Turk J Math
28 (2004) , 281 – 293.

Fuzzy β-Compactness and Fuzzy β-Closed Spaces

I. M. Hanafy

Abstract

The concepts of β-compactness and β- closed spaces in the fuzzy setting are defined and investigated. Fuzzy filterbases are used to characterize these concepts. A comparison between these types and some different forms of compactness in fuzzy topology is established.

Key Words: Fuzzy topological spaces, fuzzy β-compactness, fuzzy β-closed spaces, fuzzy filterbases.

1. Introduction

Compactness occupies a very important place in fuzzy topology and so do some of its forms. In [1] , Abd El- Monsef et al introduced the concepts of β-open sets and β-continuous functions in general topology and Fath Alla in [8] introduced these concepts in fuzzy setting. In [5], some interesting properties of fuzzy β- compactness are investigated. The purpose of this paper is devoted to introduce and study the concepts of β-compactness and β-closed spaces in fuzzy setting. These notions generalize basic classical results (see [1], [2], [3], [4], [7] and [11]).Using fuzzy filterbases, we characterize fuzzy β-compactness and fuzzy β-closed spaces. We also explore some expected basic properties of these concepts.

AMS Subject Classification Code: 54A40, 54D30 and 54A20

2. Preliminaries

Throughout this paper X and Y mean fuzzy topological spaces (fts, for short). A fuzzy point x_t in X is a fuzzy set having support $x \in X$ and value $t \in (0,1]$ [13]. The complement and the support of a fuzzy set u denoted by $\acute{u}$ and $S(u)$, respectively. For two fuzzy sets u and v, we shall write uqv ($u\widetilde{q}v$) to mean that u is quasi coincident (not quasi coincident) with v, *i.e.*, there exists $x \in X$ such that $u(x) + v(x) > 1$ ($u(x) + v(x) \leq 1$) [13].

Definition 2.1 *A fuzzy set u in a fts X is said to be:*

(a) semiopen fuzzy set if $u \leq cl\ int\ u$ [4];

(b) preopen fuzzy set if $u \leq int\, cl\ u$ [15];

(c) β-open fuzzy set if $u \leq cl\ int\ cl\ u$ [8], *equivalently, if there exists a preopen fuzzy set A such that $A \leq u \leq \overline{A}$.*

It is obvious that each semiopen and preopen fuzzy set implies β-open.

Definition 2.2 [4, 10]. *Let u be a fuzzy set in a fts X, the fuzzy pre-closure (resp. semi-closure, pre-interior and semi-interior) of u denoted by $pcl\ u$ (resp. $scl\ u$, $pint\ u$ and $\sin t\ u$) are defined as follows:*

$pcl\ u\,(scl\ u) = \wedge\ \{A :\ u \leq A,\ A$ *is preclosed (semiclosed)*$\}$;

$pint\ u\,(\sin t\ u) = \vee\ \{A :\ u \geq A,\ A$ *is preopen (semiopen)*$\}$.

Definition 2.3 *Let u be a fuzzy set in a fts X. The fuzzy β-closure (βcl) and β-interior (βint) of u are defined as follows:*

$\beta cl\ u\ = \wedge\ \{A :\ u \leq A,\ A$ *is β-closed*$\}$;

$\beta int\ u = \vee\ \{A :\ u \geq A,\ A$ *is β-open*$\}$.

It is obvious that $\beta cl\ \acute{u} = (\beta int\ u)'$ and $\beta int\ \acute{u} = (\beta cl\ \ u)'$.

Definition 2.4 *A function $f : X \to Y$ is said to be fuzzy β-continuous* [8] *(resp. $M\beta$-continuous) if the inverse image of every open (resp. β- open) fuzzy set in Y is β-open (resp. β-open) fuzzy set in X.*

Lemma 2.5 *Let $f : X \to Y$ be a function, then the following are equivalent:*

(a) f is fuzzy $M\beta$-continuous.

(b) $f(\beta cl\, u) \leq \beta cl\ f(u)$, for every fuzzy set u inX.

Proof. (a)⇒(b): Let u be a fuzzy set of X, then $\beta cl\, f(u)$ is β-closed. By (a) $f^{-1}(\beta cl\, f(u))$ is β-closed and so $f^{-1}(\beta cl\, f(u)) = \beta cl\, f^{-1}(\beta cl\, f(u))$. Since $u \leq f^{-1}(f(u))$, we have $\beta cl\, u \leq \beta cl\ \ f^{-1}(f(u)) \leq \beta cl\, f^{-1}(\beta cl\, f(u)) = f^{-1}(\beta cl\, f(u))$. Hence $f(\beta cl\, u) \leq \beta cl\, f(u)$.

(b)⇒(a): Let v be a β-closed fuzzy set in Y. By (b) if $u = f^{-1}(v)$, then $\beta cl\, f^{-1}(v) \leq f^{-1}(\beta cl\, f(f^{-1}(v))) \leq f^{-1}(\beta cl\, v) = f^{-1}(v)$. Since $f^{-1}(v) \leq \beta cl\, f^{-1}(v)$, then $f^{-1}(v) = \beta cl\, f^{-1}(v)$. Hence $f^{-1}(v)$ is β-closed fuzzy set in X. Hence f is fuzzy $M\beta$- continuous. □

Lemma 2.6 *Let $f : X \to Y$ be a function, then the following are equivalent:*

(a) f is fuzzy β-continuous.

(b) $f(\beta cl\, u) \leq cl\, f(u)$, for every fuzzy set u in X.

Proof. Obvious □

Theorem 2.7 [16]. *If $f : X \to Y$ is fuzzy open function, then $f^{-1}(cl(u)) \leq cl(f^{-1}(u))$, for every fuzzy set u in Y.*

Definition 2.8 [9]. *A collection of fuzzy subsets ξ of a fts X is said to form a fuzzy filterbases iff for every finite collection $\{A_j : j = 1, ..., n\}$, $\bigwedge_{j=1}^{n} A_j \neq 0_X$.*

Definition 2.9 [9]. *A collection μ of fuzzy sets in a fts X is said to be cover of a fuzzy set u of X iff $(\bigvee_{A \in \mu} A)(x) = 1_X$, for every $x \in S(u)$. A fuzzy cover μ of a fuzzy set u in a fts X is said to have a finite subcover iff there exists a finite subcollection $\eta = \{A_1, ..., A_n\}$ of μ such that $(\bigvee_{j=1}^{n} A_j)(x) \geq u(x)$, for every $x \in S(u)$.*

Definition 2.10 *A fts X is said to be strongly compact* [14] *(resp. semicompact* [12] *) iff every preopen (resp. semiopen) cover of X has a finite subcover.*

Definition 2.11 *A fts X is said to be almost compact* [7] *(resp. S-closed* [6]*, s-closed* [11]*, P-closed* [17]*) iff every open (resp. semiopen, semiopen, preopen) cover of X has a finite subcollection whose closures (resp. closures, semi-closures, pre-closures) cover X.*

3. Fuzzy β-Compact Space

Definition 3.1 [5]. *A fts X is said to be fuzzy β-compact iff for every family μ of β-open fuzzy sets such that $\underset{A\in\mu}{\vee} A = 1_X$ there is a finite subfamily $\eta \subseteq \mu$ such that $\underset{A\in\eta}{\vee} A = 1_X$.*

Definition 3.2 *A fuzzy set u in a fts X is said to be fuzzy β-compact relative to X iff for every family μ of β-open fuzzy sets such that $\underset{A\in\mu}{\vee} A \geq u(x)$ there is a finite subfamily $\eta \subseteq \mu$ such that $\underset{A\in\eta}{\vee} A \geq u(x)$ for every $x \in S(u)$.*

Remark 3.3 *Since each of semiopen and preopen fuzzy set implies β-open, it is clear that every fuzzy β-compact space implies each of fuzzy strongly compact space and fuzzy semicompact space. But the converse need not be true.*

Theorem 3.4 *A fts X is β-compact iff for every collection $\{A_j : j \in J\}$ of β-closed fuzzy sets of X having the finite intersection property, $\underset{j\in J}{\wedge} A_j \neq 0_X$.*

Proof. Let $\{A_j : j \in J\}$ be a collection of β-closed fuzzy sets with the finite intersection property. Suppose that $\underset{j\in J}{\wedge} A_j = 0_X$. Then $\underset{j\in J}{\vee} \acute{A}_j = 1_X$. Since $\{\acute{A}_j : j \in J\}$ is a collection of β-open fuzzy sets cover of X, then from the β-compactness of X it follows that there exists a finite subset $F \subseteq J$ such that $\underset{j\in J}{\vee} \acute{A}_j = 1_X$. Then $\underset{j\in F}{\wedge} A_j = 0_X$ which gives a contradiction and therefore $\underset{j\in J}{\wedge} A_j \neq 0_X$.

Conversely, Let $\{A_j : j \in J\}$ be a collection of β- open fuzzy sets cover of X. Suppose that for every finite subset $F \subseteq J$, we have $\underset{j\in F}{\vee} A_j \neq 1_X$. Then $\underset{j\in F}{\wedge} \acute{A}_j \neq 0_X$. Hence $\{\acute{A}_j : j \in J\}$ satisfies the finite intersection property. Then from the hypothesis we have $\underset{j\in J}{\wedge} \acute{A}_j \neq 0_X$ which implies $\underset{j\in F}{\wedge} A_j \neq 1_X$ and this contradicting that $\{A_j : j \in J\}$ is a β-open cover of X. Thus X is fuzzy β-compact.

Now, we give some results of fuzzy β-compactness in terms of fuzzy filterbases. □

Theorem 3.5 *A fts X is fuzzy β-compact iff every filterbases ξ in X, $\underset{G\in\xi}{\wedge} \beta cl\, G \neq 0_X$.*

Proof. Let μ be a β-open fuzzy set cover of X and μ has no a finite subcover. Then for every finite subcollection $\{A_1, ..., A_n\}$of μ, there exists $x \in X$ such that $A_j(x) < 1$ for every $j = 1, ..., n$. Then $\acute{A}_j(x) > 0$, so that $\overset{n}{\underset{j=1}{\wedge}} \acute{A}_j(x) \neq 0_X$. Thus $\{\acute{A}_j(x) : A_j \in \mu\}$ forms a filterbases in X. Since μ is β-open fuzzy set cover of X , then $(\underset{A_j\in\mu}{\vee} A_j)(x) = 1_X$ for every $x \in X$ and hence $\underset{A_j\in\mu}{\wedge} \beta cl\, \acute{A}_i(x) = \underset{A_j\in\mu}{\wedge} \acute{A}_j(x) = 0_X$, which is a contradiction. Then every β-open fuzzy set cover of X has a finite subcover and hence X is fuzzy β-compact.

Conversely, suppose there exists a filterbases ξ such that $\underset{G\in\xi}{\wedge} \beta cl\, G = 0_X$, so that $(\underset{G\in\xi}{\vee} (\beta cl\, G)')(x) = 1_X$ for every $x \in X$ and hence $\mu = \{(\beta cl\, G)' : G \in \xi\}$ is a β-open fuzzy set cover of X. Since X is fuzzy β-compact, then μ has a finite subcover. Then $(\overset{n}{\underset{j=1}{\vee}} (\beta cl\, G_j)')(x) = 1_X$ and hence $(\overset{n}{\underset{j=1}{\vee}} G'_j)(x) = 1_X$, so that $\overset{n}{\underset{j=1}{\wedge}} G_j = 0_X$ which is a contradiction, since the G_j are members of filterbases ξ. Therefore $\underset{G\in\xi}{\wedge} \beta cl\, G \neq 0_X$ for every filterbases ξ. □

Theorem 3.6 *A fuzzy set u in a fts X is fuzzy $\beta-$compact relative to X iff for every filterbases ξ such that every finite of members of ξ is quasi coincident with u, $(\underset{G\in\xi}{\wedge} \beta cl\, G) \wedge u \neq 0_X$.*

Proof. Let u not be fuzzy β-compact relative to X , then there exists a β-open fuzzy set μ cover of u such that μ has no finite subcover η. Then $(\underset{A_j\in\eta}{\vee} A_j)(x) < u(x)$ for some $x \in S(u)$, so that $(\underset{A_j\in\eta}{\wedge} \acute{A}_j)(x) > \acute{u}(x) \geq 0$ and hence $\xi = \{\acute{A}_j : A_j \in \mu\}$ forms a filterbases and $\underset{A_j\in\eta}{\wedge} \acute{A}_j\, qu$. By hypothesis $(\underset{A_j\in\eta}{\wedge} \beta cl\, \acute{A}_j) \wedge u \neq 0_X$ and hence $(\underset{A_j\in\eta}{\wedge} \acute{A}_j) \wedge u \neq 0_X$. Then for some $x \in S(u)$, $(\underset{A_j\in\mu}{\wedge} \acute{A}_j)(x) > 0_X$, that is $(\underset{A_j\in\mu}{\vee} A_j)(x) < 1_X$, which is a contradiction. Hence u is fuzzy β-compact relative to X.

Conversely, suppose that there exists a filterbases ξ such that every finite of members of ξ is quasi coincident with u and $(\underset{G\in\xi}{\wedge} \beta cl\, G) \wedge u \neq 0_X$. Then for every $x \in S(u)$, $(\underset{G\in\xi}{\wedge} \beta cl\, G)(x) = 0_X$ and hence $(\underset{G\in\xi}{\vee} (\beta cl\, G)')(x) = 1_X$ for every $x \in S(u)$. Thus $\mu = \{(\beta cl\, G)' : G \in \xi\}$ is β-open fuzzy set cover of u. Since u is fuzzy β-compact relative to X, then there exists a finite subcover, say $\{(\beta cl\, G_1)', ..., (\beta cl\, G_n)'\}$, such that $(\overset{n}{\underset{j=1}{\vee}} (\beta cl\, G_j)')(x) \geq u(x)$ for every $x \in S(u)$. Hence $(\overset{n}{\underset{j=1}{\wedge}} (\beta cl\, G_j))(x) \leq \acute{u}(x)$ for every $x \in S(u)$, so that $\overset{n}{\underset{j=1}{\wedge}} (\beta cl\, G_j) \,\widetilde{q}\, u$, which is a contradiction. Therefore for every filterbases ξ such that every finite of members of ξ is quasi coincident with u, $(\underset{G\in\xi}{\wedge} \beta cl\, G) \wedge u \neq 0_X$.

□

Theorem 3.7 *Every β-closed fuzzy subset of a fuzzy β-compact space is fuzzy β-compact relative to X.*

Proof. Let ξ be a fuzzy filterbases in X such that $uq \wedge \{G : G \in \lambda\}$ holds for every finite subcollection λ of ξ and a β-closed fuzzy set u. Consider $\xi^* = \{u\} \cup \xi$. For any finite subcollection λ^* of ξ^*, if $u \notin \lambda^*$, then $\wedge\lambda^* \neq 0_X$. If $u \in \lambda^*$ and since $uq \wedge \{G : G \in \lambda^* - u\}$, then $\wedge\lambda^* \neq 0_X$. Hence λ^* is a fuzzy filterbases in X. Since X is fuzzy β-compact, then $\underset{G\in\xi^*}{\wedge} \beta cl\, G \neq 0_X$, so that $(\underset{G\in\xi}{\wedge} \beta cl\, G) \wedge u = (\underset{G\in\xi}{\wedge} \beta cl\, G) \wedge \beta cl\, u \neq 0_X$. Hence by Theorem 3.6, we have u is fuzzy β-compact relative to X. □

Theorem 3.8 *If a function $f : X \to Y$ is fuzzy $M\beta$-continuous and u is fuzzy β-compact relative to X, then so is $f(u)$.*

Proof. Let $\{A_j : j \in J\}$ be a β-open fuzzy set cover of $S(f(u))$. For $x \in S(u)$, $f(x) \in f(S(u)) = S(f(u))$. Since f is fuzzy $M\beta$-continuous, then $\{ f^{-1}(A_j) : j \in J\}$ is β-open fuzzy set cover of $S(u)$. Since u is fuzzy β-compact relative to X, there is a finite subfamily $\{f^{-1}(A_j) : j = 1, ..., n\}$ such that $S(u) \leq \overset{n}{\underset{j=1}{\vee}} f^{-1}(A_j)$ which implies $S(u) \leq f^{-1}(\overset{n}{\underset{j=1}{\vee}} A_j)$ and then $S(f(u)) = f(S(u)) \leq ff^{-1}(\overset{n}{\underset{j=1}{\vee}} A_j) \leq \overset{n}{\underset{j=1}{\vee}} A_j$. Therefore $f(u)$ is fuzzy β-compact relative to Y. □

Lemma 3.9 *If $f : X \to Y$ is fuzzy open and fuzzy continuous function , then f is fuzzy $M\beta$-continuous.*

Proof. Let v be an β-open fuzzy set in Y, then $v \leq cl\,int\,cl\,v$. So $f^{-1}(v) \leq f^{-1}(cl\,int\,cl\,v) \leq cl\,(f^{-1}(int\,cl\,v))$. Since f is fuzzy continuous, then $f^{-1}(int\,cl\,v) = int\,(f^{-1}(cl\,v))$. Also by Theorem 2.7, $f^{-1}(int\,cl\,v) = int\,(f^{-1}(int\,cl\,v)) \leq int\,(f^{-1}(cl\,v)) \leq int\,cl\,(f^{-1}(v))$. Thus $f^{-1}(v) \leq cl\,(f^{-1}(int\,cl\,v)) \leq cl\ int\,cl(f^{-1}(v))$. Hence the result. □

Corollary 3.10 *Let $f : X \to Y$ be fuzzy open and fuzzy continuous function and X is fuzzy β-compact , then $f(X)$ is fuzzy β-compact .*

Proof. It is follows directly from Lemma 3.9 and Theorem 3.8. □

Definition 3.11 *A function $f : X \to Y$ is said to be fuzzy $M\beta$-open iff the image of every β-open fuzzy set in X is β-open in Y.*

Theorem 3.12 *Let $f : X \to Y$ be a fuzzy $M\beta$-open bijective function and Y is fuzzy β-compact, then X is fuzzy β-compact.*

Proof. Let $\{A_j : j \in J\}$ be a collection of β-open fuzzy set cover of X, then $\{f(A_j) : j \in J\}$ is β-open fuzzy set covering of Y. Since Y is fuzzy β-compact, there is a finite subset $F \subseteq J$ such that $\{f(A_j) : j \in F\}$ is an cover of Y. But $1_X = f^{-1}(1_Y) = f^{-1}f(\underset{j\in F}{\vee} A_j) = \underset{j\in F}{\vee} A_j$ and therefore X is fuzzy β-compact. □

4. Fuzzy β-Closed Spaces

Definition 4.1 *A fuzzy set u in a fts X is said to be a $\beta q - nbd$ of a fuzzy point x_t in X if there exists a β-open fuzzy set $A \leq u$ such that $x_t q A$.*

Theorem 4.2 *Let x_t be a fuzzy point in a fts X and u be any fuzzy set of X, then $x_t \in \beta cl\, u$ iff for every βq-nbd H of x_t, Hqu.*

Proof. Let $x_t \in \beta cl\, u$ and there exists a $\beta q - nbd$ H of x_t, $H \,\widetilde{q}\, u$. Then there exists a β-open fuzzy set $A \leq H$ in X such that $x_t q A$, which implies $A \,\widetilde{q}\, u$ and hence $u \leq \acute{A}$. Since $\acute{A}$ is β-closed fuzzy set, then $\beta cl\, u \leq \acute{A}$. Since $x_t \notin \acute{A}$, then $x_t \notin \beta cl\, u$, which is a contradiction.

Conversely, let $x_t \notin \beta cl\, u = \wedge\{A : A$ is β-closed in $X, A \geq u\}$. Then there exists a β-closed fuzzy set $A \geq u$ such that $x_t \notin A$. Hence $x_t q \acute{A} = H$, where H is a β-open fuzzy set in X and $H \ \widetilde{q}\ u$. Then there exists a $\beta q-nbd$ H of x_t with $H \ \widetilde{q}\ u$. Hence the result. □

Definition 4.3 *A fts X is said to be $\beta-$closed iff for every family μ of β-open fuzzy set such that $\underset{A\in\mu}{\vee} A = 1_X$ there is a finite subfamily $\eta \subseteq \mu$ such that $(\underset{A\in\eta}{\vee} \beta cl\, A)(x) = 1_X$, for every $x \in X$.*

Remark 4.4 *From the above definition and other types of fuzzy compactness, one can draw the following diagram:*

$$\begin{array}{ccccc} F\text{-}semicompact & \rightarrow & Fs\text{-}closed & \rightarrow & FS\text{-}closed \\ \uparrow & & \uparrow & & \\ F\beta\text{-}compact & \rightarrow & F\beta\text{-}closed & & \\ \downarrow & & \downarrow & & \\ F\text{-}strongly\,compact & \rightarrow & FP\text{-}closed & & \end{array}$$

where F =fuzzy.

Example 4.5 *Let $X \neq 0_X$ be a set and $u_n(x) = 1 - \frac{1}{n}$ for every $x \in X$ and $n \in N^+$. The collection $\{u_n : n \in N^+\}$ is a base for a fuzzy topology on X. The collection $\{u_n : n \in N^+\}$ is obviously a $\beta-$open fuzzy set cover of X. On the other hand we have $\beta cl\, u = 1_X$ for every $n \geq 3$. Hence X is β-closed but not fuzzy β-compact, (see* [6]*).*

Remark 4.6 *Example 4.5 also shows that:*

(i) Each of the concepts $Fs-$closed, $FS-$closed and $FP-$closed spaces does not imply $F\beta$-compact.

(ii) Since the collection $\{u_n : n \in N^+\}$ is also semiopen (resp. preopen) fuzzy sets cover of X, then X is $F\beta$-closed space but not $F-$semicompact space (resp. $F-$strongly compact space).

Example 4.7 *Let $X = I = [0, 1]$ and consider the following fuzzy sets*

$U_1(x) = \frac{1.7}{\sqrt{3}}$, $U_2(x) = \frac{1.73}{\sqrt{3}}$, $U_3(x) = \frac{1.732}{\sqrt{3}}$, ...,$\forall x \in I$.

Let $\sigma = \{u_j : j \in N^+\} \cup \{0_X,\ 1_X\}$*. It is clear that* σ *is a fuzzy topology on* X*. Now, the collection* $\{u_j : j \in N^+\}$ *is a semiopen (resp. preopen) fuzzy set cover of* X *but not has a finite subcover. So* X *is not* $F-$*semicompact space (resp.* $F-$*strongly compact space). Since the semi-closure (resp. pre-closure) of every semiopen (resp. preopen) fuzzy set of* X *is* 1_X*, then* X *is* $Fs-$*closed (resp.* $FP-$*closed).*

Remark 4.8 *Example 4.7 is also shows that each of the concepts* $FS-$*closed and* $FP-$*closed spaces does not imply each of* $F-$*semicompact and* $F-$*strongly compact spaces.*

Remark 4.9 *From Remark 3.3, Example 4.5, Remark 4.6, Example 4.7 and Remark 4.8, it is clear that:*

(i) $FS-$*closed and* $FP-$*closed spaces are independent notions.*

(ii) $FS-$*closed and F-strongly compact spaces are independent notions.*

(iii) $FP-$*closed and F-semicompact spaces are independent notions.*

(iv) $F\beta$*-compact, F-semicompact and F-strongly compact spaces are independent notions.*

Theorem 4.10 *A fts* X *is* β*-closed iff for every fuzzy* β*-open filterbases* ξ *in* X, $\underset{G\in\xi}{\wedge}\beta cl\, G \neq 0_X$.

Proof. Let μ be a β-open fuzzy set cover of X and let for every finite subfamily η of μ, $(\underset{A\in\eta}{\vee}\beta cl\, A)(x) < 1_X$ for some $x \in X$. Then $(\underset{A\in\eta}{\wedge}\beta cl\, \acute{A})(x) > 0_X$ for some $x \in X$. Thus $\{(\beta cl\, A)' : A \in \mu\} = \xi$ forms a fuzzy β-open filterbases in X. Since μ is a β- open fuzzy set cover of X, then $\underset{A\in\mu}{\wedge}\acute{A} = 0_X$ which implies $\underset{A\in\mu}{\wedge}\beta cl(\beta cl\, A)' = 0_X$, which is a contradiction. Then every β-open fuzzy set μ cover of X has a finite subfamily η such that $(\underset{A\in\eta}{\vee}\beta cl\, A)(x) = 1_X$ for every $x \in X$. Hence X is β-closed.

Conversely, suppose there exists a fuzzy β-open filterbases ξ in X such that $\underset{G\in\xi}{\wedge}\beta cl\, G = 0_X$, so that $(\underset{G\in\xi}{\vee}(\beta cl\, G)')(x) = 1_X$ for every $x \in X$ and hence $\mu = \{(\beta cl\, G)' : G \in \xi\}$ is a β- open fuzzy set cover of X. Since X is β-closed, then μ has a finite subfamily η such that $(\underset{G\in\eta}{\vee}\beta cl(\beta cl\, G)')(x) = 1_X$ for every $x \in X$, and hence $\underset{G\in\eta}{\wedge}(\beta cl(\beta cl\, G)')' = 0_X$. Thus $\underset{G\in\eta}{\wedge} G = 0_X$ which is a contradiction, since all the G are members of filterbases. □

Definition 4.11 *A fuzzy set u in a fts X is said to be β-closed relative to X iff for every family μ of β-open fuzzy sets such that $\underset{A\in\mu}{\vee} A = u$, there is a finite subfamily $\eta \subseteq \mu$ such that $(\underset{A\in\eta}{\vee} \beta cl\, A)(x) \geq u(x)$ for every $x \in S(u)$.*

Theorem 4.12 *A fuzzy subset u in a fts X is β- closed relative to X iff every fuzzy β-open filterbases ξ in X, $(\underset{G\in\xi}{\wedge} \beta cl\, G) \wedge u = 0_X$, there exists a finite subfamily λ of ξ such that $(\underset{G\in\lambda}{\wedge} G)\ \widetilde{q}\ u$.*

Proof. Let u be a β-closed relative to X, suppose ξ is a fuzzy β-open filterbases in X such that for every finite subfamily λ of ξ, $(\underset{G\in\lambda}{\wedge} G) q u$, but $(\underset{G\in\xi}{\wedge} \beta clG) \wedge u = 0_X$. Then for every $x \in S(u)$, $(\underset{G\in\xi}{\wedge} \beta clG)(x) = 0_X$ and hence $(\underset{G\in\xi}{\vee} (\beta clG)')(x) = 1_X$ for every $x \in S(u)$. Then $\mu = \{(\beta clG)' : G \in \xi\}$ is a β-open fuzzy set cover of u and hence there exists a finite subfamily $\lambda \subseteq \xi$ such that $\underset{G\in\lambda}{\vee} \beta cl(\beta clG)' \geq u$, so that $\underset{G\in\lambda}{\wedge} (\beta cl(\beta clG)')' = \underset{G\in\lambda}{\wedge} \beta int(\beta clG) \leq \acute{u}$ and hence $\underset{G\in\lambda}{\wedge} G \leq \acute{u}$. Then $\underset{G\in\lambda}{\wedge} G\ \widetilde{q}\ u$ which is a contradiction.

Conversely, let u not be a β-closed fuzzy set relative to X, then there exists a β-open fuzzy set μ cover of u such that every finite subfamily $\eta \subseteq \mu$, $(\underset{A\in\eta}{\vee} \beta clA)(x) \leq u(x)$ for some $x \in S(u)$ and hence $(\underset{A\in\eta}{\wedge} (\beta clA)')(x) > \acute{u}(x) \geq 0$ for some $x \in S(u)$. Thus $\xi = \{(\beta clA)' : A \in \mu\}$ forms a fuzzy β-open filterbases in X. Let there exists a finite subfamily $\{(\beta clA)' : A \in \eta\}$ such that $(\underset{A\in\eta}{\wedge} (\beta clA)')\ \widetilde{q}\ u$. Then $u \leq \underset{A\in\eta}{\vee} \beta clA$. So there exists a finite subfamily $\eta \subseteq \mu$ such that $\underset{A\in\eta}{\vee} \beta clA \geq u$ which is a contradiction. Then for each finite subfamily $\lambda = \{(\beta clA)' : A \in \eta\}$ of ξ, we have $(\underset{A\in\eta}{\wedge} (\beta clA)') q u$. Hence by the given condition $(\underset{A\in\mu}{\wedge} \beta cl\, (\beta clA)') \wedge u \neq 0_X$, so there exists $x \in S(u)$ such that $(\underset{A\in\mu}{\wedge} \beta cl\, (\beta clA)')(x) > 0_X$. Then $(\underset{A\in\mu}{\vee} (\beta cl\, (\beta clA)')')(x) = (\underset{A\in\mu}{\vee} \beta int\, (\beta clA))(x) < 1_X$, and hence $(\underset{A\in\mu}{\vee} A)(x) < 1_X$ which contradicts the fact that μ is a $\beta-$open fuzzy set cover of u. Therefore u is fuzzy β-closed relative to X. □

Definition 4.13 *A fuzzy set u of X is said to be fuzzy $\beta - regular$ if it is both β-open* *and β-closed fuzzy set.*

Proposition 4.14 *If u is β-open fuzzy set in X, then $\beta cl\, u$ is $\beta - regular$.*

Proof. Since $\beta cl\, u$ is β-closed, we must show that $\beta cl\, u$ is β-open. Since u is β-open in X, $v \leq u \leq cl\, v$ holds for some preopen fuzzy set v in X. Therefore, we have $v \leq \beta cl\, v \leq \beta cl\, u \leq cl\, v$, and hence $\beta cl\, u$ is β-open. □

Theorem 4.15 *For a fts X, the following are equivalent:*

(a) X is β-closed space.

(b) Every cover of X by fuzzy $\beta - regular$ sets has a finite subcover.

(c) For every collection $\{A_j : j \in J\}$ of fuzzy $\beta - regular$ sets such that $\underset{j\in J}{\wedge} A_j = 0_X$, there exists a finite subset $F \subseteq J$ such that $\underset{j\in F}{\wedge} A_j = 0_X$.

Proof. It is obvious from Proposition 4.14 and from the facts that, for every collection $\{A_j : j \in J\}$, $(\underset{j\in J}{\vee} A_j)' = \underset{j\in J}{\wedge} \acute{A}_j$, $(\underset{j\in J}{\wedge} A_j)' = \underset{j\in J}{\vee} \acute{A}_j$ and

A is β-open fuzzy set iff $\acute{A}$ is β-closed fuzzy set. □

Theorem 4.16 *Let $f : X \to Y$ be a fuzzy β-continuous surjection function. If X is β-closed space, then Y is almost compact.*

Proof. Let $\{A_j : j \in J\}$ be an open fuzzy set cover of Y. Then $\{f^{-1}(A_j) : j \in J\}$ is a β-open fuzzy set cover of X. By hypothesis, there exists a finite subset $F \subseteq J$ such that $\underset{j\in F}{\vee} \beta cl\, f^{-1}(A_j) = 1_X$. From the surjectivity of f and by Lemma 2.6, $1_Y = f(1_X) = f(\underset{j\in F}{\vee} \beta cl\, f^{-1}(A_j)) \leq \underset{j\in F}{\vee} cl f(f^{-1}(A_j)) = \underset{j\in F}{\vee} cl A_j$. Hence Y is almost compact. □

Using Lemma 2.5, we have also the following theorem which can proved similarly to Theorem 4.16.

Theorem 4.17 *If $f : X \to Y$ is fuzzy $M\beta$-continuous surjection function and X is fuzzy $\beta - closed$ space, then Y is so.*

Acknowledgement

It is a pleasure to thank the referees for their comments which resulted in an improved presentation of the paper.

References

[1] M.E. Abd El-Monsef and S.N. El-Deeb and R.A. Mahmoud, β-open sets and β-continuous mappings, Bull. Fac. Sci., Assiut Univ., 12, 77-90 (1983).

[2] M.E. Abd El-Monsef and A.M. Kozae, Some generalized forms of compactness and closedness, Delta J. Sci. 9(2), 257-269 (1985).

[3] A. Andrijevic, Semi-preopen sets. Mat. Vesnik 38, 24-32 (1986).

[4] K.K. Azad, On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity, J. Math. Anal. Appl., 82, 14-32 (1981).

[5] Balasubramanian, Ganesan, On fuzzy β-compact spaces and fuzzy β- extremally disconnected spaces, Kybernetika [cybernetics] 33, no. 3, 271-277 (1997).

[6] D. Coker and A. Haydar Es, On fuzzy S-closed spaces, Doga Tu J. Math., 145-152 (1987).

[7] A. Di Concillio and G. Gerla, Almost compactness in fuzzy topological spaces, Fuzzy Sets and Systems, 13, 187-192 (1984).

[8] M.A. Fath Alla, On fuzzy topological spaces, Ph. D. Thesis, Assiut Univ., Sohag, Egypt (1984).

[9] S. Ganguly and S. Saha, A note on compactness in fuzzy setting, Fuzzy Sets and Systems, 34, 117-124 (1990).

[10] B. Ghosh, Semicontinuous and semiclosed mappings and semi-connectedness in fuzzy setting, Fuzzy Sets and Systems, 35, 345-355 (1990).

[11] G. Di. Maio and T. Noiri, On s-closed spaces, Indian J. Pure Appl. Math., 18(3), 226-233 (1987).

[12] A.S. Mashhour, A.A. Allam and K.M. Abd El-Hakeim, On fuzzy semicompact spaces, Bull. Fac. Sci., Assiut Univ., 16(1), 277-285 (1987).

[13] Pu Pao-Ming and Liu Ying-Ming, Fuzzy topology I. Neighborhood structure of a fuzzy point and Moore-smith convergence, J. Math. Anal. Appl., 76, 571-599 (1980).

[14] S. Nanda, Strongly compact fuzzy topological spaces, Fuzzy Sets and Systems, 42, 259-262 (1991).

[15] M.K. Singal and N. Prakash, Fuzzy preopen sets and fuzzy preseparation axioms, Fuzzy Sets and Systems, 44, 273-281 (1991).

[16] T.H. Yalvac, Fuzzy sets and functions on fuzzy topological spaces, J. Math. Anal. Appl., 126, 409-423 (1987).

[17] A.M. Zahran, Strongly compact and P-closed fuzzy topological spaces, J. Fuzzy Math., 3(1), 97-102 (1995).

I. M. HANAFY
Department of Mathematics,
Faculty of Education,
Suez Canal University,
El-Arish-EGYPT
e-mail: ihanafy@hotmail.com

Received 14.04.2003