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Splitting of Sharply 2-Transitive Groups of Characteristic 3

Seyfi Türkelli

Abstract

We give a group theoretic proof of the splitting of sharply 2-transitive groups of characteristic 3.

Key Words: Sharply 2-transitive groups, Permutation groups.

A *sharply* 2*-transitive group* is a pair (G, X), where G is a group acting on the set X in such a way that for all x, y, z, $t \in X$ such that $x \neq y$ and $z \neq t$ there is a unique $g \in G$ for which $gx = z$ and $gy = t$. From now on, (G, X) will stand for a sharply 2-transitive group with $|X| \geq 3$. We fix an element $x \in X$. We let $H := \{g \in G : gx = x\}$ denote the stabilizer of x. Finally we let I denote the set of involutions (elements of order 2) of G.

It follows easily from the definition that the group G has an involution; in fact any element of G that sends a distinct pair (y, z) of X to the pair (z, y) is an involution by sharp transitivity. It is also known that I is one conjugacy class and the nontrivial elements of I^2 cannot fix any point (See Lemma 1 and Lemma 4). Then one can see that I^2 cannot have an involution if H has an involution.

In case H has no involution, one says that $\mathrm{char}(G) = 2$.

Let us assume that $\mathrm{char}(G) \neq 2$. Then $I^2 \setminus \{1\}$ is one conjugacy class [1, Lemma 11.45]. Since I^2 is closed under power taking, either the nontrivial elements of I^2 all have order p for some prime $p \neq 2$ or I^2 has no nontrivial torsion element. One writes $\mathrm{char}(G) = p$ or $\mathrm{char}(G) = 0$ depending on the case.

One says that G *splits* if the one point stabilizer H has a normal complement in G. It is not known whether or not an infinite sharply 2-transitive group splits, except for those of characteristic 3. Results in this direction for some special cases can be found in [1, §11.4] and [2, ch 2]. We will prove that if $\mathrm{char}(G) = 3$ then G splits, a result of W. Kerby [2, Theorem 8.7]. But Kerby's proof is in the language of near domains and is not easily accessible. Here, we give a much simpler proof of this fact, in fact an experienced reader can directly go to the proof the Theorem, which contains only a simple computation (all the lemmas are well-known facts).

All the results of this short and elementary paper can be found in [1, §11.4], except for the final theorem.

Lemma 1 *I is one conjugacy class.*

Proof. Let $i, j \in I$ and $x \in X$ be such that $jx \neq x$ and $ix \neq x$. Since G is 2-transitive, there exists a $g \in G$ such that $gx = x$ and $gjx = ix$. Then $i^g jx = x$ and $i^g j(jx) = jx$. By double sharpness of G, $i^g j = 1$. Hence, $i^g = j$ and we are done. □

Lemma 2 *If N is a nontrivial normal subgroup of G then $G = NH$.*

Proof. Let $g \in G \setminus H, a \in N, y \in X \setminus \{x\}$ be such that $ay \neq y$ and $h \in G$ be such that $hx = y$ and $hgx = ay$. Then $(a^{-1})^h g \in H$ and $g \in NH$. Since $1 \in N$, it holds for all $g \in G$. □

Lemma 3 *H has at most one involution.*

Proof. Let $i, j \in H \cap I$, $y \in X \setminus \{x\}$, $g \in G$ be such that $gjy = iy$ and $gy = y$. Then $ji^g(y) = y$ and $ji^g(jy) = jy$. Since ji^g fixes two different points and G is sharply 2-transitive, $ji^g = 1$ and $j = i^g$. One can easily see that $H \cap H^z \neq \{1\}$ if and only if $z \in H$. Therefore $g \in H$ as $j \in H \cap H^g$. Since g fixes two points, namely x and y, $g = 1$. Hence $i = j$ and we are done. □

Lemma 4 *A nontrivial element of I^2 cannot fix any element of X.*

Proof. Assume not. Then, there are distinct involutions i, j such that ij fixes a point. Since G is transitive, we may assume $ij \in H$. It follows from Lemma 3 that $j \notin H$ otherwise $i \in H$, hence a contradiction. On the other hand, $(ij)^{-1} = (ji) = (ij)^j$ and

$(ij)^j \in H \cap H^j$. Therefore, $j \in H$, a contradiction. $\square$

Lemma 5 *If the elements of Ii commute with each other for some $i \in I$, then I^2 is a normal subgroup of G.*

Proof. It suffices to prove that I^2 is closed under multiplication. Let i, j, k, $w \in I$. We claim that $ijkw \in I^2$. By Lemma 1, we may assume that the elements of Ii commute with each other. Noting that $Ii = iI$, we have $(ijk)^2 = ijkijk = kiijjk = 1$. So, $ijk \in I \cup \{1\}$. If $ijk \in I$, we are done. Assume $ijk = 1$. If H has an involution, by Lemma 1, $(ij)^g = k^g \in H$ for some $g \in G$, i.e. $(ij)^g$ fixes x, contradicting Lemma 4. If H has no involution, $ij = k \in I$ and, by Lemma 1, $I \subseteq I^2$. Therefore, $ijkw = w \in I^2$. $\square$

Lemma 6 *If H has an involution, then the action of G on X is equivalent to the action of G on I by conjugation.*

Proof. Let $i \in H$ be an involution. It is easy to see that the action of G on X is equivalent to the action of G on the left coset space G/H. So we may assume that the set X is the left coset space G/H. Consider the map from G/H to I defined as $\bar{g} \mapsto i^{g^{-1}}$ for $g \in G$. One can easily see that this is the required equivalence. $\square$

Theorem 1 *If* $\mathrm{char}(G) = 3$ *then G splits.*

Proof. We claim that $G = I^2 \rtimes H$. If I^2 is a normal subgroup of G, then we know that $H \cap I^2 = \{1\}$ by Lemma 4 and $G = I^2 H$ by Lemma 2. Therefore, we just need to prove that I^2 is a normal subgroup of G. By lemma 5, it is enough to show that the elements of Ii commute with each other for some $i \in I$. Let $i \in H \cap I$ be the (unique) involution of H and let $ji, ki \in Ii$. We may assume that $j \neq k$. By double sharpness of G, it suffices to prove that $jiki$ and $kiji$ agree on two different points. By Lemma 6, we can take X to be I and the action to be the conjugation. We now claim that $jiki$ and $kiji$ agree on j and k i.e. that $j^{jiki} = j^{kiji}$ and $k^{jiki} = k^{kiji}$. By symmetry of the situation, it is enough to prove one of the equalities. Since $\mathrm{char}(G) = 3$, $i^j = j^i$ for all $i, j \in I$ and so we have

$$j^{jiki} = j^{(k^i)} = (k^i)^j = k^{ij} = k^{jiji} = (k^j)^{iji} = (j^k)^{iji} = j^{kiji}.$$

$\square$

References

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Seyfi TÜRKELLİ
Park Rheyngaerde 100 D 16
3545 NE Utrecht The Netherlands
e-mail: turkelli@wisc.edu

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