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Determination of a Fractional-Linear Pencil of Sturm-Liouville Operators by Two of Its Spectra

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Abstract

In this paper we consider the Sturm-Liouville equations on a finite interval which is fractional-linear in the spectral parameter. The inverse spectral problem consisting of the recovering of the operator from the two spectra is investigated and a uniqueness theorem for solution of the inverse problem is proved.

Key Words: Eigenvalue, scattering function.

1. Introduction

Consider the second order differential equation

$$-y'' + q(x)y = \begin{cases} \lambda y, & a < x < b, \\ -\frac{1}{\lambda}y, & 0 < x < a, \end{cases} \tag{1}$$

where a and b are fixed positive numbers such that $0 < a < b$, $q(x)$ is a real-valued continuous function on $[0, a]$, and λ is a complex parameter (spectral parameter).

Now we attach to Equation (1) two boundary conditions

$$y'(0) - hy(0) = 0, \quad y(b) = 0, \tag{2}$$

and

$$y'(0) - hy(0) = 0, \quad y'(b) = 0, \tag{3}$$

where h is a fixed real number.

Let us denote the eigenvalues of boundary-value problems (1),(2) and (1),(3) by $\{\lambda_n\}$ and $\{\mu_n\}$, respectively.

In this paper we study the inverse spectral problem consisting in determination of the coefficient $q(x)$ of Equation (1) and the number h in boundary conditions (2),(3) from two spectra $\{\lambda_n\}$ and $\{\mu_n\}$. The main result of this paper is a theorem proved in Section 2 which states that the function $q(x)$ and number h are uniquely determined by two spectra $\{\lambda_n\}$ and $\{\mu_n\}$. We prove this theorem by reducing the inverse problem from two spectra to the inverse scattering problem investigated earlier in [1,2].

2. Investigation of the Inverse Spectral Problem

Put $\lambda = k^2$ in the Equation (1) and denote by $\varphi(x,k)$ and $\psi(x,k)$ its solutions satisfying the initial conditions

$$\varphi(0,k) = 1, \ \varphi'(0,k) = h, \ \psi(0,k) = 0, \ \psi'(0,k) = 1.$$

The eigenvalues $\{\lambda_n\}$ and $\{\mu_n\}$ of boundary-value problems (1),(2) and (1),(3) coincide with zeros of functions

$$\Phi_1(k) = \varphi(b,k) \text{ and } \Phi_2(k) = \varphi'(b,k), \tag{4}$$

respectively.

Let's consider the auxiliary boundary-value problem on the half axis:

$$-y'' + \tilde{q}(x)y = \begin{cases} k^2 y, & a < x < +\infty, \\ -\frac{1}{k^2} y, & 0 < x < a, \end{cases} \tag{5}$$

$$y'(0) - hy(0) = 0, \tag{6}$$

where

$$\widetilde{q}(x) = \begin{cases} q(x), & 0 < x < b, \\ 0, & b < x < \infty \end{cases}$$

We derive the formula by means of which we can determine the scattering data of problem (5),(6) from two known spectra $\{\lambda_n\}$ and $\{\mu_n\}$ of problems (1),(2) and (1),(3). So the solution of the inverse problem from two spectra will be reduced to the solution of inverse problem from scattering data which has already been investigated (see [1], [2]).

Denote by $f(x,k)$ the solution of Equation (5) that satisfies the condition

$$\lim_{x\to+\infty} f(x,k)e^{-ikx} = 1.$$

Lemma 1 *The formula*

$$f'(0,k) - hf(0,k) = \frac{1}{h}e^{ikb}\{\varphi'(b,k) - ik\varphi(b,k)\} \tag{7}$$

holds.

Proof. Since functions $\varphi(x,k)$ and $\psi(x,k)$ form a fundamental system of solutions of Equation (5), the solution $f(x,k)$ of the same equation is equal to their linear combination:

$$f(x,k) = c_1\varphi(x,k) + c_2\psi(x,k).$$

Since the potential $\widetilde{q}(x)$ is equal to zero when $x > b$ we have $f(x,k) = e^{ikx}$ if $x > b$. Therefore the continuity of $f(x,k)$ and $f'(x,k)$ at $x = b$ yields

$$e^{ikb} = c_1\varphi(b,k) + c_2\psi(b,k),$$

$$ike^{ikb} = c_1\varphi'(b,k) + c_2\psi'(b,k)$$

from which we find the coefficients c_1 and c_2:

$$c_1 = \frac{1}{h} e^{ikb} \{ik\varphi(b,k) - \varphi'(b,k)\},$$

$$c_2 = \frac{1}{h} e^{ikb} \{\varphi'(b,k) - ik\varphi(b,k)\}.$$

Hence

$$f(x,k) = \frac{1}{h} e^{ikb} \{ik\varphi(b,k) - \varphi'(b,k)\} \varphi(x,k) + \frac{1}{h} e^{ikb} \{\varphi'(b,k) - ik\varphi(b,k)\} \psi(x,k)$$

and the formula (7) follows. The lemma is proved. □

Theorem 2 *The function $q(x)$ and number h are uniquely determined by two spectra $\{\lambda_n\}$ and $\{\mu_n\}$.*

Proof. It is known that the scattering function of problem (5),(6) has the form ([1],[2]):

$$S(k) = \frac{f'(0,-k) - hf(0,-k)}{f'(0,k) - hf(0,k)}.$$

So by virtue of (7) we get

$$S(k) = \frac{\varphi'(b,k) + ik\varphi(b,k)}{\varphi'(b,k) - ik\varphi(b,k)} e^{-2kbi}.$$

Hence, by (4) we find

$$S(k) = e^{-2kbi} \frac{\Phi_2(k) + ik\Phi_1(k)}{\Phi_2(k) - ik\Phi_1(k)}. \tag{8}$$

Let us set

$$F(k) = \Phi_2(k) - ik\Phi_1(k).$$

We can prove in a standard way that $\{\lambda_n\}$ and $\{\mu_n\}$ are intermittent, so that the argument of $F(k)$ as k describes the real axis once is zero (see [3]). Hence, $F(k)$ is non-zero in the upper halfplane, i.e. problem (5),(6) has no eigenvalues. Thus the scattering data of problem (5),(6) consist of the function $S(k)$ only. On the other hand in [2] it has been proved that the coefficient $\widetilde{q}(x)$ of Equation (5) and the number h in the boundary condition (6) are uniquely determined by the scattering data of boundary value problem (5),(6). Since The scattering function $S(k)$ is uniquely determined by (8) from two spectra $\{\lambda_n\}$ and $\{\mu_n\}$, the statement of the theorem follows. □

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