Q) \cong S/A$, so that the ring $R/(P+Q)$ contains an infinite number of minimal prime ideals. If N is the submodule $P \oplus Q$ of the R-module $M = R \oplus R$ then N has a prime decomposition but the R-module M/N contains an infinite number of minimal prime submodules by Theorem 3.10.

To find a commutative domain S and an ideal A with the above properties we proceed as follows. Let T be any commutative von Neumann regular ring which is not Artinian. Then every prime ideal of T is maximal and T contains an infinite number of (minimal) prime ideals. Let $S = \mathbb{Z}[X]$ denote the polynomial ring in the set $X = \{x_t : t \in T\}$ of indeterminates. Then S is a commutative domain and there exists a ring epimorphism $\phi : S \to T$ such that $\phi(x_t) = t$ $(t \in T)$. Let A denote the kernel of ϕ. Then A is an ideal of S such that $S/A \cong T$.

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