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On Simultaneous Approximation by a Linear Combination of a New Sequence of Linear Positive Operators

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Abstract

In [1] we introduced a new sequence of linear positive operators M_n to approximate unbounded continuous functions of exponential growth on $[0,\infty)$. As this sequence is saturated with $O(n^{-1})$, to accelerate the rate of convergence we applied the technique of linear combination introduced by May [3] and Rathore et al. [4] to these operators. The object of the present paper is to study the phenomena of simultaneous approximation (approximation of derivatives of functions by the corresponding order derivatives of operators) by the linear combination $M_n(\,.\,,k,x)$ of M_n. First, we establish a Voronovskaja-type asymptotic formula and then proceed to obtain an estimate of error in terms of modulus of continuity in simultaneous approximation by this sequence of operators.

Key words and phrases: Simultaneous approximation, Linear positive operators, Linear combination, Voronovskaja-type asymptotic formula, Modulus of continuity.

1. Introduction

We [1] introduced a new sequence of linear positive operators M_n given as follows:

Let $\alpha > 0$ and $f \in C_\alpha[0,\infty) := \{\, f \in C[0,\infty) : |f(t)| \leq M\, e^{\alpha\, t}\, for some M > 0 \,\}$. Then,

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$$M_n(f(t);x) = n\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\int_0^{\infty} q_{n,\nu-1}(t)f(t)\,dt + (1+x)^{-n}f(0), \tag{1.1}$$

where $p_{n,\nu}(x) = \binom{n+\nu-1}{\nu} x^{\nu}(1+x)^{-n-\nu}$ and $q_{n,\nu}(t) = \frac{e^{-nt}(nt)^{\nu}}{\nu\,!}$, $x\,,\ t\ \in [0,\infty)$.

We may also write operators (1.1) as $M_n(f(t);x) = \int\limits_0^{\infty} W_n(t,x)f(t)\,dt$, where the kernel $W_n(t,x) = n\sum\limits_{\nu=1}^{\infty} p_{n,\nu}(x)\,q_{n,\nu-1}(t) + (1+x)^{-n}\delta(t)$, $\delta(t)$ being the Dirac-delta function. The space $C_\alpha[0,\infty)$ is normed by $\|f\|_{C_\alpha} := \sup\limits_{0\le t<\infty} |f(t)|\,e^{-\alpha\,t}$, $f \in C_\alpha[0,\infty)$.

In [1], we observed that the order of approximation by the operators (1.1) is, at best, $O(n^{-1})$ however smooth the function may be. May [3] and Rathore et al. [4] have described a method for forming linear combinations of a sequence of linear positive operators so as to improve the order of approximation. Following their method, in [1] we established some direct theorems for a linear combination of the operators (1.1) (i.e. Voronovskaja-type asymptotic formula and an error estimate in terms of higher order modulus of continuity of the function involved by the operators $M_n(\,.\,,k,x)$). The approximation process is described as follows.

For $k \in N^0$ (the set of nonnegative integers) and $f \in C_\alpha[0,\infty)$, the linear combination $M_n(f,k,x)$ of the operators $M_{d_j n}(f;x)$, $j = 0\,,\ 1\,,\ \ldots\,,\ k$ is defined as:

$$M_n(f,k,x) = \sum_{j=0}^{k} C(j,k)\,M_{d_j n}(f;x),$$

where $d_0\,,\ d_1,\ \ldots\,, d_k \in N$ (the set of positive integers) are arbitrary and distinct but fixed and $C(j,k) = \prod\limits_{i=0\,,\,i\neq j}^{k} \frac{d_j}{d_j-d_i}$, $k \neq 0$ and $C(0,0) = 1$.

Throughout this paper, we denote by $C[a,b]$ the space of all continuous functions on the interval $[a,b]$, $\|\ .\ \|_{C[a,b]}$ denotes the sup norm on the space $C[a,b]$ and C denotes a constant not necessarily the same in different cases.

The object of the present paper is to obtain a Voronovskaja-type asymptotic formula and an error estimate in terms of the modulus of continuity of the function approximated by the operators $M_n^{(r)}(. , k, x)$, where $r \in N$.

2. Preliminaries

In the sequel, we shall require the following results:

For $m \in N^0$, let the $m-th$ order moment for the Lupas operators be defined by $\mu_{n,m}(x) = \sum\limits_{\nu=0}^{\infty} p_{n,\nu}(x)\ \left(\frac{\nu}{n} - x\right)^m$.

Lemma 1 [2] *For the function $\mu_{n,m}(x)$, we have $\mu_{n,0}(x) = 1$, $\mu_{n,1}(x) = 0$ and there holds the recurrence relation*

$n\mu_{n,m+1}(x) = x(1+x)[\mu'_{n,m}(x) + m\mu_{n,m-1}(x)]$, *for* $m \geq 1$.

Consequently, we have that

(i) $\mu_{n,m}(x)$ *is a polynomial in x of degree atmost m;*

(ii) for every $x \in [0, \infty)$, $\mu_{n,m}(x) = O(n^{-[(m+1)/2]})$, where $[\beta]$ denotes the integer part of β.

Let the $m-th$ order moment ($m \in N^0$) for the operators (1.1) be defined by:

$$T_{n,m}(x) = M_n((t-x)^m; x) = n\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\int\limits_0^{\infty} q_{n,\nu-1}(t)(t-x)^m dt + (-x)^m(1+x)^{-n}.$$

Lemma 2 [1] *For the function $T_{n,m}(x)$, there follow $T_{n,0}(x) = 1$, $T_{n,1}(x) = 0$ and* $nT_{n,m+1}(x) = x(1+x)T'_{n,m}(x) + mT_{n,m}(x) + mx(x+2)T_{n,m-1}(x)$, $m \geq 1$.

Further, we have the following consequences of $T_{n,m}(x)$:

(i) $T_{n,m}(x)$ is a polynomial in x of degree m, $m \neq 1$;

(ii) for every $x \in [0, \infty)$, $T_{n,m}(x) = O(n^{-[(m+1)/2]})$;

(iii) the coefficients of n^{-k} in $T_{n,2k}(x)$ and $T_{n,2k-1}(x)$ are $C_1\,\{x(x+2)\}^k$ and $C_2\,x^{k-1}(x+2)^{k-2}(x^2+3x+3)$, respectively, where C_1 and C_2 are constants dependent on k.

Lemma 3 *Let $\delta$and γ be any two positive real numbers and $[a,b] \subset (0,\infty)$. Then, for any $m > 0$ we have,*

$$\sup_{x\in[a,b]} \left| n\sum_{\nu=1}^{\infty} p_{n,v}(x) \int\limits_{|t-x|\geq\delta} q_{n,v-1}(t)\, e^{\gamma\, t} dt \right| = O(n^{-m}).$$

Making use of Taylor's expansion, Schwarz inequality for integration and then for summation and Lemma 2, we easily prove Lemma 3 (hence the details are omitted).

Lemma 4 [2] *There exist the polynomials $q_{i,j,r}(x)$ independent of n and ν such that*

$$p_{n,\nu}^{(r)}(x) = \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} n^i(\nu - nx)^j \frac{q_{i,j,r}(x)}{x^r(1+x)^r}\, p_{n,\nu}(x).$$

Theorem 1 [1] *Suppose that $f \in C_\alpha[0,\infty)$ for some $\alpha > 0$ and $f^{(2k+2)}$ exists at a point $x \in [0,\infty)$, then*

$$\lim_{n\to\infty} n^{k+1}[\, M_n(f,k,x) - f(x)] = \sum_{j=k+2}^{2k+2} \frac{f^{(j)}(x)}{j\,!}\, Q(j,k,x)$$

and

$$\lim_{n\to\infty} n^{k+1}[\, M_n(f,k+1,x) - f(x)] = 0,$$

where $Q(j,k,x)$ are certain polynomials in x of degree j.

3. Main Results

Theorem 2 *Let $r \in N$ and $f \in C_\alpha[0,\infty)$ for some $\alpha > 0$, admitting a derivative of order $(2k+2+r)$ at a point $x \in (0,\infty)$. Then we have*

$$\lim_{n\to\infty} n^{k+1}\Big[\, M_n^{(r)}(f,k,x) - f^{(r)}(x)\Big] = \sum_{m=r}^{2k+2+r} f^{(m)}(x)\; Q(m,k,r,x) \qquad (3.1)$$

and

$$\lim_{n\to\infty} n^{k+1}\Big[M_n^{(r)}(f,k+1,x)-f^{(r)}(x)\Big]=0, \tag{3.2}$$

where $Q(m,k,r,x)$ are certain polynomials in x.

Further, if $f^{(2k+2+r)}$ exists and is continuous on $(a-\eta,b+\eta)\subset(0,\infty)$, $\eta>0$, then (3.1) and (3.2) hold uniformly on $[a,b]$.

Proof. Since $f^{(2k+2+r)}$ exists at $x\in(0,\infty)$, it follows that

$$f(t)=\sum_{m=0}^{2k+2+r}\frac{f^{(m)}(x)}{m!}\,(t-x)^m+\varepsilon(t,x)\,(t-x)^{2k+2+r},$$

where $\varepsilon(t,x)\to 0$ as $t\to x$.

Thus, we can write

$$\begin{aligned} M_n^{(r)}(f(t),k,x) &= \sum_{m=0}^{2k+2+r}\frac{f^{(m)}(x)}{m!}\,M_n^{(r)}((t-x)^m,k,x)\\ &\quad+\sum_{j=0}^{k}C(j,k)\,M_{d_jn}^{(r)}(\varepsilon(t,x)\,(t-x)^{2k+2+r};x):=\sum_1+\sum_2. \end{aligned}$$

Now, with $D\equiv\frac{d}{dx}$ by Lemma 2 and Theorem 1 we obtain

$$\begin{aligned} \sum_1 &= \sum_{m=r}^{2k+2+r}\frac{f^{(m)}(x)}{m!}\,M_n^{(r)}\left((t-x)^m,k,x\right)\\ &= \sum_{m=r}^{2k+2+r}\frac{f^{(m)}(x)}{m!}\sum_{i=0}^{m}\binom{m}{i}(-x)^{m-i}M_n^{(r)}(t^i,k,x)\\ &= \sum_{m=r}^{2k+2+r}\frac{f^{(m)}(x)}{m!}\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}(x)^{m-i} \end{aligned}$$

$$\times \left\{ D^r x^i + n^{-(k+1)} \left[\sum_{j=k+2}^{2k+2} D^r \left(\frac{D^j x^i}{j\,!} Q(j,k,x) \right) + o(1) \right] \right\}.$$

$$= \sum_{m=r}^{2k+2+r} \frac{f^{(m)}(x)}{m\,!} r! \sum_{i=0}^{m} \binom{m}{i} \binom{i}{r} (-1)^{m-i} (x)^{m-r}$$

$$+ n^{-(k+1)} \sum_{m=r}^{2k+2+r} Q(m,k,r,x)\, f^{(m)}(x) + o(n^{-(k+1)}).$$

By using the identities

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} \binom{i}{r} = \begin{cases} 0 & , \quad m > r \\ (-1)^r & , \quad m = r \end{cases},$$

we get

$$\sum_1 = f^{(r)}(x) + n^{-(k+1)} \sum_{m=r}^{2k+2+r} Q(m,k,r,x)\, f^{(m)}(x) + o(n^{-(k+1)}).$$

Hence, in order to prove (3.1) it is sufficient to show that $n^{k+1} \sum_2 \to 0$ as $n \to \infty$ i.e. $n^{k+1} M_n^{(r)}(\,\varepsilon(t,x)\,(t-x)^{2k+2+r}\,;x) \to 0$ as $n \to \infty$.

Now,

$$\begin{aligned} \sum \quad &\equiv \quad M_n^{(r)}(\,\varepsilon(t,x)\,(t-x)^{2k+2+r}\,;x) \\ &= \quad n \sum_{\nu=1}^{\infty} p_{n,\nu}^{(r)}(x) \int_0^{\infty} q_{n,\nu-1}(t) \varepsilon(t,x)\,(t-x)^{2k+2+r} dt \\ &\quad + (-1)^r \frac{(n+r-1)!}{(n-1)!} (1+x)^{-n-r} \varepsilon(0,x)(-x)^r. \end{aligned}$$

Therefore, by using Lemma 4 we have

$$\left|\sum\right| \leq \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \frac{|q_{i,j,r}(x)|}{x^r(1+x)^r}\, n\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\,|\nu-nx|^j \int_0^{\infty} q_{n,\nu-1}(t)\,|\varepsilon(t,x)|\,|t-x|^{\,2k+2+r}dt$$

$$+\frac{(n+r-1)!}{(n-1)!}(1+x)^{-n-r}\,|\varepsilon(0,x)|\,x^{2k+2+r} := J_1+J_2.$$

Since $\varepsilon(t,x)\to 0$ as $t\to x$, for a given $\varepsilon>0$, there exists a $\delta>0$ such that $|\varepsilon(t,x)|<\varepsilon$, whenever $0<|t-x|<\delta$. For $|t-x|\geq\delta$, there exists a constant $C>0$ such that $|\varepsilon(t,x)(t-x)^r|\leq C\,e^{\alpha\,t}$. Hence,

$$J_1 \leq \left(\sup_{\substack{2i+j\leq r\\ i,j\geq 0}} \frac{|q_{i,j,r}(x)|}{x^r(1+x)^r}\right) \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^{i+1}\sum_{\nu=1}^{\infty} p_{n,\nu}(x)|\nu-nx|^j$$

$$\left[\int_{|t-x|<\delta} q_{n,\nu-1}(t)\varepsilon\,|t-x|^{2k+2+r}dt+\int_{|t-x|\geq\delta} q_{n,\nu-1}(t)\,C\,e^{\alpha\,t}dt\right] := J_3+J_4.$$

Let $\sup\limits_{\substack{2i+j\leq r\\ i,j\geq 0}}$, $\frac{|q_{i,j,r}(x)|}{x^r(1+x)^r}=M(x)$, $x\in(0,\infty)$ but fixed. Applying Schwarz inequality for integration and then for summation, we are led to

$$J_3 \leq \varepsilon\,C \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^{i+1}\sum_{v=1}^{\infty} p_{n,\nu}(x)|\nu-nx|^j\left[\int_0^{\infty} q_{n,\nu-1}(t)dt\right]^{1/2}\left[\int_0^{\infty} q_{n,\nu-1}(t)(t-x)^{4k+4+2r}dt\right]^{1/2}$$

$$\leq \varepsilon\,C \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^{i}\left[\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\,(\nu-nx)^{2j}\right]^{1/2}\left[n\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\int_0^{\infty} q_{n,\nu-1}(t)(t-x)^{4k+4+2r}dt\right]^{1/2}$$

(in view of $\int_0^{\infty} q_{n,\nu-1}(t)\,dt=n^{-1}$).

From Lemma 1, we have

$$\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\,(\nu-nx)^{2j} = n^{2j}\left[\sum_{\nu=0}^{\infty} p_{n,\nu}(x)\left(\frac{\nu}{n}-x\right)^{2j} - (1+x)^{-n}(-x)^{2j}\right]$$

$$= n^{2j}\big[O(n^{-j})+O(n^{-s})\big] = O(n^{j}) \text{ (for any } s>0). \tag{3.3}$$

Similarly, Lemma 2 yields us

$$n\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\int_0^{\infty} q_{n,\nu-1}(t)(t-x)^{2s}dt = T_{n,2s}(x) - (1+x)^{-n}(-x)^{2s}$$

$$= O(n^{-s})+O(n^{-m}) = O(n^{-s}) \text{ (for any } m>0). \tag{3.4}$$

Therefore, $J_3 \le \varepsilon\, C \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \;\; O(n^{j/2})O(n^{-(2k+2+r)/2}) = \varepsilon O(n^{-(k+1)})$.

Next, again using Schwarz inequality for integration and then for summation, (3.3) and Lemma 3, we have

$$J_4 \le C\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^{i+1}\sum_{\nu=1}^{\infty} p_{n,\nu}(x)|\nu-nx|^j\left[\int_{|t-x|\ge\delta} q_{n,\nu-1}(t)dt\right]^{1/2}\left[\int_{|t-x|\ge\delta} q_{n,\nu-1}(t)e^{2\alpha t}dt\right]^{1/2}$$

$$\le C\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^{i}\left[\sum_{\nu=1}^{\infty} p_{n,\nu}(x)|\nu-nx|^{2j}\right]^{1/2}\left[n\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\int_{|t-x|\ge\delta} q_{n,\nu-1}(t)e^{2\alpha t}dt\right]^{1/2}$$

$$\le C\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^{i}\, O(n^{j/2})O(n^{-s}) \text{ (for any } s>0)$$

$$= O(n^{(r/2)-s}) = o(n^{-(k+1)})(\text{for } s>k+1+\frac{r}{2}).$$

Combining the estimate of J_3 and J_4 we get $J_1 = \varepsilon\, O(n^{-(k+1)})$. Clearly, $J_2 = O(n^{-s})$, for any $s>0$. Choosing $s>k+1$, we have $J_2 = o(n^{-(k+1)})$. Hence, due to the arbitrariness of $\varepsilon>0$, $n^{k+1}\sum \to 0$ as $n\to\infty$.

This completes the proof of the assertion (3.1).

The assertion (3.2) can be proved along similar lines by noting that $M_n((t-x)^m, k+1, x) = O(n^{-(k+2)})$, for $m = k+3\,,\ k+4\,,\ \ldots$ (cf. [1], p.61).

The uniformity assertion follows easily from the fact that $\delta(\varepsilon)$ in the above proof can be chosen to be independent of $x \in [a,b]$ and all the other estimates hold uniformly on $[a,b]$. $\square$

For $r \in N$, the next result provides an estimate of the degree of approximation in $M_n^{(r)}(f,k,\,.x) \to f^{(r)}(x)$, $n \to \infty$.

Theorem 3. Let $1 \le p \le 2k+2$ and $f \in C_\alpha[0,\infty)$ for some $\alpha > 0$. If $f^{(p+r)}$ exists and is continuous on $(a-\eta, b+\eta) \subset (0,\infty)$, $\eta > 0$, then for sufficiently large n,

$$\left\| M_n^{(r)}(f,k,\,.\,) - f^{(r)} \right\|_{C[a,b]} \le Max\left\{ C_1 n^{-p/2}\, \omega_{f^{(p+r)}}(n^{-1/2})\,,\ \ C_2\, n^{-(k+1)} \right\},$$

where $C_1 = C_1(k,p,r)$, $C_2 = C_2(k,p,r,f)$ and $\omega_{f^{(p+r)}}(n^{-1/2})$denotes the modulus of continuity of $f^{(p+r)}$on $(a-\eta, b+\eta)$.

Proof. By the hypothesis

$$f(t) = \sum_{i=0}^{p+r} \frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{f^{(p+r)}(\xi) - f^{(p+r)}(x)}{(p+r)!}(t-x)^{p+r}\chi(t) + h(t,x)(1-\chi(t)),$$

where ξ lies between t and x, and $\chi(t)$ is the characteristic function of the interval $(a-\eta, b+\eta)$.

For $t \in (a-\eta, b+\eta)$ and $x \in [a,b]$, we get

$$f(t) = \sum_{i=0}^{p+r} \frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{f^{(p+r)}(\xi) - f^{(p+r)}(x)}{(p+r)!}(t-x)^{p+r}.$$

For $t \in [0,\infty)\backslash(a-\eta, b+\eta)$and $x \in [a,b]$, we define

$$h(t,x) = f(t) - \sum_{i=0}^{p+r} \frac{f^{(i)}(x)}{i!}(t-x)^i.$$

Now,

$$\begin{aligned} M_n^{(r)}(f(t),k,x) &= \sum_{i=0}^{p+r} \frac{f^{(i)}(x)}{i!} M_n^{(r)}\left((t-x)^i,k,x\right) \\ &+ \frac{1}{(p+r)\,!} M_n^{(r)}\left(\left(f^{(p+r)}(\xi)-f^{(p+r)}(x)\right)(t-x)^{p+r}\chi(t),k,x\right) \\ &+ M_n^{(r)}\left(h(t,x)(1-\chi(t),k,x\right) := \sum_1 + \sum_2 + \sum_3 . \end{aligned}$$

Proceeding along the lines of the proof of $\sum_1$ in Theorem 2, we get

$$\sum_1 = f^{(r)}(x) + O(n^{-(k+1)}),\ \text{uniformly for all } x \in [a,b].$$

For every $\delta > 0$, we have

$$\left|f^{(p+r)}(\xi) - f^{(p+r)}(x)\right| \le \omega_{f^{(p+r)}}\left(|\xi - x|\right) \le \omega_{f^{(p+r)}}\left(|t-x|\right) \le \left(1 + \frac{|t-x|}{\delta}\right)\omega_{f^{(p+r)}}(\delta)\,.$$

Hence, we have

$$\left|\sum_2\right| \le \frac{1}{(p+r)\,!} \sum_{j=o}^{k} |C(j,k)| \int_0^\infty \left|W_{d_j n}^{(r)}(t,x)\right| \left(1 + \frac{|t-x|}{\delta}\right) \omega_{f^{(p+r)}}(\delta)\ |t-x|^{p+r}\,dt$$

$$\le \frac{\omega_{f^{(p+r)}}(\delta)}{(p+r)\,!} \left[\sum_{j=o}^{k} |C(j,k)|\, d_j n \sum_{\nu=1}^{\infty} \left|p_{d_j n,\nu}^{(r)}(x)\right| \int_0^\infty q_{d_j n,\nu-1}(t)\left(|t-x|^{p+r} + \delta^{-1}|t-x|^{p+r+1}\right)\ dt\right.$$

$$\left. +(-1)^r \frac{(d_j n + r - 1)\,!}{(d_j n - 1)\,!} (1+x)^{-d_j n - r}\left(|x|^{p+r} + \delta^{-1}|x|^{p+r+1}\right)\right].$$

Now, in order to estimate $\sum_2$, we proceed as follows:

Using Schwarz inequality for integration and then for summation, (3.3) and (3.4), for $s = 0\,,\ 1\,,\ \ldots,$ we have

$$
\begin{aligned}
& n\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\,|\nu - nx|^{j}\int_0^{\infty} q_{n,\nu-1}(t)\,|t-x|^{s}dt \\
& \leq n\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\,|\nu - nx|^{j}\left[\left(\int_0^{\infty} q_{n,\nu-1}(t)dt\right)^{1/2}\left(\int_0^{\infty} q_{n,\nu-1}(t)(t-x)^{2s}dt\right)^{1/2}\right] \\
& \leq \left[\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\,(\nu - nx)^{2j}\right]^{1/2}\left[n\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\int_0^{\infty} q_{n,\nu-1}(t)(t-x)^{2s}dt\right]^{1/2} \\
& = O(n^{j/2})O(n^{-s/2}) = O(n^{(j-s)/2}),\ \text{uniformly in } x\in[a,b].
\end{aligned}
$$

Therefore, by Lemma 4, we get

$$
\begin{aligned}
& \sum_{j=0}^{k}|C(j,k)|\,d_j n\sum_{\nu=1}^{\infty}\left|p^{(r)}_{d_jn,\nu}(x)\right|\int_0^{\infty} q_{d_jn,\nu-1}(t)\,|t-x|^{s}dt \\
& \leq \sum_{j=0}^{k}|C(j,k)|\,d_j n\sum_{\nu=1}^{\infty}\sum_{\substack{2i+m\leq r\\ i,m\geq 0}}(d_jn)^i|\nu-d_jnx|^m\frac{|q_{i,m,r}(x)|}{x^r(1+x)^r}p_{d_jn,\nu}(x)\int_0^{\infty} q_{d_jn,\nu-1}(t)|t-x|^{s}dt \\
& \leq C\sum_{j=0}^{k}|C(j,k)|\sum_{\substack{2i+m\leq r\\ i,m\geq 0}}(d_jn)^i\left[d_jn\sum_{\nu=1}^{\infty}p_{d_jn,\nu}(x)|\nu-d_jnx|^m\int_0^{\infty} q_{d_jn,\nu-1}(t)|t-x|^{s}dt\right] \\
& \qquad\qquad\qquad\qquad \left(C=\sup_{\substack{2i+m\leq r\\ i,m\geq 0}}\ \sup_{x\in[a,b]}\frac{|q_{i,m,r}(x)|}{x^r(1+x)^r}\right) \\
& = \sum_{\substack{2i+m\leq r\\ i,m\geq 0}} n^i\,O(n^{(m-s)/2}) = O(n^{(r-s)/2}),\ \text{uniformly in } x\in[a,b]. \qquad (3.5)
\end{aligned}
$$

Choosing $\delta = n^{-1/2}$ and applying (3.5), we are led to

$$
\begin{aligned}
\left|\sum_{2}\right| &\leq \frac{\omega_{f^{(p+r)}}(n^{-1/2})}{(p+r)!}\Big[O(n^{-p/2})+n^{1/2}O(n^{-(p+1)/2})+O(n^{-m})\Big] \text{ (for any } m>0) \\
&\leq C\,n^{-p/2}\,\omega_{f^{(p+r)}}(n^{-1/2}), \text{ choosing } m>p/2.
\end{aligned}
$$

Since $t \in [0,\infty)\backslash(a-\eta, b+\eta)$, we can choose $\delta > 0$ in such a way that $|t-x| \geq \delta$ for all $x \in [a,b]$.

Thus, by Lemma 4, we obtain

$$
\begin{aligned}
\left|\sum_{3}\right| \leq \sum_{j=0}^{k}|C(j,k)| &\Bigg[d_j n \sum_{\nu=1}^{\infty} \sum_{\substack{2i+m\leq r\\ i,m\geq 0}} (d_j n)^i |\nu-d_j nx|^m \frac{|q_{i,m,r}(x)|}{x^r(1+x)^r} p_{d_j n,\nu}(x) \\
&\times \int\limits_{|t-x|\geq\delta} q_{d_j n,\nu-1}(t)\,|h(t,x)|dt + \frac{(d_j n+r-1)!}{(d_j n-1)!}(1+x)^{-d_j n-r}|h(0,x)|\Bigg].
\end{aligned}
$$

For $|t-x| \geq \delta$, we can find a constant $C>0$ such that $|\,h(t,x)| \leq C\,e^{\alpha\, t}$. Finally using Schwarz inequality for integration and then for summation, (3.3), and Lemma 3, it easily follows that

$$
\sum_{3} = O(n^{-s}) \text{ for any } s>0, \text{ uniformly on } [a,b].
$$

Choosing $s>k+1$ and then combining the estimates of $\sum_1$, $\sum_2$ and $\sum_3$, the required result is immediate.

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