Turk J Math
28 (2004) , 335 – 352.

Coisotropic Submanifolds of a Semi-Riemannian Manifold*

Erol Kılıç, Bayram Şahin, H. Bayram Karadağ, Rıfat Güneş

Abstract

In this paper, we study coisotropic submanifolds of a semi-Riemannian manifold. We investigate the integrability condition of the screen distribution and give a necessary and sufficient condition on Ricci tensor of a coisotropic submanifold to be symmetric. Finally, we present some new theorems and results about totally umbilical coisotropic submanifolds of a semi-Riemannian manifold.

Key words and phrases: Semi-Riemannian manifold, Lightlike submanifolds, Coisotropic submanifolds.

1. Introduction

The geometry of lightlike submanifolds of a semi-Riemannian manifold is one of the interesting topics of differential geometry. In [2], Bejancu-Duggal have constructed a transversal vector bundle of a lightlike submanifold. D. N. Kupeli [5], using the canonical projection, has investigated the properties of these submanifolds. On the other hand, Duggal and Jin have studied totally umbilical half-lightlike submanifolds in semi-Riemannian manifolds, of codimension 2 [4].

In this paper, we consider coisotropic submanifolds which were proposed as a research problem by Duggal and Jin in [4]. We obtain a necessary and sufficient condition for integrability of the screen distribution. Also, we investigate Ricci tensor of a coisotropic sub-

Mathematics Subject Classification (2000): 53C50, 53C15.
*This work was supported by İnönü University.

manifold and give a necessary and sufficient condition on the Ricci tensor of a coisotropic submanifold to be symmetric. Moreover, we prove that the null sectional curvatures of an ambient space and of a coisotropic submanifold are the same for a totally umbilical coisotropic submanifold.

2. Preliminaries

Let $(\overline{M}, \overline{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1,\ 1 \leq q \leq m+n-1$ and (x^i) be a local coordinate system at a point $x \in \overline{M}$. Then the associated quadratic form of $\overline{g}$ is a mapping $h : T_x(\overline{M}) \rightarrow R$ given by $h(X) = \overline{g}(X, X)$ for any $X \in T_x(\overline{M})$. Using a well-known result from linear algebra, we have the following canonical form for h (with respect to a local basis of $T_x(\overline{M})$):

$$h = -\sum_{I=1}^{q}(w^I)^2 + \sum_{A=q+1}^{m+n}(w^A)^2,$$

where $w^1, \cdots, w^{m+n}$ are linearly independent local differential 1-forms on $\overline{M}$. With respect to the local coordinate system (x^i), by replacing in above each $w^I = w^I_i dx^i$ and each $w^A = w^A_i dx^i$, we obtain

$$h = \overline{g}_{ij} dx^i dx^j,\ rank|\overline{g}_{ij}| = m+n,$$

$$\overline{g}_{ij} = \overline{g}(\partial_i, \partial_j) = -\sum_{I=1}^{q} w^I_i w^I_j + \sum_{A=q+1}^{m+n} w^A_i w^A_j,$$

where q is the index of $\overline{g}$.

Now, let M be an m-dimensional submanifold of $\overline{M}$ and g the induced metric of $\overline{g}$ on M. In this paper, we suppose that all manifolds are paracompact and smooth. M is called a lightlike (degenerate) submanifold of $\overline{M}$, if $\overline{g}$ is degenerate on the tangent bundle TM of M, [3]. We suppose that g is degenerate. Then, for each tangent space T_xM, $x \in M$,

$$T_xM^{\perp} = \{u \in T_x\overline{M} : \overline{g}(u, v) = 0, \forall v \in T_xM\}$$

is a degenerate n-dimensional subspace of $T_x\overline{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $Rad\ T_xM = T_xM \cap T_xM^{\perp}$ which is called radical (null) subspace. If the mapping

$$Rad\ TM : x \in M \longrightarrow Rad\ T_xM$$

defines a smooth distribution on M of rank $r > 0$, then the submanifold M of $\overline{M}$ is called r-lightlike (r-degenerate) submanifold and $Rad\ TM$ is called the radical (lightlike, null) distribution on M [3]. Following are four possible cases:

Case 1. r-lightlike submanifold. $1 \leq r < min\{m, n\}$.

Case 2. Coisotropic submanifold. $1 \leq r = n < m$.

Case 3. Isotropic submanifold. $1 \leq r = m < n$.

Case 4. Totally lightlike submanifold. $1 \leq r = m = n$.

For Case 1, there exists a non-degenerate screen distribution $S(TM)$ which is a complementary vector subbundle to $Rad\ TM$ in TM. Therefore,

$$TM = Rad\ TM \perp S(TM), \tag{1}$$

where $\perp$ denotes orthogonal direct sum. Although $S(TM)$ is not unique, it is isomorphic to the factor bundle $TM/Rad\ TM$. Denote an r-lightlike submanifold by $(M, g, S(TM), S(TM^{\perp}))$, where $S(TM^{\perp})$ is a complementary vector subbundle to $Rad\ TM$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vectors bundles to TM in $T\overline{M}\mid_M$ and to $Rad\ TM$ in $S(TM^{\perp})$, respectively. Then we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}), \tag{2}$$

$$\begin{aligned} T\overline{M}\mid_M &= TM \oplus tr(TM) \\ &= (RadTM \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}), \end{aligned} \tag{3}$$

where $\oplus$ denotes direct sum, but it is not orthogonal.

Now, we suppose that $\mathcal{U}$ is a local coordinate neighborhood of M. We consider the following local quasi-orthonormal field of frames of $\overline{M}$ along M, on $\mathcal{U}$:

$$\{\xi_1, ..., \xi_r, W_1, ..., W_{m-r}, N_1, ..., N_r, U_1, ..., U_{n-r}\}, \tag{4}$$

where $\{\xi_1,\ ...,\ \xi_r\}$, $\{N_1, ..., N_r\}$ are local lightlike bases of $\Gamma(Rad\, TM\mid_{\mathcal{U}})$, $\Gamma(ltr(TM)\mid_{\mathcal{U}})$ and $\{W_1, ..., W_{m-r}\}$ and $\{U_1, ..., U_{n-r}\}$ are local orthonormal bases of $\Gamma(S(TM)\mid_{\mathcal{U}})$ and $\Gamma(S(TM^{\perp})\mid_{\mathcal{U}})$, respectively.

For Case 2, we have $Rad\ TM = TM^{\perp}$. Therefore $S(TM^{\perp}) = \{0\}$ and from (2), $tr(TM) = ltr(TM)$. From (3) and (4), we can write

$$\begin{aligned} T\overline{M}\mid_M &= (Rad\ TM \oplus ltr(TM))\perp S(TM) \\ &= (TM^{\perp} \oplus ltr(TM))\perp S(TM), \end{aligned} \tag{5}$$

$$\{\xi_1, ..., \xi_r, W_1, ..., W_{m-r}, N_1, ..., N_r\}, \tag{6}$$

where $\{\xi_1,\ ...\ ,\xi_r\}$, $\{N_1, ..., N_r\}$ are local lightlike bases of $\Gamma(Rad\, TM\mid_{\mathcal{U}})$, $\Gamma(ltr(TM)\mid_{\mathcal{U}})$ and $\{W_1, ..., W_{m-r}\}$ is a local orthonormal basis of $\Gamma(S(TM)\mid_{\mathcal{U}})$, respectively.

For Case 3, we have $Rad\ TM = TM$. Thus $S(TM) = \{0\}$. Therefore, from (3) and (4), we have

$$T\overline{M}\mid_{\mathcal{M}} = (TM \oplus ltr(TM))\perp S(TM^{\perp}) \tag{7}$$

$$\{\xi_1, ..., \xi_r, N_1, ..., N_r, U_1, ..., U_{n-r}\}, \tag{8}$$

where $\{\xi_1,\ ...\ ,\xi_r\}$, $\{N_1, ..., N_r\}$ are local lightlike bases of $\Gamma(Rad\, TM\mid_{\mathcal{U}})$, $\Gamma(ltr(TM)\mid_{\mathcal{U}})$ and $\{U_1, ..., U_{n-r}\}$ is a local orthonormal basis of $\Gamma(S(TM^{\perp})\mid_{\mathcal{U}})$, respectively.

For Case 4, we have $Rad\ TM = TM = TM^{\perp}$, $S(TM) = S(TM^{\perp}) = \{0\}$. Therefore, from (3) and (4), we have

$$T\overline{M}\mid_M = (TM \oplus ltr(TM)) \tag{9}$$

$$\{\xi_1, ..., \xi_r, N_1, ..., N_r\}, \tag{10}$$

where $\{\xi_1, ..., \xi_r\}$, and $\{N_1, ..., N_r\}$ are local lightlike bases of $\Gamma(Rad\ TM\ \mid_{\mathcal{U}})$, and $\Gamma(ltr(TM)\mid_{\mathcal{U}})$, respectively.

For the dependence of all the induced geometric objects, of M, on $\{S(TM), S(TM^{\perp})\}$ we refer to [3].

Now, let $(\overline{M}, \overline{g})$ be an $(m+n)$-dimensional semi-Riemannian manifold with index $q \geq 1$ and M a coisotropic submanifold of $\overline{M}$, of codimension n. Then, there exists lightlike vector fields on a local coordinate neighborhood $\mathcal{U}$ of M, also denoted by ξ_i, such that

$$\overline{g}(\xi_i, X) = 0,\ \overline{g}(\xi_i, \xi_j) = 0,\ i, j = 1, ..., n,$$

for any $X \in \Gamma(TM\ |_{\mathcal{U}})$. Therefore, an n-dimensional radical distribution $Rad\ TM$ of the coisotropic submanifold M is locally spanned by $\{\xi_1, ..., \xi_n\}$. Then, there exists local lightlike vector fields N_i on $\mathcal{U}$, such that

$$\overline{g}(\xi_i, N_i) = 1,\ \overline{g}(\xi_i, N_j) = 0,\ i \neq j,\ \overline{g}(N_i, N_j) = 0,\ i, j = 1, ..., n,$$

where N_i are not tangent to M.

If we choose $\xi_i^* = \alpha_i \xi_i$, i=1,...,n, on another neighborhood of coordinates then we obtain $N_i^* = \frac{1}{\alpha_i} N_i$. Thus, the vector bundle $ltr(TM)$ is defined over M which is the canonical affine normal bundle of M with respect to the screen distribution $S(TM)$, where $ltr(TM)$ is a n-dimensional vector bundle locally spanned by $\{N_1, ..., N_n\}$.
Now, we give two examples for coisotropic submanifolds.

Example 2.1 *Suppose M is a submanifold of R_2^5 given by the equations*

$$x^3 = \frac{1}{\sqrt{2}}(x^2 + x^1), \quad x^4 = \frac{1}{\sqrt{2}}(x^2 - x^1).$$

Then

$$\begin{aligned} TM = Sp\{U_1 &= \frac{\partial}{\partial x^1} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x^3} - \frac{1}{\sqrt{2}}\frac{\partial}{\partial x^4},\ U_2 = \frac{\partial}{\partial x^2} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x^3} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x^4}, \\ U_3 &= \frac{\partial}{\partial x^5}\}, \end{aligned}$$

and

$$TM^\perp = Sp\{\xi_1 = U_1, \quad \xi_2 = U_2\}.$$

Thus, $Rad\ TM = TM^\perp \subset TM$, and M is an 3-dimensional coisotropic submanifold of R_2^5. Let $S(TM)$ be spanned by the spacelike vector field U_3. Then, a lightlike transversal *vector bundle $ltr(TM)$ is spanned by*

$$\{N_1 = -\frac{1}{2}\frac{\partial}{\partial x^1} + \frac{1}{2\sqrt{2}}\frac{\partial}{\partial x^3} - \frac{1}{2\sqrt{2}}\frac{\partial}{\partial x^4}, \quad N_2 = -\frac{1}{2}\frac{\partial}{\partial x^2} + \frac{1}{2\sqrt{2}}\frac{\partial}{\partial x^3} + \frac{1}{2\sqrt{2}}\frac{\partial}{\partial x^4}\}.$$

Example 2.2 *(Duggal and Bejancu, p. 152 in [3]) Consider in R^5_2 the submanifold M given by the equations*

$$x^2 = \{(x^3)^2 + (x^5)^2\}^{1/2}, \quad x^4 = x^1, \quad x^3 > 0, \quad x^5 > 0.$$

Then we have

$$TM = Sp\{U_1 = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^4}, \quad U_2 = x^3\frac{\partial}{\partial x^2} + x^2\frac{\partial}{\partial x^3}, \quad U_3 = x^5\frac{\partial}{\partial x^2} + x^2\frac{\partial}{\partial x^5}\},$$

and

$$TM^\perp = Sp\{\xi_1 = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^4}, \quad \xi_2 = x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} + x^5\frac{\partial}{\partial x^5}\}.$$

It follows that Rad $TM = TM^\perp \subset TM$. Hence M is an 3-dimensional coisotropic submanifold of R^5_2. Let $S(TM)$ be spanned by the spacelike vector field U_3 and the complementary vector bundle F of $TM^\perp$ in $S(TM)^\perp$ be spanned by

$$\{V_1 = \frac{\partial}{\partial x^1}, \quad V_2 = \frac{\partial}{\partial x^3}\}.$$

Moreover, $ltr(TM)$ is spanned by

$$\{N_1 = \frac{1}{2}(\frac{\partial}{\partial x^4} - \frac{\partial}{\partial x^1}), \quad N_2 = \frac{1}{2(x^3)^2}(-x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} - x^5\frac{\partial}{\partial x^5})\}.$$

Let us denote by P the projection of TM on $S(TM)$ with respect to the decomposition (5), then we can write

$$X = PX + \sum_{i=1}^{n} \eta_i(X)\xi_i, \tag{11}$$

for any $X \in \Gamma(TM)$, where $\eta_i,\ i = 1, ..., n$, are local differential 1-forms on M given by

$$\eta_i(X) = \overline{g}(X, N_i),\ i = 1, ..., n. \tag{12}$$

Let $\overline{\nabla}$ be the Levi-Civita connection on $\overline{M}$. Then, according to (1) and (5), the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{13}$$

$$\overline{\nabla}_X N_i = -A_{N_i} X + \nabla^{\perp}_X N_i,\ i = 1, ..., n, \tag{14}$$

for any $X, Y \in \Gamma(TM)$, where $\nabla_X Y$, $A_{N_i} X$ belong to $\Gamma(TM)$, while $h(X,Y)$, and $\nabla^{\perp}_X N_i$, $i = 1, ..., n$ belong to $\Gamma(ntr(TM))$. Moreover, it is easy to check that ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ which is called the second fundamental form, A_{N_i}, $i = 1, ..., n$ are linear operators on M which are called shape operators.

We define symmetric bilinear forms D_i and 1-forms ρ_{ij}, $i, j = 1, ..., n$, on a local coordinate neighborhood U of M by

$$D_i(X,Y) = \overline{g}(h(X,Y), \xi_i),$$
$$\rho_{ij}(X) = \overline{g}(\nabla^{\perp}_X N_i, \xi_j),\ i, j = 1, ..., n,$$

for any $X, Y \in \Gamma(TM)$. Since $ltr(TM)$ is spanned by $N_1, ..., N_n$, we get

$$h(X,Y) = \sum_{i=1}^{n} D_i(X,Y) N_i, \tag{15}$$

$$\nabla^{\perp}_X N_i = \sum_{j=1}^{n} \rho_{ij}(X) N_j,\ i = 1, ..., n, \tag{16}$$

for any $X, Y \in \Gamma(TM)$, where D_i, $i = 1, ..., n$, are called the lightlike second fundamental forms of M with respect to $ltr(TM)$.

From (13) and (15), we have

$$\overline{\nabla}_X \xi_i = \nabla_X \xi_i + \sum_{j=1}^{n} D_j(X, \xi_i) N_j,\ i = 1, ..., n.$$

Hence, we obtain

$$D_i(X, \xi_i) = 0. \tag{17}$$

Moreover, since $\overline{g}(\xi_i, \xi_j) = 0$, we have

$$D_i(X, \xi_j) + D_j(X, \xi_i) = 0. \tag{18}$$

Similarly, for the lightlike transversal vector fields $N_i,\ i = 1, ..., n$, we get

$$\overline{g}(A_{N_i} X, N_i) = 0,\ i = 1, ..., n. \tag{19}$$

$$\overline{g}(A_{N_i} X, N_j) + \overline{g}(A_{N_j} X, N_i) = 0,\ i \neq j,\ i, j = 1, ..., n. \tag{20}$$

Now, by using (12)–(16), we obtain

$$\rho_{ij}(X) = -\eta_i(\nabla_X \xi_j),\ i, j = 1, ..., n. \tag{21}$$

Since $\overline{\nabla}$ is a metric connection and by using (13)-(16), we arrive at

$$(\nabla_X g)(Y, Z) = \sum_{i=1}^{n} D_i(X, Y)\eta_i(Z) + D_i(X, Z)\eta_i(Y), \tag{22}$$

for any $X, Y, Z \in \Gamma(TM)$.

Now, we consider the decomposition (1). Then, we can write

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY), \tag{23}$$

$$\nabla_X \xi_i = -A^*_{\xi_i} X + \nabla^{*\perp}_X \xi_i,\ i = 1, ..., n, \tag{24}$$

for any $X, Y \in \Gamma(TM)$, where $\nabla^*_X PY$ and $A^*_{\xi_i} X$ belong to $\Gamma(S(TM))$ while $h^*(X, PY)$ and $\nabla^{*\perp}_X \xi_i$ belong to $\Gamma(Rad\ TM)$. Furthermore, ∇^* and $\nabla^{*\perp}$ are linear connections on the screen and radical distribution, respectively, A_{ξ_i} are linear operators on $\Gamma(TM)$, h^* is a bilinear form on $\Gamma(TM) \times \Gamma(S(TM))$. We note that ∇^* is a metric connection on $S(TM)$, but it is not free torsion. We define

$$E_i(X, PY) = \overline{g}(h^*(X, PY), N_i),$$

$$u_{ij}(X) = \overline{g}(\nabla^{*\perp}_X \xi_i, N_j),\ i, j = 1, ..., n,$$

for any $X, Y \in \Gamma(TM)$. Thus, (23) and (24) become

$$\nabla_X PY = \nabla^*_X PY + \sum_{i=1}^{n} E_i(X, PY)\xi_i, \tag{25}$$

$$\nabla_X \xi_i = -A^*_{\xi_1} X + \sum_{j=1}^{n} u_{ij}(X)\xi_j, \ i = 1, ..., n. \tag{26}$$

Using (13)–(16), (25) and (26) we have

$$E_i(X, PY) = \overline{g}(A_{N_i} X, PY), \ i = 1, ..., n, \tag{27}$$

$$D_i(X, PY) = g(A^*_{\xi_i} X, PY), \ i = 1, ..., n, \tag{28}$$

$$u_{ij}(X) = -\rho_{ji}(X) \ , i, j = 1, ..., n. \tag{29}$$

Hence (26) becomes

$$\nabla_X \xi_1 = -A^*_{\xi_1} X - \sum_{j=1}^{n} \rho_{ji}(X)\xi_j, \ i = 1, ..., n. \tag{30}$$

From (17) and (28), we get

$$A^*_{\xi_i} \xi_i = 0, \ i = 1, ..., n. \tag{31}$$

3. Some Properties of Coisotropic Submanifolds

It is known that lightlike submanifolds whose screen distribution is integrable have interesting properties. Therefore, we investigate the integrability of the screen distribution. On the other hand, the Ricci tensor of a lightlike submanifold is not symmetric, in general. In this section, we will show that the Ricci tensor of a coisotropic submanifold is symmetric under certain conditions.

Now, taking $\xi^*_i = \alpha_i \xi_i$, it follows that $N^*_i = \frac{1}{\alpha_i} N_i, \quad i = 1, ..., n$. Hence we obtain

$$\rho_{ij}(X) = \rho^*_{ij}(X) + X(\log \alpha_i), \ i = 1, ..., n,$$

for any $X \in \Gamma(TM)$, where we note that ρ_{ij} depends on the section $\xi_i \in \Gamma(Rad\ TM)$. The exterior derivative of 1-form ρ_{ij} is given by

$$d\rho_i(X, Y) = \frac{1}{2}\{X(\rho_i(Y)) - Y(\rho_i(X)) - \rho_i([X, Y])\}, \quad i = 1, ..., n.$$

Thus we have the following theorem.

Theorem 3.1 *Let M be a coisotropic submanifold of a semi-Riemannian manifold $(\overline{M}, \overline{g})$, of codimension n. Suppose ρ_{ij} and ρ^*_{ij} are the 1-forms on U associated to ξ_i and ξ^*_i, respectively. Then, $d\rho^*_{ij} = d\rho_{ij},\ i, j = 1, ..., n\ \ ,\ \ on\ U$.*

Theorem 3.2 *Let M be a coisotropic submanifold of $\overline{M}$, of codimension n. The screen distribution $S(TM)$ is integrable if and only if η_i, $i = 1, ..., n$, are closed forms on $S(TM)$.*

Proof. Since ∇ is a torsion-free linear connection, by using (11), (23) and (30) we obtain

$$\begin{aligned}[X, Y] &= \nabla^*_X PY - \nabla^*_Y PX + \sum_{i=1}^{n} \eta_i(X) A^*_{\xi_i} Y - \eta_i(Y) A^*_{\xi_i} X \\ &+ \sum_{i=1}^{n} \{E_i(X, PY) - E_i(Y, PX) + X(\eta_i(Y)) - Y(\eta_i(X)) \qquad (32) \\ &+ \sum_{j=1}^{n} \eta_j(X)\rho_{ij}(Y) - \eta_j(Y)\rho_{ij}(X)\}\xi_j.\end{aligned}$$

Taking the scalar product of the last equation with N_i, $i = 1, ..., n$, we obtain

$$\begin{aligned}\overline{g}([X, Y], N_i) &= E_i(X, PY) - E_i(Y, PX) + X(\eta_i(Y)) - Y(\eta_i(X)) \qquad (33) \\ &+ \sum_{j=1}^{n} \eta_j(X)\rho_{ij}(Y) - \eta_j(Y)\rho_{ij}(X),\ i = 1, ..., n.\end{aligned}$$

Hence we get

$$\begin{aligned}2d\eta_i(X, Y) &= E_i(Y, PX) - E_i(X, PY) \qquad (34) \\ &+ \sum_{j=1}^{n} \eta_j(Y)\rho_{ij}(X) - \eta_j(X)\rho_{ij}(Y),\ i = 1, ..., n.\end{aligned}$$

From (12) and (34) we obtain

$$2d\eta_i(PX, PY) = E_i(PY, PX) - E_i(PX, PY),\ i = 1, ..., n, \qquad (35)$$

or

$$\eta_i([PX, PY]) = E_i(PY, PX) - E_i(PX, PY),\ i = 1, ..., n. \qquad (36)$$

Thus, we have the assertion of the theorem. □

The Riemannian curvature tensor $\overline{R}$ of an arbitrary differentiable manifold $\overline{M}$ is given by $\overline{R}(X,Y)Z = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z$, for any $X, Y, Z \in \Gamma(T\overline{M})$.

Now, let M be a coisotropic submanifold of an $(m+n)$-dimensional semi-Riemannian manifold $\overline{M}$, of codimensional n. Denote by $\overline{R}$ and R the curvature tensors of $\overline{\nabla}$ and ∇, respectively. Then by straightforward calculations, we have

$$\begin{aligned}
\overline{R}(X,Y)Z &= R(X,Y)Z + \sum_{i=1}^{n} D_i(X,Z)A_{N_i}Y - D_i(Y,Z)A_{N_i}X \\
&+ \sum_{i=1}^{n}\{(\nabla_X D_i)(Y,Z) - (\nabla_Y D_i)(X,Z)\}N_i \\
&+ \sum_{j=1}^{n}\sum_{i=1}^{n}\{\rho_{ij}(X)D_i(Y,Z) - \rho_{ij}(Y)D_i(X,Z)\}N_j,
\end{aligned} \tag{37}$$

$$\begin{aligned}
\overline{R}(X,Y)\xi_k &= R(X,Y)\xi_k + \sum_{i=1}^{n} D_i(X,\xi_k)A_{N_i}Y - D_i(Y,\xi_k)A_{N_i}X \\
&+ \sum_{i=1}^{n}\{(\nabla_X D_i)(Y,\xi_k) - (\nabla_Y D_i)(X,\xi_k)\}N_i \\
&+ \sum_{j=1}^{n}\sum_{i=1}^{n}\{\rho_{ij}(X)D_i(Y,\xi_k) - \rho_{ij}(Y)D_i(X,\xi_k)\}N_j,
\end{aligned} \tag{38}$$

$$\begin{aligned}
R(X,Y)\xi_k &= \nabla^*_Y(A^*_{\xi_k}X) - \nabla^*_X(A^*_{\xi_k}Y) + A^*_{\xi_k}[X,Y] + \sum_{i=1}^{n}\rho_{ik}(Y)A^*_{\xi_i}X - \rho_{ik}(X)A^*_{\xi_i}Y \\
&+ \sum_{i=1}^{n}\{E_i(Y,A^*_{\xi_k}X) - E_i(X,A^*_{\xi_k}Y) - 2d\rho_{ik}(X,Y)\}\xi_i \\
&+ \sum_{j=1}^{n}\sum_{i=1}^{n}\{\rho_{ik}(Y)\rho_{ji}(X) - \rho_{ik}(X)\rho_{ji}(Y)\}\xi_j,
\end{aligned} \tag{39}$$

for any $X, Y, Z \in \Gamma(TM)$. From (37)–(39), we have Gauss and Codazzi equations:

$$\begin{aligned}\overline{g}(\overline{R}(X,Y)PZ,PW) &= g(R(X,Y)PZ,PW) \\ &+ \sum_{i=1}^{n} D_i(X,PZ)E_i(Y,PW) - D_i(Y,PZ)E_i(X,PW),\end{aligned} \tag{40}$$

$$\begin{aligned}\overline{g}(\overline{R}(X,Y)\xi_k,N_k) &= \overline{g}(R(X,Y)\xi_k,N_k) \\ &+ \sum_{i=1}^{n} \eta_k(A_{N_i}Y)D_i(X,\xi_k) - \eta_k(A_{N_i}X)D_i(Y,\xi_k),\end{aligned} \tag{41}$$

$$\begin{aligned}\overline{g}(R(X,Y)\xi_k,N_k) &= E_k(Y,A^*_{\xi_k}X) - E_k(X,A^*_{\xi_k}Y) - 2d\rho_{kk}(X,Y) \\ &+ \sum_{i=1}^{n} \rho_{ik}(Y)\rho_{ki}(X) - \rho_{ik}(X)\rho_{ki}(Y).\end{aligned} \tag{42}$$

Thus, from (37) we have the following theorem.

Theorem 3.3 *Let (M,g) be a coisotropic submanifold of $(\overline{M},\overline{g})$, of codimension n. If M is totally geodesic in $\overline{M}$, then*

$$\overline{R}(X,Y) = R(X,Y)$$

for any $X, Y \in \Gamma(TM)$.

Now, we consider the Ricci tensor of a coisotropic submanifold. The Ricci tensor Ric of an arbitrary manifold M is defined by

$$Ric(X,Y) = trace\{Z \longrightarrow R(X,Z)Y\}$$

for any $X, Y \in \Gamma(TM)$ [6]. Then, the Ricci tensor of a coisotropic submanifold M of an $(m+n)$-dimensional semi-Riemannian manifold $\overline{M}$, of codimension n, is given by

$$Ric(X,Y) = \sum_{i=1}^{m-n} \epsilon_i g(R(X,W_i)Y,W_i) + \sum_{i=1}^{n} \overline{g}(R(X,\xi_i)Y,N_i),$$

where $\{W_1, W_2, ..., W_{m-n}\}$ is an orthonormal basis of $\Gamma(S(TM))$. Using first Bianchi identity we have

$$\begin{aligned} Ric(X,Y) - Ric(Y,X) &= \sum_{i=1}^{m-n} \epsilon_i g(R(X,Y)W_i, W_i) \\ &+ \sum_{i=1}^{n} \overline{g}(R(X,Y)\xi_i, N_i). \end{aligned} \tag{43}$$

Moreover, from (27) and (28) we derive

$$E_j(X, A^*_{\xi_i} Y) = \sum_{k=1}^{m-n} \epsilon_k D_j(Y, W_k) E_i(X, W_k), \; j = 1, ..., n. \tag{44}$$

Using the structure equations given with (43) and (44), we obtain

$$Ric(X,Y) - Ric(Y,X) = -2\sum_{k=1}^{n} d\rho_{kk}(X,Y),$$

for any $X, Y \in \Gamma(TM)$. So, we have the following theorem.

Theorem 3.4 *Let (M, g) be a coisotropic submanifold of $(\overline{M}, \overline{g})$, of codimension n. Then the Ricci tensor Ric of M is symmetric if and only if on M*

$$\sum_{k=1}^{n} d\rho_{kk} = 0.$$

Corollary 3.5 *Let (M, g) be a coisotropic submanifold of $(\overline{M}, \overline{g})$, of codimension n. Then Ricci tensor Ric of M is symmetric, if ρ_{kk}, $k = 1, ..., n$, are closed form.*

Let $\overline{M}(c)$ be a semi-Riemannian manifold with constant sectional curvature c. Then curvature tensor of $\overline{M}(c)$ is given by

$$\overline{R}(X,Y)Z = c\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y\}. \tag{45}$$

Let M be a coisotropic submanifold of $\overline{M}(c)$, of codimension n. Then, from (40), (41) and (45) we get

$$\begin{aligned} Ric(X,Y) &= (1-m)cg(PX,PY) \\ &+ \sum_{k=1}^{m-n}\sum_{i=1}^{n}\epsilon_k\{D_i(W_k,Y)E_i(X,W_k)-D_i(X,Y)E_i(W_k,W_k)\} \\ &+ \sum_{j=1}^{n}\sum_{i=1}^{n}\eta_j(A_{N_i}X)D_i(\xi_j,Y)-\eta_j(A_{N_i}\xi_j)D_i(X,Y). \end{aligned} \tag{46}$$

Then we have the following theorem

Theorem 3.6 *Let M be a coisotropic submanifold of an $(m+n)$-dimensional semi-Riemannian space form $(\overline{M}(c),\overline{g})$, of codimension n. If M is total geodesic, then M is an Einstein manifold.*

The rest of this section we consider totally umbilical coisotropic submanifolds. A coisotropic submanifold M is said to be totally umbilical in $\overline{M}$ if there is a smooth affine normal vector field $Z \in \Gamma(tr((TM))$ on M such that

$$h(X,Y) = Z\overline{g}(X,Y)$$

for all $X,Y \in \Gamma(TM)$ [4]. From (15) it is easy to see that M is totally umbilical if and only if there exist smooth functions $H_i,\ i=1,...,n$, on each coordinate neighborhood U such that

$$D_i(X,Y) = H_i\overline{g}(X,Y),\ i=1,...,n, \tag{47}$$

for any $X,Y \in \Gamma(TM)$. From (28), we have

$$A^*_{\xi_i}X = H_iPX, \quad i=1,...,n, \tag{48}$$

for any $X \in \Gamma(TM)$. Moreover, we have

$$D_i(X,\xi_j) = 0, \quad A^*_{\xi_i}\xi_j = 0, \quad i,j=1,...,n. \tag{49}$$

From (22), we derive

$$\nabla_{\xi_i}g = 0, \quad i=1,...,n. \tag{50}$$

From (46) and (49), we get

$$Ric(X, \xi_i) = 0, \ i = 1, ..., n,$$

for any $X \in \Gamma(TM)$.

Corollary 3.7 *Let M be a coisotropic submanifold of a semi-Riemannian space form $\overline{M}(c)$. Then the Ricci tensor of M is degenerate.*

Theorem 3.8 *Let M be a totaly umbilical coisotropic submanifold of an $(m+n)$-dimensional semi-Riemannian manifold $(\overline{M})$, of codimension n. Then, the radical distribution $Rad\,TM$ is parallel in M.*

Proof. Since $\overline{\nabla}$ is a metric connection, we obtain

$$\overline{g}(\overline{\nabla}_\xi \xi', X) = -\overline{g}(\overline{\nabla}_\xi X, \xi'),$$

for any $\xi, \xi' \in \Gamma(Rad\,TM)$ and $X \in \Gamma(TM)$. By using Gauss formula, we get

$$\overline{g}(\overline{\nabla}_\xi \xi', X) = -\overline{g}(h(\xi, X), \xi').$$

Thus, since M is totally umbilical coisotropic submanifold, we have $h(\xi, X) = 0$. Hence

$$\overline{g}(\overline{\nabla}_\xi \xi', X) = g(\nabla_\xi \xi', X) = 0,$$

i.e., $\nabla_\xi \xi' \in \Gamma(Rad\,TM)$. Thus $Rad\,TM$ is parallel in M.

Theorem 3.9 *Let M be a totaly umbilical coisotropic submanifold of an $(m+n)$-dimensional semi-Riemannian manifold of constant curvature $(\overline{M}(c), \overline{g})$, of codimension n. Then the functions $H_i, \ i = 1, ..., n$, satisfies the following partial differential equation:*

$$\xi_i(H_i) + \sum_{j=1}^{n} \rho_{ji}(\xi_i) H_j - H_i^2 = 0, \quad i, j = 1, ..., n. \tag{51}$$

Proof. Taking $X = \xi_i$ in (37) and using (49), (50) and the fact that $\overline{M}$ is a space of constant curvature, we have the assertion of the Theorem 3.9.

Let σ be a null plane spanned by ξ and X. Then the null sectional curvature of a semi-Riemannian manifold with respect to ξ is given by

$$\overline{K}_{\xi}(\sigma) = \frac{\overline{R}(X, \xi, \xi, X)}{\overline{g}(X, X)},$$

where X is an arbitrary non-null vector field in $\Gamma(TM)$ and $\xi \in Rad\ T_M$ [1]. Similarly the null sectional curvature is given by

$$K_{\xi}(\sigma) = \frac{R(X, \xi, \xi, X)}{g(X, X)}.$$

□

Then, from (37) and (49) we have the next theorem.

Theorem 3.10 *Let M be a totally umbilical coisotropic submanifold of an $(m+n)$-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$, of codimension n. Then,*

$$\overline{K}_{\xi}(\sigma) = K_{\xi}(\sigma).$$

The screen distribution $S(TM)$ is called totally umbilical in M if there exists a smooth vector field $\omega \in \Gamma(Rad\ TM)$ on M such that

$$h^*(X, PY) = \omega g(X, PY),$$

for all $X, Y \in \Gamma(TM)$, (see [3]). Hence $S(TM)$ is totally umbilical if and only if, on any coordinate neighborhood $U \subset M$, there exists a smooth functions $K_i,\ i = 1, ..., n$, such that

$$E_i(X, PY) = K_i g(X, PY), \quad i = 1, ..., n, \tag{52}$$

for any $X, Y \in \Gamma(TM)$. From (27) we have

$$E_i(\xi_j, PY) = 0,\ i = 1, ..., n.$$

Using (34), we obtain

$$2d\eta_i(X, Y) = \sum_{j=1}^{n} \eta_j(Y)\rho_{ij}(X) - \eta_j(X)\rho_{ij}(Y).$$

Hence, we have the following corollary.

Corollary 3.11 *Let M be a coisotropic submanifold of an $(m+n)$-dimensional semi-Riemannian manifold $\overline{M}$, of codimension n, such that screen distribution $S(TM)$ is totally umbilical. If $\rho_{ij}=0$, then $d\eta_i=0,\ i,j=1,...,n$.*

From (36), we have the following corollary.

Corollary 3.12 *Let M be a coisotropic submanifold of an $(m+n)$-dimensional semi-Riemannian manifold $\overline{M}$, of codimension n. If $S(TM)$ is totally umbilical, then $S(TM)$ is integrable.*

Acknowledgement

The authors would like to express their deep thanks to the referee for suggestions that have led to improvements in the paper.

References

[1] Beem, J.K.,Ehrlic, P.E. and Easley, K.L., Global Lorentzian Geometry, Marcel Dekker, Inc. New York, Second Edition, (1996).

[2] Bejancu, A., and Duggal, K.L., Lightlike Submanifolds of Semi-Riemannian Manifolds, Acta. Applic. Math., 38, (1995), 197-215.

[3] Duggal, K.L. and Bejancu, A., Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications, Kluwer Acad. Publishers, Dordrecht, 1966.

[4] Duggal, K.L. and Jin, D.H, Half Lightlike Submanifolds of codimension 2, Toyama Univ. Vol.22, (1999), 121-161.

[5] Kupeli, D.N., Singular Semi-Riemannian Geometry, Kluwer, Dortrecht, (1996).

[6] O'Neill, B., Semi-Riemannian Geometry with Applications to Relativity, Academic Press, New York, (1983).

Erol KILIÇ, Bayram ŞAHİN,
H. Bayram KARADAĞ, Rıfat GÜNEŞ
University of İnönü,
Faculty of Science and Art,
Department of Mathematics,
Malatya, 44069, TURKEY
e-mail: ekilic@inonu.edu.tr, bsahin@inonu.edu.tr,
hbkaradag@inonu.edu.tr, rgunes@inonu.edu.tr,

Received 10.04.2003