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On Local Hörmander-Beurling Spaces

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Abstract

In this paper we aim to extend a result of Hörmander's, that $\mathcal{B}^{loc}_{p,k}(\Omega) \subset \mathcal{C}^m(\Omega)$ if $\frac{(1+|\cdot|)^m}{k} \in L_{p'}$, to the setting of vector valued local Hörmander-Beurling spaces, as well as to show that the space $\bigcap_{j=1}^{\infty} \mathcal{B}^{loc}_{p_j,k_j}(\Omega, E)$ $(1 \le p_j \le \infty,\ k_j = e^{j\omega},\ j = 1, 2, \dots)$ is topologically isomorphic to $\mathcal{E}_\omega(\Omega, E)$. Moreover, it is well known that the union of Sobolev spaces $\mathcal{H}^{loc}_s(\Omega)$ $(= \mathcal{B}^{loc}_{2,(1+|\cdot|^2)^{s/2}}(\Omega))$ coincides with the space $\mathcal{D}'^F(\Omega)$ of finite order distributions on Ω. We show that this is also verified in the context of vector valued Beurling ultradistributions.

Key Words: Hörmander space, Hörmander-Beurling space, Beurling ultradistributions, local space, Fourier-Laplace transform.

1. Introduction

It is well-known that Hörmander spaces $\mathcal{B}_{p,k}$ play a crucial role in the theory of linear partial differential operators (see [4], [5]). In [10] we have extended these spaces to the vector-valued case and have studied some of their properties.

In this paper we continue on our study of these vector valued spaces, introducing and analyzing certain properties of such local spaces $\mathcal{B}^{loc}_{p,k}(\Omega, E)$. We show that

$$\bigcup_{k \in \mathcal{K}_\omega} \mathcal{B}^{loc}_{p,k}(\Omega, E) = \mathcal{D}'_{\omega,F}(\Omega, E),$$

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and if $k_j = e^{j\omega}$, $j = 1, 2, \ldots$ space $\bigcap_{j=1}^{\infty} \mathcal{B}_{p_j,k_j}^{loc}(\Omega, E)$ is topologically isomorphic to $\mathcal{E}_\omega(\Omega, E)$, thereby extending Theorem 10.1.25 of [5] and Theorems 2.3.9 and 2.3.11 of [1] to the vector valued context. Applications of spaces $\bigcap_{j=1}^{\infty} \mathcal{B}_{p_j,k_j}^{loc}(\Omega)$ to analysis of linear partial differential operators can be found, for example, in sections 10.5, 10.6, 11.1 and 12.8 of [5].

The motivating theme for this work is the study of linear partial differential operator and apply it to the solution of vector valued differential equations and dependence of parameters of solutions of scalar valued equations.

Notation. The linear spaces we use are defined over $\mathbb{C}$. Let E and F be locally convex spaces. Then $\mathcal{L}_b(E, F)$ is the locally convex space of all continuous linear operators equipped with the bounded convergence topology. We write $E \hookrightarrow F$ if E is a linear subspace of F and the canonical injection is continuous. We replace $\hookrightarrow$ by $\overset{d}{\hookrightarrow}$ if E is also dense in F. The topological dual of E is denoted by E' and is given the strong topology so that $E' = \mathcal{L}_b(E, \mathbb{C})$. $\mathcal{C}^m$, $\mathcal{D}$, $\mathcal{S}$, $\mathcal{D}'$ and $\mathcal{S}'$ have the usual meaning (see [8]). In the vector valued case we write $\mathcal{C}^m(E)$, $\mathcal{D}(E)$, $\mathcal{S}(E)$, $\mathcal{D}'(E)$ and $\mathcal{S}'(E)$ (see [9]). Let $1 \leq p \leq \infty$, $k : \mathbb{R}^n \longrightarrow]0, \infty[$ a Lebesgue measurable function, and E a Banach space. Then $L_p(E)$ is the set of all Bochner measurable functions $f : \mathbb{R}^n \longrightarrow E$ for which $\|f\|_p = \left(\int_{\mathbb{R}^n} \|f(x)\|_E^p \, dx\right)^{1/p}$ is finite (if $p = \infty$ we assume $\|f\|_\infty = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} \|f\|_E < \infty$). $L_{p,k}(E)$ denotes the set of all measurable Bochner functions $f : \mathbb{R}^n \longrightarrow E$ such that $kf \in L_p(E)$. Putting $\|f\|_{L_{p,k}(E)} = \|kf\|_p$ for $f \in L_{p,k}(E)$, $L_{p,k}(E)$ becomes a Banach space isometrically isomorphic to $L_p(E)$. When E is the field $\mathbb{C}$, we simply write L_p and $L_{p,k}$. If $f \in L_1(E)$ the Fourier transformation of f, $\hat{f}$ or $\mathcal{F}f$, is defined by $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-i\xi x} dx$.

2. Spaces of vector-valued ultradistributions

In this section we collect some basic facts about vector-valued ultradistributions. The results are “elementary” in the sense that the usual scalar proofs carry over to the vector-valued setting by using obvious modifications only. Comprehensive treatments of the theory of (scalar or vector-valued) ultradistributions can be found in [1], [2], [3] and [6]. Our notations are based on [1].

Let $\mathcal{M}$ be the set of all continuous real-valued functions $\omega(x)$ on $\mathbb{R}^n$ such that

$\omega(x) = \sigma(|x|)$ where $\sigma(t)$ is an increasing continuous concave function on $[0,\infty[$ with the following properties:
(i) $\sigma(0) = 0$, (ii) $\int_0^\infty \frac{\sigma(t)}{1+t^2}dt < \infty$ (Beurling's condition), (iii) there exists a real number a and a positive number b such that $\sigma(t) \geq a + b\log(1+t)$ for $t \geq 0$. The main assumption is (ii), which is essentially the Denjoy-Carleman non-quasi-analyticity condition (see [1, Sec.1.5]).

If $\omega \in \mathcal{M}$, then $\mathcal{K}_\omega$ is the set of all positive functions k on $\mathbb{R}^n$ for which there exists a constant $\lambda > 0$ such that $k(x+y) \leq e^{\lambda\omega(x)}k(y)$ for all x and y in $\mathbb{R}^n$. If $k, k_1, k_2 \in \mathcal{K}_\omega$ and s is a real number then $\log k$ is uniformly continuous, $k^s \in \mathcal{K}_\omega$, $k_1k_2 \in \mathcal{K}_\omega$ and $M_k(x) = \sup_{y\in\mathbb{R}^n} \frac{k(x+y)}{k(y)} \in \mathcal{K}_\omega$ (see [1, Th. 2.1.3]).

If $\omega \in \mathcal{M}$ and E is a Banach space, we denote by $\mathcal{D}_\omega(E)$ the set of all functions $f \in L_1(E)$ with compact support, such that $\|f\|_\lambda := \int_{\mathbb{R}^n} \|\hat{f}(x)\|_E e^{\lambda\omega(x)}dx < \infty$, for all $\lambda > 0$. For each compact subset K of $\mathbb{R}^n$, $\mathcal{D}_\omega(K,E) = \{f \in \mathcal{D}_\omega(E) : \mathrm{supp} f \subset K\}$, equipped with the topology induced by the family of norms $\{\|\cdot\|_\lambda : \lambda > 0\}$, is a Frechet space and $\mathcal{D}_\omega(E)$ with the inductive limit topology $\mathcal{D}_\omega(E) = \underset{K}{\underrightarrow{\mathrm{ind}}}\mathcal{D}_\omega(K,E)$ becomes a strict (LF)-space. Let $S_\omega(E)$ be the set of all functions $f \in L_1(E)$ such that both f and $\hat{f}$ are infinitely differentiable functions on $\mathbb{R}^n$ with $\vec{p}_{\alpha,\lambda}(f) = \sup_{x\in\mathbb{R}^n} e^{\lambda\omega(x)} \|\partial^\alpha f(x)\|_E < \infty$ and $\vec{q}_{\alpha,\lambda}(f) = \sup_{x\in\mathbb{R}^n} e^{\lambda\omega(x)}\|\partial^\alpha \hat{f}(x)\|_E < \infty$ for all multi-indices α and all positive numbers λ. $\mathcal{S}_\omega(E)$ with the topology induced by the family of seminorms $\{\vec{p}_{\alpha,\lambda}, \vec{q}_{\alpha,\lambda}\}$ is a Frechet space and the Fourier transformation $\mathcal{F}$ is an automorphism of $\mathcal{S}_\omega(E)$. A continuous linear operator from $\mathcal{D}_\omega$ into E is said to be a ultradistribution with values in E. We write $\mathcal{D}'_\omega(E)$ for the space of all E-valued ultradistributions endowed with the bounded convergence topology. If Ω is any open set in $\mathbb{R}^n$, $\mathcal{D}_\omega(\Omega, E)$ is the subspace of $\mathcal{D}_\omega(E)$ consisting of all functions f with $\mathrm{supp} f \subset \Omega$. $\mathcal{D}_\omega(\Omega, E)$ is endowed with the corresponding inductive limit topology. $\mathcal{D}'_\omega(\Omega, E) = \mathcal{L}_b(\mathcal{D}_\omega(\Omega), E)$ is the space of all ultadistributions on Ω with values in E. A continuous linear operator from $\mathcal{S}_\omega$ into E is said to be an E-valued tempered ultradistribution. $\mathcal{S}'_\omega(E)$ is the space of all E-valued tempered ultradistributions equipped with the bounded convergence topology. The Fourier transformation $\mathcal{F}$ is an automorphism of $\mathcal{S}'_\omega(E)$. If $u \in L_1^{loc}$ and $\int_{\mathbb{R}^n} \varphi(x)u(x)dx = 0$ for all $\varphi \in \mathcal{D}_\omega$, then $u = 0$ a.e (see [1]).

Now, let Ω be any open subset of $\mathbb{R}^n$, ω any weight in $\mathcal{M}$ and E a Banach space. Generalizing the definition of $\mathcal{E}_\omega(\Omega)$ (see [1, Def. 1.5.1]), we are led to consider the space

$\mathcal{E}_\omega(\Omega, E)$ of all functions $f : \Omega \longrightarrow E$ such that, for every compact set K in Ω, there exists $g \in \mathcal{D}_\omega(\Omega, E)$ satisfying $g = f$ on K (see [6]). Obviously, all functions $\mathcal{E}_\omega(\Omega, E)$ are infinitely differentiable on Ω and the function $f : \Omega \longrightarrow E$ belongs to $\mathcal{E}_\omega(\Omega, E)$ if and only if $\varphi f \in \mathcal{D}_\omega(\Omega, E)$, whenever $\varphi \in \mathcal{D}_\omega(\Omega)$. Here $\varphi f(x) = \begin{cases} \varphi(x)f(x) & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$.

$\mathcal{E}_\omega(\Omega, E)$ is considered endowed with the topology generated by the fundamental system of seminorms $\{\|\cdot\|_{\lambda,\varphi} : \lambda > 0, \varphi \in \mathcal{D}_\omega(\Omega)\}$ (if $f \in \mathcal{E}_\omega(\Omega, E), \|f\|_{\lambda,\varphi} = \|\varphi f\|_\lambda = \int_{\mathbb{R}^n} \|\widehat{\varphi f}(x)\|_E e^{\lambda\,\omega(x)} dx$). With this topology, $\mathcal{E}_\omega(\Omega, E)$ is a Frechet space. By using the restriction $g \to g\,|_{\Omega}$, $\mathcal{D}_\omega(\Omega, E)$ is continuously injected in $\mathcal{E}_\omega(\Omega, E)$. $\mathcal{D}_\omega(\Omega, E)$ is dense in $\mathcal{E}_\omega(\Omega, E)$ and $f : \Omega \to E$ belongs to $\mathcal{E}_\omega(\Omega, E)$ if and only if $e' \circ f \in \mathcal{E}_\omega(\Omega)$ for every $e' \in E'$.

Finally, let us recall that if $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$, $1 \le p \le \infty$ and E is a Banach space, we denote by $\mathcal{B}_{p,k}(E)$ (see [7] and [10]) the set of all E-valued tempered ultradistributions T for which there exists a function $f \in L_{p,k}(E)$ such that $\langle u, \widehat{T}\rangle = \int_{\mathbb{R}^n} u(x) f(x)\, dx$, $u \in \mathcal{S}_\omega$. $\mathcal{B}_{p,k}(E)$ with the norm

$$\|T\|_{p,k} = \begin{cases} \left((2\pi)^{-n} \int_{\mathbb{R}^n} \|k(x)\widehat{T}(x)\|_E^p dx\right)^{1/p} & \text{if } p < \infty \\ \\ \operatorname{ess\,sup}_{x\in\mathbb{R}^n} \|k(x)\widehat{T}(x)\|_E & \text{if } p = \infty \end{cases}$$

becomes a Banach space isometrically isomorphic to $L_{p,k}(E)$ and, therefore, to $L_p(E)$. (In the previous formulae we have written $\widehat{T}(x)$ instead of $f(x)$ we shall frequently commit this abuse of notation.) Spaces $\mathcal{B}_{p,k}(E)$ are called Hörmander-Beurling spaces with values in E.

3. Some properties of local Hörmander-Beurling spaces

In this section we will define local Hörmander-Beurling spaces $\mathcal{B}_{p,k}^{loc}(\Omega, E)$ generalizing local spaces $\mathcal{B}_{p,k}^{loc}(\Omega)$ considered by Björck in [1] and vector-valued local Hörmander-Beurling spaces discussed in [7].

Definition 3.1 *Let $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$, Ω an open set in $\mathbb{R}^n$, $1 \le p \le \infty$ and E a Banach space. We denote by $\mathcal{B}_{p,k}^{loc}(\Omega, E)$ the space of all vector valued ultradistributions*

$T \in \mathcal{D}'_\omega(\Omega, E)$ such that, for every $\varphi \in \mathcal{D}_\omega(\Omega)$, the map $\varphi T : \mathcal{S}_\omega \longrightarrow E$ defined by $\langle u, \varphi T\rangle = \langle u\,\varphi, T\rangle$, $u \in \mathcal{S}_\omega$, belongs to $\mathcal{B}_{p,k}(E)$. Space $\mathcal{B}^{loc}_{p,k}(\Omega, E)$ is endowed with the topology generated by seminorms $\{\|\cdot\|_{p,k,\varphi} : \varphi \in \mathcal{D}_\omega(\Omega)\}$, where $\|T\|_{p,k,\varphi} = \|\varphi T\|_{p,k}$ for $T \in \mathcal{B}^{loc}_{p,k}(\Omega, E)$.

Remark 3.2 *1. $\mathcal{B}^{loc}_{p,k}(\Omega, \mathbb{C})$ is the local space $\mathcal{B}^{loc}_{p,k}(\Omega)$ considered by Björck in [1], and our definition coincides with Definition 7 of [7] when $\omega(x) = \log(1 + |x|)$.*

2. The topology on $\mathcal{B}^{loc}_{p,k}(\Omega, E)$ defined by seminorms above is metrizable. In fact, if $T \in \mathcal{B}^{loc}_{p,k}(\Omega, E) \setminus \{0\}$ and $\varphi \in \mathcal{D}_\omega(\Omega)$ is such that $\langle \varphi, T\rangle \neq 0$ it suffices to take $\theta \in \mathcal{D}_\omega(\Omega)$ so that $\theta \equiv 1$ on suppφ to have $\|T\|_{p,k,\theta} > 0$, showing that the considered topology is Hausdorff. On the other hand, if $(K_v)_{v=1}^{\infty}$ is any fundamental sequence of compact subsets of Ω and $\varphi_v \in \mathcal{D}_\omega(\Omega)$ is such that $\varphi_v \equiv 1$ on K_v and supp$\varphi_v \subset \overset{\circ}{K}_{v+1}$, $v = 1, 2, \cdots$ then the topology on $\mathcal{B}^{loc}_{p,k}(\Omega, E)$ is also generated by seminorms $\{\|\cdot\|_{p,k,\varphi_v} : v = 1, 2, \cdots\}$. Indeed, if $\varphi \in \mathcal{D}_\omega(\Omega)$ and K_v is such that supp$\varphi \subset K_v$, it follows that $\varphi T = \varphi(\varphi_v T)$ for $T \in \mathcal{B}^{loc}_{p,k}(\Omega, E)$; and using [10, Prop.4.1] we have

$$\|T\|_{p,k,\varphi} = \|\varphi T\|_{p,k} = \|\varphi(\varphi_v T)\|_{p,k} \leq \|\varphi\|_{1,M_k} \|\varphi_v T\|_{p,k} = \|\varphi\|_{1,M_k} \|T\|_{p,k,\varphi_v}$$

establishing our assertion. (Note then the estimates
$\|\cdot\|_{p,k,\varphi_v} \leq \|\varphi_v\|_{1,M_k} \|\cdot\|_{p,k,\varphi_{v+1}}$, $v = 1, 2, \cdots$.)

3. It can be shown that if $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$, Ω an open set in $\mathbb{R}^n$, E is a Banach space and $1 \leq p \leq \infty$, the space $\mathcal{B}^{loc}_{p,k}(\Omega, E)$ is a Frechet space. Moreover, $\mathcal{E}_\omega(\Omega, E) \hookrightarrow \mathcal{B}^{loc}_{p,k}(\Omega, E) \hookrightarrow \mathcal{D}'_\omega(\Omega, E)$ y $\mathcal{D}_\omega(\Omega, E)$ is dense in $\mathcal{B}^{loc}_{p,k}(\Omega, E)$ if $p < \infty$.

In [5, Th. 10.1.25] it is proved that if k is a Hörmander weight such that $\frac{(1+|\cdot|)^m}{k} \in L_{p'}$, it follows that $\mathcal{B}^{loc}_{p,k}(\Omega) \subset \mathcal{C}^m(\Omega)$ for every open subset Ω of $\mathbb{R}^n$. In the following proposition, we show that this is also true here when $\omega \in \mathcal{M}$ and $k \in \mathcal{K}_\omega$. First, we will establish a well-known lemma.

Lemma 3.3 *Let $\omega \in \mathcal{M}$, Ω an open set in $\mathbb{R}^n$, $m \geq 0$ and E a Banach space. If $T \in \mathcal{D}'_\omega(\Omega, E)$ is such that, for every $\varphi \in \mathcal{D}_\omega(\Omega)$, there exists $f_\varphi \in \mathcal{C}^m(\Omega, E)$ with $\varphi T = f_\varphi$ in $\mathcal{D}'_\omega(\Omega, E)$, it follows $T \in \mathcal{C}^m(\Omega, E)$.*

Proof. Let $(K_i)_{i=1}^{\infty}$ be any fundamental sequence of compact subsets of Ω and let $\varphi_i \in \mathcal{D}_\omega(\Omega)$ such that $\varphi_i = 1$ in K_i, $i = 1, 2, \cdots$. By hypothesis, there is for each i a function $f_i := f_{\varphi_i} \in \mathcal{C}^m(\Omega, E)$ with $\varphi_i T = f_i$ in $\mathcal{D}'_\omega(\Omega, E)$. We check that $f_j = f_i$ for $j > i$: In fact, for every $\varphi \in \mathcal{D}_\omega(\overset{\circ}{K}_i)$, we have

$$\begin{aligned}\int_{\overset{\circ}{K}_i} [f_j(x) - f_i(x)]\varphi(x)dx &= \langle \varphi,\, f_j\rangle - \langle \varphi,\, f_i\rangle \\ &= \langle \varphi,\, \varphi_j\, T\rangle - \langle \varphi,\, \varphi_i\, T\rangle \\ &= \langle \varphi\, \varphi_j - \varphi\, \varphi_i,\, T\rangle = 0,\end{aligned}$$

and it follows (see Section 2) that $f_j = f_i$ in $\overset{\circ}{K}_i$.

We can then define a function f on Ω putting $f(x) := f_i(x)$ if $x \in \overset{\circ}{K}_i$. Obviously $f \in \mathcal{C}^m(\Omega, E)$. Let us see that f and T coincide on $\mathcal{D}'_\omega(\Omega, E)$. Take any $\mathcal{D}_\omega(\Omega)$-locally finite partition of unity $(\theta_i)_{i=1}^{\infty}$, for which $\operatorname{supp}\theta_i \subset \overset{\circ}{K}_i$, $i = 1, 2, \cdots$. Then, for every $\varphi \in \mathcal{D}_\omega(\Omega)$, there is a positive integer r such that $\varphi = \sum_{i=1}^{r} \varphi\, \theta_i$ and so $\langle \varphi, T\rangle = \sum_{i=1}^{r} \langle \varphi\, \theta_i, T\rangle = \sum_{i=1}^{r} \langle \varphi\, \theta_i \varphi_i, T\rangle = \sum_{i=1}^{r} \langle \varphi\, \theta_i, \varphi_i T\rangle = \sum_{i=1}^{r} \langle \varphi\, \theta_i, f_i\rangle = \sum_{i=1}^{r} \langle \varphi\, \theta_i, f\rangle = \langle \varphi, f\rangle$ and the proof is complete. □

Remark 3.4 *Let $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$, E a Banach space, $1 \le p \le \infty$ and $\frac{(1+|\cdot|)^m}{k} \in L_{p'}$ (m is a non-negative integer and $\frac{1}{p} + \frac{1}{p'} = 1$). Then $\mathcal{B}_{p,k}(E)$ is continuously embedded in $\mathcal{C}_0^b(E)$. In fact, if $u \in \mathcal{B}_{p,k}(E)$ functions $t \mapsto \widehat{u}(t)\, t^\alpha\, e^{itx}$, $|\alpha| \le m$ belong to $L_1(E)$ for each $x \in \mathbb{R}^n$ (using Hölder's inequality, we have that $\int_{\mathbb{R}^n} \left\|\widehat{u}(t)\, t^\alpha\, e^{itx}\right\|_E dt \le (2\pi)^{n/p} \|u\|_{p,k} \left\|\frac{(1+|\cdot|)^m}{k}\right\|_{p'}$). Therefore the function $x \mapsto (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{u}(t) e^{itx} dt$ is in $\mathcal{C}_0^b(E)$. But the vector ω-tempered ultradistribution associated to this function coincides with u since for each $\theta \in \mathcal{S}_\omega$, we have by virtue of Fubini's theorem for vector valued functions that*

$$
\begin{aligned}
(2\pi)^{-n}\int_{\mathbb{R}^n}\theta(x)\left(\int_{\mathbb{R}^n}\widehat{u}(t)e^{itx}dt\right)dx &= (2\pi)^{-n}\int_{\mathbb{R}^n}\widehat{u}(t)\left(\int_{\mathbb{R}^n}\theta(x)e^{itx}dx\right)dt \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n}\widehat{u}(t)\widetilde{\widehat{\theta}}(t)dt \\
&= (2\pi)^{-n}\left\langle\widetilde{\widehat{\theta}},\widehat{u}\right\rangle = \langle\theta,u\rangle\,.
\end{aligned}
$$

Hence, finally $u\in\mathcal{C}_0^b(E)$ *and* $max_{|\alpha|}\left\|\partial^\alpha u\right\|_\infty \le (2\pi)^{-n/p'}\left\|u\right\|_{p,k}\left\|\frac{(1+|\cdot|)^m}{k}\right\|_{p'}$.

Proposition 3.5 *Let* $\omega\in\mathcal{M}$, $k\in\mathcal{K}_\omega$, $1\le p\le\infty$, *m non-negative integer and E a Banach space. If* $\frac{(1+|\cdot|)^m}{k}\in L_{p'}$, *then* $\mathcal{B}^{loc}_{p,k}(\Omega,E)\subset\mathcal{C}^m(\Omega,E)$.

Proof. By using the above remark, it is quite easy to see that $\mathcal{B}_{p,k}(E)\hookrightarrow\mathcal{C}_0^m(E)$. Hence, if $T\in\mathcal{B}^{loc}_{p,k}(\Omega,E)$, we have that, for every $\varphi\in\mathcal{D}_\omega(\Omega)$, φT coincides on $\mathcal{S}'_\omega(E)$ and a fortiori on $\mathcal{D}'_\omega(\Omega,E)$, with the ultradistribution associated to some function in $\mathcal{C}_0^m(E)$. By applying the above lemma, the conclusion that $T\in\mathcal{C}^m(\Omega,E)$ follows easily. □

Theorem 3.6 *Let* Ω *be any open subset of* $\mathbb{R}^n$, *E a Banach space,* $\omega\in\mathcal{M}$, $k_j\in\mathcal{K}_\omega$ *and* $1\le p_j\le\infty$, $(j=1,2,\ldots)$. *If the space*

$$\mathcal{H}=\bigcap_{j=1}^{\infty}\mathcal{B}^{loc}_{p_j,k_j}(\Omega,E)$$

is endowed with the topology given by seminorms $\|\cdot\|_{p_j,k_j,\varphi}$, $j=1,2,\ldots$, $\varphi\in\mathcal{D}_\omega(\Omega)$, *then* $\mathcal{H}$ *becomes a Frechet space. In particular, if* $k_j=e^{j\omega}$, $j=1,2,\ldots$, $\mathcal{H}$ *is naturally topologically isomorphic to* $\mathcal{E}_\omega(\Omega,E)$.

Proof. If $(K_v)_{v=1}^\infty$ is any fundamental sequence of compact subsets of Ω and $\varphi_v\in\mathcal{D}_\omega(\Omega)$ is such that $\varphi_v=1$ on K_v and $\operatorname{supp}\varphi_v\subset\overset{\circ}{K}_{v+1}$, $v=1,2,\ldots$, then the topology on $\mathcal{H}$ is generated by the collection of seminiorms $\{\|\cdot\|_{p_j,k_j,\varphi_v}: j,\,v=1,2,\cdots\}$ and it is therefore metrizable. Completeness of $\mathcal{H}$ follows immediately from completeness of $\mathcal{B}^{loc}_{p_j,k_j}(\Omega,E)$. Let us prove now that if $k_j=e^{j\omega}$, $j\ge1$, then $\mathcal{H}\simeq\mathcal{E}_\omega(\Omega,E)$. From the inequality $a + b\log(1+|x|) \le \omega(x)$ ($a \in \mathbb{R}$, $b > 0$) it follows that for each j, there is an ℓ such that $\frac{(1+|\cdot|)^j}{e^{\ell\,\omega}} \in L_{p'_j}$. Now, applying the above proposition, we see that $T \in \mathcal{H}$ is also in $\mathcal{C}^\infty(\Omega, E)$. Moreover, if $T \in \mathcal{H}$ it is also easily verified that $e' \circ T \in \mathcal{E}_\omega(\Omega)$ for all $e' \in E'$ (use [1, Th. 2.3.9]). Consequently, every $T \in \mathcal{H}$ coincides with the ultradistribution corresponding to an element of $\mathcal{E}_\omega(\Omega, E)$ (see Section 2). We deduce that natural application

$$\begin{array}{rcl} \mathcal{E}_\omega(\Omega,\, E) & \longrightarrow & \mathcal{H} \\ f & \longmapsto & \{\varphi \in \mathcal{D}_\omega(\Omega) \to \int_\Omega \varphi(x) f(x) dx \in E\} \end{array}$$

is continuous. Since we deal with Frechet spaces this isomorphism is topological and the proof is complete. □

Remark 3.7 *1. If $E = \mathbb{C}$ the above proposition reduces to Theorem 2.3.9 of [1].*

2. Applications of spaces $\bigcap_{j=1}^\infty \mathcal{B}^{loc}_{p_j,k_j}(\Omega)$ to analysis of linear partial differential operators can be found, for example, in sections 10.5, 10.6, 11.2 and 12.8 of [5].

It is well known that the union of local Sobolev spaces $\mathcal{H}^{loc}_s(\Omega)$ $(= \mathcal{B}^{loc}_{2,(1+|\cdot|^2)^{s/2}}(\Omega))$ coincides with space $\mathcal{D}'^F(\Omega)$ of all finite order distributions on Ω. We will see now that this is also true in the context of vector-valued Beurling's ultradistributions, thereby generalizing [1, Th. 2.3.11]. For this, we put

$$\begin{aligned} \mathcal{D}'_{\omega,F}(\Omega, E) := \{T \in \mathcal{D}'_\omega(\Omega, E) : \exists\lambda > 0,\ \forall K \Subset \Omega\ \exists C_K > 0 \\ \|\langle\varphi, T\rangle\|_E \le C_K \|\varphi\|_\lambda\ \ \forall\varphi \in \mathcal{D}_\omega(K)\}. \end{aligned}$$

Theorem 3.8 *Let $\omega \in \mathcal{M}$, $p \in [1, \infty]$, Ω an open set in $\mathbb{R}^n$ and E a Banach space. Then we have*

$$\bigcup_{k \in \mathcal{K}_\omega} \mathcal{B}^{loc}_{p,k}(\Omega, E) = \mathcal{D}'_{\omega,F}(\Omega, E).$$

Proof. Assume that $k \in \mathcal{K}_\omega$ and $T \in \mathcal{B}^{loc}_{p,k}(\Omega, E)$. Now let then K be a compact subset of Ω and $\varphi \in \mathcal{D}_\omega(\Omega)$ such that $\varphi = 1$ on K (see [1, Th.1.3.7]). Since $\varphi T \in \mathcal{B}_{p,k}(E)$

(see [10, Prop. 4.1]) and $\frac{1}{k(x)} \le ce^{\lambda\,\omega(x)}$ ($c,\ \lambda > 0$), we have for every $\theta \in \mathcal{D}_\omega(K)$, $\psi := (2\,\pi)^{-n}\tilde{\hat{\theta}}$ that

$$\begin{aligned}
\|\langle\theta,\,T\rangle\|_E &= \|\langle\theta\varphi, T\rangle\|_E = \|\langle\theta,\,\varphi T\rangle\|_E \\
&= \left\|\left\langle\hat{\psi},\,\varphi T\right\rangle\right\|_E = \left\|\left\langle\psi,\,\widehat{\varphi T}\right\rangle\right\|_E = \left\|\int_{\mathbb{R}^n}\widehat{\varphi T}(x)\psi(x)dx\right\|_E \\
&\le (2\,\pi)^{-n}\int_{\mathbb{R}^n}\left\|\widehat{\varphi T}(x)\right\|_E\left|\hat{\theta}(-x)\right|dx \\
&\le c\int_{\mathbb{R}^n}\|\widehat{\varphi T}(x)k(x)\|_E|\hat{\theta}(-x)|e^{\lambda\,\omega(x)}dx \\
&\overset{\{\beta\ge 0\}}{=} c\int_{\mathbb{R}^n}\left\|\widehat{\varphi T}(x)k(x)\right\|_E e^{-\beta\,\omega(x)}\left|\hat{\theta}(-x)\right|e^{(\lambda+\beta)\,\omega(x)}dx \\
&\overset{\star}{\le} c\,|||\theta|||_{\lambda+\beta}\int_{\mathbb{R}^n}\left\|\widehat{\varphi T}(x)k(x)\right\|_E e^{-\beta\,\omega(x)}dx,
\end{aligned}$$

and using Hölder inequality, obtain

$$\|\langle\theta,\,T\rangle\|_E \le c\,|||\theta|||_{\lambda+\beta}\,\|\varphi T\|_{p,k}\left(\int_{\mathbb{R}^n}e^{-p'\beta\,\omega(x)}dx\right)^{1/p'}.$$

$\star$ (If $\omega \in \mathcal{M}$, $\varphi \in L_1$, $\lambda \in \mathbb{R}$, then (according to [1, Def.1.3.25]), $|||\varphi|||_\lambda = \sup_{\xi\in\mathbb{R}^n}|\hat{\varphi}(\xi)|e^{\lambda\omega(\xi)}$). The last integral is finite for a sufficiently large β so that we finally obtain $\|\langle\theta,\,T\rangle\|_E \le C_K\,\|\theta\|_{\lambda+\beta}$ (by [1, Cor. 1.4.3] we know that the families of norms $\{\|\cdot\|_\lambda : \lambda > 0\}$ and $\{|||\cdot|||_\lambda : \lambda > 0\}$ are equivalent on $\mathcal{D}_\omega(K)$ but examining the proof of Theorem 1.4.1 of [1] we deduce that $\forall\lambda > 0\ \exists C = C(n,\lambda,K)$ such that $|||\varphi|||_\lambda \le C\,\|\varphi\|_\lambda$, $\forall\varphi \in \mathcal{D}_\omega(K)$), where constant C_K depends only on the compact set K. Therefore $T \in \mathcal{D}'_{\omega,F}(\Omega, E)$.
Let us prove now the reverse inclusion. So let $T \in \mathcal{D}'_{\omega,F}(\Omega,E)$ and $\lambda > 0$ such that, for every compact $K \subset \Omega$, there exists $C_K > 0$ with $\|\langle\varphi,\,T\rangle\|_E \le C_K\,\|\varphi\|_\lambda$, for all $\varphi \in \mathcal{D}_\omega(K)$. We should find $k \in \mathcal{K}_\omega$ such that $T \in \mathcal{B}^{loc}_{p,k}(\Omega,E)$. For the time being let us take $\beta > 0$ and write $k(x) = e^{-\beta\,\omega(x)}$. We will show that $\varphi T \in \mathcal{B}_{p,k}(E)$, for every $\varphi \in \mathcal{D}_\omega(\Omega)$ if β is chosen sufficiently large. Given such φ we know that the Fourier-Laplace transform of φT, denoted here by $\widehat{\varphi T^L}$ ($\widehat{\varphi T^L}(\zeta) = \left\langle e^{-i\,\zeta(\cdot)}, \varphi T\right\rangle = \left\langle\varphi e^{-i\,\zeta(\cdot)}, T\right\rangle$, $\zeta \in \mathbb{C}^n$), is an entire analytic function with values in E (see [6]). Moreover, if $\operatorname{supp}\varphi \subset K \Subset \Omega$, we have that $\left\|\widehat{\varphi T^L}(x)\right\|_E = \left\|\left\langle\varphi e^{-i\,x(\cdot)}, T\right\rangle\right\|_E \le C_K\left\|\varphi e^{-i\,x(\cdot)}\right\|_\lambda = C_K\,\|\varphi\|_\lambda \le C_K\,\|\varphi\|_\lambda\,e^{\lambda\,\omega(x)}$ $\quad(x \in$

$\mathbb{R}^n$) and it follows that $\widehat{\varphi T^L}|_{\mathbb{R}^n} \in L_{p,k}(E)$, since,

$$\left(\int_{\mathbb{R}^n} \left\|\widehat{\varphi T^L}(x)\right\|_E^p k^p(x)dx\right)^{1/p} \leq C_K \|\varphi\|_\lambda \left(\int_{\mathbb{R}^n} e^{p(\lambda-\beta)\,\omega(x)}dx\right)^{1/p} < \infty,$$

if β is large enough (usual modification if $p = \infty$). Finally, from

$$\begin{aligned} \left\langle \theta, \widehat{\varphi T^L}|_{\mathbb{R}^n} \right\rangle &= \int_{\mathbb{R}^n} \theta(x) \left\langle e^{-i\,x(\cdot)}\varphi, T\right\rangle dx = \int_{\mathbb{R}^n} \left\langle \theta(x)e^{-i\,x(\cdot)}\varphi, T\right\rangle \\ &\overset{\dagger}{=} \left\langle \int_{\mathbb{R}^n} \theta(x)e^{-i\,x(\cdot)}\varphi dx, T\right\rangle = \left\langle \hat{\theta}\varphi, T\right\rangle \\ &= \left\langle \hat{\theta}, \varphi T\right\rangle = \left\langle \theta, \widehat{\varphi T}\right\rangle, \quad (\theta \in \mathcal{S}_\omega), \end{aligned}$$

it follows that $\widehat{\varphi T^L}|_{\mathbb{R}^n} = \widehat{\varphi T}$ in $\mathcal{S}'_\omega(E)$. Therefore, $\varphi T \in \mathcal{B}_{p,k}(E)$ with $k = e^{-\beta\,\omega}$ and the proof is complete. □

† (The function $\mathbb{R}^n \to \mathcal{D}_\omega(K) : x \mapsto \theta(x)e^{-i\,x(\cdot)}\varphi(\cdot)$ is Bochner integrable (since it is continuous and for each $\gamma > 0$, $\left\|\theta(x)e^{-i\,x(\cdot)\varphi(\cdot)}\right\|_\gamma \in L_1$ because $\theta \in \mathcal{S}_\omega$))).

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