

Rate of Convergence of Durrmeyer Type Baskakov-Bezier Operators for Locally Bounded Functions

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Abstract

In the present paper, we introduce the Durrmeyer variant of Baskakov-Bezier operators $B_{n,\alpha}(f, x)$, which is the modified form of Baskakov-Beta operators. Here we obtain an estimate on the rate of convergence of $B_{n,\alpha}(f, x)$ for functions of bounded variation in terms of Chanturiya's modulus of variation. In the end we also propose an open problem for the readers.

Key Words: Locally Bounded Functions, Rate of Convergence, Modulus of Variation.

1. Introduction

Let $I \equiv [0, \infty)$ and let $M_{loc}(I)$ be the class of all measurable complex valued locally bounded functions on I . For $f \in M_{loc}(I)$, the Baskakov-Durrmeyer type operators B_n ($n \in \mathbb{N}$) applied to f are defined as

$$B_n(f, x) = \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} b_{n,k}(t) f(t) dt, x \in I \quad (1)$$

where

$$p_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \quad , \quad b_{n,k}(t) = \frac{t^k}{B(k+1, n)(1+t)^{n+k+1}}$$

and $B(k+1, n) = k!(n-1)/(n+k)!$.

The operator defined by (1) were introduced by the author (see[5],[6]). Some approximation properties of these operators were studied by Gaur and Sharma [4] and Argawal and Thamer [2].

Recently Zeng and Gupta [7] estimated the rate of convergence of the discrete Baskakov-Bezier operators, which are defined by

$$P_{n,\alpha}(f, x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) Q_{n,k}^{(\alpha)}(x),$$

where $Q_{n,k}^{(\alpha)}(x) = J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)$, $\alpha \geq 1$, $\sum_{j=k}^{\infty} p_{n,j}(x) = J_{n,k}(x)$ and $p_{n,k}(x)$ is the

Baskakov basis function. As the operator (1) are the Baskakov Durrmeyer type operators and they have many interesting properties, this motivated us to study further on such operators. For $\alpha \geq 1$, we now introduce the Durrmeyer variant of the Baskakov Bezier operators as

$$B_{n,\alpha}(f, x) = \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \int_0^{\infty} b_{n,k}(t) f(t) dt . \tag{2}$$

Obviously, $B_{n,\alpha}(1, x) = 1$ and particularly when $\alpha = 1$, the operators (2) reduce to the operators (1). For further properties of $Q_{n,k}^{(\alpha)}(x)$ and $J_{n,k}(x)$, we refer the readers to [7]. Clearly if $f \in M_{loc}(I)$, and if for every $t > 0$, $|f(t)| \leq M(1+t)^\gamma$ with some $M > 0$, $\gamma \geq 0$, then $B_{n,\alpha}(f, x)$ are well defined for $n > \gamma$.

In the present paper, we obtain the rate of convergence for the Durrmeyer type Baskakov-Bezier operators $B_{n,\alpha}(f, x)$ in terms of Chanturiya's modulus of variation of the auxiliary function g_x defined by

$$g_x(t) = \begin{cases} f(t) - f(x_-) & , \quad 0 \leq t < x \\ 0 & , \quad t = x \\ f(t) - f(x_+) & , \quad x < t < \infty . \end{cases}$$

The Chanturiya's modulus of variation of j^{th} order for the function g , bounded on a finite or infinite interval Y contained in I is denoted by $v_j(g, Y)$ and defined as the upper bound of the set of all numbers $\sum_{k=1}^j |g(b_k) - g(a_k)|$ over all systems of j non-overlapping intervals (a_k, b_k) , $k = 1, 2, 3, \dots, j$ contained in Y . In particular, if $j = 0$ we have $v_0(g, Y) = 0$, the sequence $\{v_j(g, Y)\}_{j=0}^{\infty}$ is called the modulus of variation we refer to the readers [3].

2. Basic Results

In this section we give certain results which are necessary to prove the main result.

Recently Zeng [9] estimated the exact bounds for Bernstein basis functions and Meyer Konig Zeller basis functions. For $k \in \mathbb{N}$ and $t \in (0, 1]$, the author of [9] obtained the inequality

$$\binom{n+k-1}{k} t^k (1-t)^n < \frac{1}{\sqrt{2e}} \frac{1}{\sqrt{nt}}. \tag{3}$$

Replacing the variable t with $\frac{x}{1+x}$ in eq. (3), we get the following result:

Lemma 1 For all $x \in (0, \infty)$ and $k \in \mathbb{N}$, we have

$$Q_{n,k}^{(\alpha)}(x) \leq \alpha \cdot p_{n,k}(x) < \frac{\alpha \sqrt{1+x}}{\sqrt{2e} \sqrt{nx}},$$

where the constant $\frac{1}{\sqrt{2e}}$ is the best possible.

Lemma 2 [5] Let the m^{th} order moment be defined by

$$B_{n,1}((t-x)^m, x) \equiv \mu_{n,m}(x) = \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} b_{n,k}(t) (t-x)^m dt$$

then, we have

$$\mu_{n,0}(x) = 1, \quad \mu_{n,1}(x) = \frac{1+x}{n-1},$$

and

$$(n - m - 1)\mu_{n,m+1}(x) = x(1 + x) \left\{ \mu_{n,m}^{(1)}(x) + 2m \mu_{n,m-1}(x) \right\} + \\ + \{(m + 1)(1 + 2x) - x\} \mu_{n,m}(x), \quad n > m + 1.$$

From the above recurrence relation, we have

$$\mu_{n,2}(x) = \frac{2(n + 1)x^2 + 2(n + 2)x + 2}{(n - 1)(n - 2)}.$$

Remark 1 In particular, given any number $\lambda > 2$ and $x > 0$, there is an integer $N(\lambda, x) > 2$, such that

$$\mu_{n,2}(x) = \frac{\lambda x(1 + x)}{n}.$$

Following along the lines of the proof of Lemma in [1] and in view of the inequality $|a^\alpha - b^\alpha| \leq \alpha |a - b|$, $0 \leq a, b \leq 1$; $\alpha \geq 1$, we can easily obtain the following lemma:

Lemma 3 Let $x \in (0, \infty)$, $h \neq 0$ and f be a function of the class $M_{loc}(I)$. Put $I_x(h) = [x + h, x] \cap I$ if $h < 0$ and $I_x(h) = [x, x + h]$ if $h > 0$. Then, for every $n \geq 4$

$$\left| \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \int_{I_x(h)} g_x(t) b_{n,k}(t) dt \right| \leq \left(1 + \frac{8n\alpha\mu_{n,2}(x)}{h^2} \right) \\ \times \left\{ \sum_{j=1}^{m-1} \frac{1}{j^3} v_j(g_x; I_x(\frac{jh}{\sqrt{n}})) + \frac{1}{m^2} v_j(g_x; I_x(h)) \right\},$$

where $m = [\sqrt{n}]$ means the greatest integer not greater than \sqrt{n} .

3. Rate of Convergence

In this section we prove the following theorem.

Theorem 3.1 Let $f \in M_{loc}(I)$ and let there be a fixed point $x \in (0, \infty)$, the one sided limits $f(x \pm)$ exist. Also, $|f(t)| \leq M(1 + t)^\gamma$, $t > 0$ with some $M > 0$, $\gamma \geq 0$ and choose

a number $\lambda > 2$ and $\alpha \geq 1$. Then for $n \geq \max\{4, \gamma + 1, N(\lambda, x)\}$, we have

$$\begin{aligned} & \left| B_{n,\alpha}(f, x) - \left[\frac{1}{\alpha+1}f(x_+) + \frac{\alpha}{\alpha+1}f(x_-) \right] \right| \\ & \leq |f(x_+) - f(x_-)| \frac{\alpha\sqrt{1+x}}{\sqrt{2enx}} + M_0 \frac{(1+x)^\gamma}{(nx^2)^\gamma} + M_1 \frac{(1+x)^{\gamma+1}}{nx} + \\ & + \left(1 + \frac{8\alpha\lambda(1+x)}{x} \right) \left\{ \frac{\sum_{j=1}^{m-1} \frac{v_j(g_x; x - jx/\sqrt{n}, x) + v_j(g_x; x, x + jx/\sqrt{n})}{j^3} + \frac{v_m(g_x; 0, x) + v_m(g_x; x, 2x)}{m^3}}{m^3} \right\}, \end{aligned}$$

where M_0 and M_1 are certain constants depending on α, λ and γ .

Proof. It is easily verified [8] that

$$\begin{aligned} & \left| B_{n,\alpha}(f, x) - \left[\frac{1}{\alpha+1}f(x_+) + \frac{\alpha}{\alpha+1}f(x_-) \right] \right| \\ & \leq |B_{n,\alpha}(g_x, x)| + \frac{1}{2} |f(x_+) - f(x_-)| \left| B_{n,\alpha}(\text{sign}(t-x), x) + \frac{\alpha-1}{\alpha+1} \right|. \end{aligned} \tag{4}$$

In order to prove the theorem we need the estimate for $B_{n,\alpha}(g_x, x)$ and $B_{n,\alpha}(\text{sign}(t-x), x)$. We first estimate $B_{n,\alpha}(\text{sign}(t-x), x)$ as follows:

$$\begin{aligned} B_{n,\alpha}(\text{sign}(t-x), x) &= \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \left(\int_x^{\infty} b_{n,k}(t) dt - \int_0^x b_{n,k}(t) dt \right) \\ &= \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \left(\int_0^{\infty} b_{n,k}(t) dt - 2 \int_0^x b_{n,k}(t) dt \right) \\ &= 1 - 2 \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \int_0^x b_{n,k}(t) dt. \end{aligned}$$

Using the fact $\int_x^\infty b_{n,k}(t)dt = \sum_{j=0}^k p_{n,j}(x)$, we have

$$\begin{aligned} B_{n,\alpha}(\text{sign}(t-x), x) &= 1 - 2 \sum_{k=0}^\infty Q_{n,k}^{(\alpha)}(x) \left(1 - \sum_{j=0}^k p_{n,j}(x) \right) \\ &= -1 + 2 \sum_{k=0}^\infty Q_{n,k}^{(\alpha)}(x) \sum_{j=0}^k p_{n,j}(x) \\ &= -1 + 2 \sum_{j=0}^\infty p_{n,j}(x) \sum_{k=j}^\infty Q_{n,k}^{(\alpha)}(x) = -1 + 2 \sum_{j=0}^\infty p_{n,j}(x) J_{n,j}^\alpha(x). \end{aligned}$$

Thus

$$B_{n,\alpha}(\text{sign}(t-x), x) + \frac{\alpha-1}{\alpha+1} = 2 \sum_{j=0}^\infty p_{n,j}(x) J_{n,j}^\alpha(x) - \frac{2}{\alpha+1} \sum_{j=0}^\infty Q_{n,k}^{(\alpha+1)}(x),$$

since $\sum_{k=0}^\infty Q_{n,k}^{(\alpha)}(x) = 1$. By the mean value theorem, we have

$$Q_{n,j}^{(\alpha+1)}(x) = J_{n,j}^{\alpha+1}(x) - J_{n,j+1}^{\alpha+1}(x) = (\alpha+1)p_{n,j}(x)\gamma_{n,j}^\alpha(x)$$

where $J_{n,j}^\alpha(x) < \gamma_{n,j}^\alpha(x) < J_{n,j+1}^\alpha(x)$. Therefore

$$\begin{aligned} \left| B_{n,\alpha}(\text{sign}(t-x), x) + \frac{\alpha-1}{\alpha+1} \right| &= 2 \sum_{j=0}^\infty p_{n,j}(x) (J_{n,j}^\alpha(x) - \gamma_{n,j}^\alpha(x)) \\ &\leq 2 \sum_{j=0}^\infty p_{n,j}(x) (J_{n,j}^\alpha(x) - J_{n,j+1}^\alpha(x)) \\ &\leq 2\alpha \sum_{j=0}^\infty p_{n,j}(x) (J_{n,j}(x) - J_{n,j+1}(x)) \\ &= 2\alpha \sum_{j=0}^\infty p_{n,j}^2(x). \end{aligned}$$

Applying Lemma 1, we have

$$\left| B_{n,\alpha}(\text{sign}(t-x), x) + \frac{\alpha-1}{\alpha+1} \right| < 2\alpha \frac{\sqrt{1+x}}{\sqrt{2enx}} \sum_{j=0}^{\infty} p_{n,j}(x) = \frac{\alpha\sqrt{2(1+x)}}{\sqrt{e}\sqrt{nx}}. \tag{5}$$

Next we estimate $B_{n,\alpha}(g_x, x)$, as follows:

$$\begin{aligned} B_{n,\alpha}(g_x, x) &= \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \int_0^x g_x(t) b_{n,k}(t) dt \\ &\quad + \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \int_x^{2x} g_x(t) b_{n,k}(t) dt + \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \int_{2x}^{\infty} g_x(t) b_{n,k}(t) dt \\ &= E_1(\alpha, n, x) + E_2(\alpha, n, x) + E_3(\alpha, n, x), \text{ say.} \end{aligned} \tag{6}$$

Using Lemma 3 (with $h = -x$) and Lemma 2, we have

$$\begin{aligned} |E_1(\alpha, n, x)| &\leq \left\{ 1 + \frac{8\alpha\lambda(1+x)}{x} \right\} \\ &\quad \times \left\{ \sum_{j=1}^{m-1} \frac{1}{j^3} v_j(g_x; x, x + jx/\sqrt{n}, x) + \frac{1}{m^3} v_m(g_x; 0, x) \right\}, \end{aligned} \tag{7}$$

for all $n \geq N(\lambda, x)$. Also by Lemma 3 with $h = x$ and Lemma 2, we get the corresponding estimate of $E_2(\alpha, n, x)$ as follows:

$$\begin{aligned} |E_2(\alpha, n, x)| &\leq \left\{ 1 + \frac{8\alpha\lambda(1+x)}{x} \right\} \\ &\quad \times \left\{ \sum_{j=1}^{m-1} \frac{1}{j^3} v_j(g_x; x, x + jx/\sqrt{n}) + \frac{1}{m^3} v_m(g_x; x, 2x) \right\}. \end{aligned} \tag{8}$$

Finally for $n > \gamma$, we have

$$E_3(\alpha, n, x) \leq M \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \int_{2x}^{\infty} \{(1+t)^\gamma + (1+x)^\gamma\} b_{n,k}(t) dt.$$

Using the identity $(1+t)^\gamma - (1+x)^\gamma \leq (2^\gamma - 1) \frac{(1+x)^\gamma}{x^\gamma} (t-x)^\gamma$, $t \geq 2x$, we have

$$|E_3(\alpha, n, x)| \leq M \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \int_{2x}^{\infty} \left\{ (2^\gamma - 1) \frac{(1+x)^\gamma}{x^\gamma} (t-x)^\gamma + 2(1+x)^\gamma \right\} b_{n,k}(t) dt$$

$$\leq M(2^\gamma - 1) \alpha \frac{(1+x)^\gamma}{x^\gamma} \mu_{n,\gamma}(x) + 2M\alpha \frac{(1+x)^\alpha}{x^2} \mu_{n,2}(x).$$

Making use of Lemma 2, we get

$$|E_3(\alpha, n, x)| \leq M_0 \frac{(1+x)^\gamma}{(nx^2)^\gamma} + M_1 \frac{(1+x)^\gamma}{nx}, \tag{9}$$

where M_0, M_1 are constants depending on α, λ and γ .

Finally collecting the estimates of (4)-(9), we get the required result. This completes the proof of the theorem.

Remark 2 *It is remarked here that on the similar lines we may introduce the Durrmeyer variant of similar operators. For example, Szasz Bezier operators introduced by Zeng [10], are defined by*

$$L_n(f, x) = \sum_{k=0}^{\infty} R_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right),$$

where $R_{n,k}^{(\alpha)}(x) = \left(\sum_{j=k}^{\infty} q_{n,k}(x) \right)^\alpha - \left(\sum_{j=k+1}^{\infty} q_{n,k}(x) \right)^\alpha$, $\alpha \geq 1$ and

$q_{n,k}(x) = e^{-nx} \frac{(nx)^k}{k!}$ is the Szasz basis function. Some basic properties of $R_{n,k}^{(\alpha)}(x)$ can be found in [10].

We can define the Durrmeyer type modification of the operators L_n as

$$S_{n,\alpha}(f, x) = \sum_{k=0}^{\infty} R_{n,k}^{(\alpha)}(x) \int_0^{\infty} b_{n,k}(t) dt, \quad x \in [0, \infty), \tag{10}$$

where $f \in L_1[0, \infty)$ and $b_{n,k}(t)$ is defined as in eq.(1).

Actually the operators $S_{n,\alpha}(f, x)$ defined above are hybrid Durrmeyer type Szasz-Bezier operators, as in this case we have taken the entirely different weight functions.

We may note here that the rates of convergence in terms of Chanturiya's modulus of variation for the operators $S_{n,\alpha}(f, x)$ are not possible. The main problem is in the estimation of $S_{n,\alpha}(\text{sign}(t-x), x)$, because we can not relate easily the integration of Baskakov basis functions with the summation of Szasz basis function. The other approximation properties like direct, inverse and saturation results for the operators $S_{n,1}(f, x)$ are easier, but the analogues results for the operators $S_{n,\alpha}(f, x)$ (even for $S_{n,1}(f, x)$) are still unresolved. This may be considered as an open problem to the readers.)

Acknowledgment

The author is thankful to the referee for making valuable suggestions leading to the better presentation of the paper.

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Received 08.04.2003