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On Jordan Generalized Higher Derivations in Rings

Wagner Cortes[*], *Claus Haetinger*[†]

Abstract

I. N. Herstein proved that any Jordan derivation on a prime ring of characteristic not 2 is a derivation. M. Brešar extended this result to semiprime rings, while M. Ferrero and C. Haetinger extended the result to Jordan higher derivations. Recently, M. Ashraf and N. Rehman considered the question of Herstein for a Jordan generalized derivation.

This paper extends Ashraf's Theorem. We prove that if R is a 2-torsion-free ring which has a commutator right nonzero divisor, then every Jordan generalized higher derivation on R is a generalized higher derivation.

Key words and phrases: Higher Derivations, Generalized Higher Derivations, Commutator.

Introduction

Let R be an associative ring not necessarily with an identity element. A derivation (resp. Jordan derivation) d of R is an additive mapping d: $R \to R$ such that $d(ab) = d(a)b + ad(b)$, for every $a, b \in R$ (resp. $d(a^2) = d(a)a + ad(a)$, for every $a \in R$). As it is well-known, every derivation is a Jordan derivation and the converse is, in general, not true. If R is a 2-torsion-free semiprime ring, then by the results of I. N. Herstein and M.

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Brešar, every Jordan derivation of R is a derivation ([3], [4], [8]). It turns out that every Jordan derivation of a 2-torsion-free ring is a Jordan triple derivation ([9], Lemma 3.5). We recall that an additive mapping $d : R \to R$ is said to be a Jordan triple derivation if $d(aba) = d(a)ba + ad(b)a + abd(a)$, for every $a, b \in R$.

Also, R. Awtar extended the Herstein's Theorem to Lie ideals ([2], Theorem) by proving that if U is a Lie ideal of a prime ring R of $char(R) \neq 2$ such that $u^2 \in U$, for every $u \in U$, and d: $R \to R$ is an additive mapping such that $d \mid_U$ is a Jordan derivation of U into R, then $d \mid_U$ is a derivation of U into R.

Following B. Hvala ([10], page 1447), an additive mapping F: $R \to R$ is called a generalized derivation if there exists a derivation d: $R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. We call an additive mapping $F : R \to R$ a Jordan generalized derivation if there exists a derivation d: $R \to R$ such that $F(x^2) = F(x)x + xd(x)$ holds for all $x \in R$ ([1], page 7). In ([1], Theorem), M. Ashraf and N. Rehman showed that in a 2-torsion-free ring R which has a commutator nonzero divisor, every Jordan generalized derivation on R is a generalized derivation.

On the other hand, higher derivations have been studied in many papers mainly in commutative rings, but also in non-commutative rings. M. Ferrero and C. Haetinger extended some of the above results to the higher derivations. In particular, they pointed out that every Jordan higher derivation in a 2-torsion-free semiprime ring is a higher derivation ([6], Theorem 1.2 and [7], Theorem 2.1). Thus, it is natural to ask whether every Jordan generalized higher derivation on a ring R is a generalized higher derivation. In this paper we give the corresponding definitions and we prove a result extending ([1], Theorem), as we state more precisely in the next section.

As usual, $[x, y]$ will denote the commutator $xy - yx$ and $I\!N$ is the set of natural numbers including 0.

1. Definitions and Main Results

Throughout the paper U will denote a Lie ideal of a ring R.

In the main result of this paper we assume that the Lie ideal U verifies $u^2 \in U$, for every $u \in U$. A Lie ideal of this type will be called a *square closed* Lie ideal. We begin with the following definitions

1.1 Definition *Let* $D = (d_i)_{i \in I\!N}$ *be a family of additive mappings of* R *such that*

$d_0 = id_R$. *D is said to be:*

a higher derivation (*HD, for short*) *if for every* $n \in \mathbb{N}$ *we have* $d_n(ab) = \sum_{i+j=n} d_i(a)d_j(b)$, *for all* $a, b \in R$ (*[11], Exerc. 4, p. 540*);

a Jordan higher derivation (*JHD, for short*) *if for every* $n \in \mathbb{N}$ *we have* $d_n(a^2) = \sum_{i+j=n} d_i(a)d_j(a)$, *for all* $a \in R$;

a Jordan triple higher derivation (*JTHD, for short*) *if for every* $n \in \mathbb{N}$ *we have* $d_n(aba) = \sum_{i+j+k=n} d_i(a)d_j(b)d_k(a)$, *for all* $a, b \in R$.

If there is no possibility of misunderstanding, $D = (d_i)_{i\in\mathbb{N}}$ will always denote a HD of R.

1.2 Definition *Let* $F = (f_i)_{i\in\mathbb{N}}$ *be a family of additive mappings of* R *such that* $f_0 = id_R$. *F is said to be:*

a generalized higher derivation (*GHD, for short*) *if there exists a HD* $D = (d_i)_{i\in\mathbb{N}}$ *of* R *such that for every* $n \in \mathbb{N}$ *we have* $f_n(ab) = \sum_{i+j=n} f_i(a)d_j(b)$, *for all* $a, b \in R$ (*[13], Definition 2.3 (ii)*);

a Jordan generalized higher derivation (*JGHD, for short*) *if there exists a HD* $D = (d_i)_{i\in\mathbb{N}}$ *of* R *such that for every* $n \in \mathbb{N}$ *we have* $f_n(a^2) = \sum_{i+j=n} f_i(a)d_j(a)$, *for all* $a \in R$;

a Jordan generalized triple higher derivation (*JGTHD, for short*) *if there exists a HD* $D = (d_i)_{i\in\mathbb{N}}$ *of* R *such that for every* $n \in \mathbb{N}$ *we have* $f_n(aba) = \sum_{i+j+k=n} f_i(a)d_j(b)d_k(a)$, *for all* $a, b \in R$.

It is clear that in the case of Definition 1.2, f_1 is a Jordan generalized derivation.

Similarly, if U is a Lie ideal of R, then a family of additive mappings of R, $D = (d_i)_{i\in\mathbb{N}}$, is said to be a *HD (JHD, JTHD) of U into R* and a family of additive mappings of R, $F = (f_i)_{i\in\mathbb{N}}$, is said to be a *GHD (JGHD, JGTHD) of U into R* in case that the above corresponding conditions are satisfied for all $a, b \in U$.

As we have mentioned above, the main purpose of this paper is to prove the following result.

1.3 Theorem *Let R be a 2-torsion-free ring which has a commutator right nonzero divisor and U a square closed Lie ideal of R. Then every Jordan generalized higher* *derivation of U into R is a generalized higher derivation of U into R.*

Remark. Since $U = R$ is obviously a square closed Lie ideal of R, then Theorem 1.3 is also true for JGHD of R.

In particular, if $f_i = d_i$ for every $i \in I\!N$, we have the following

1.4 Corollary *Let R be a 2-torsion-free ring which has a commutator right nonzero divisor and U a square closed Lie ideal of R. Then every JHD of U into R is a HD of U into R.*

Note that Corollary 1.4 states the result of ([6], Corollary 1.4) without the semiprimality condition. We include an example by M. Brešar showing that the semiprimality and the right nonzero divisor commutator assumptions are independent each other.

Furthermore, we prove the next theorem, which generalizes ([6], Theorem 1.3).

1.5 Theorem *Let R be a 2-torsion-free ring and U a Lie ideal of R. Then every Jordan generalized higher derivation of U into R is a Jordan generalized triple higher derivation of U into R.*

One can ask whether the result of ([6], Theorem 1.2) is also true for Jordan generalized triple higher derivations. We were still unable to answer this question.

2. Proofs

If R is a ring, R has a Lie structure by the product $[x, y] = xy - yx$, for $x, y \in R$. A *Lie ideal* of R is any additive subgroup U of R with $[u, r] \in U$ for all $u \in U$ and $r \in R$ ([12]). Let U be a squared closed Lie ideal of R. It follows that $uv + vu \in U$ and $2uv \in U$ (see [6], page 251). This remark will be freely used in the paper.

For the proof of Theorem 1.3 we need several steps. Let U be a Lie ideal of R and F be a JGHD of U into R, where $D = (d_i)_{i \in I\!N}$ is the HD of U into R associated to F. For every fixed $n \in I\!N$ and for each $x, y \in U$ we denote by $\delta_n(x, y)$ the element of R defined by

$$\delta_n(x, y) = f_n(xy) - \sum_{i+j=n} f_i(x) d_j(y).$$

It is easy to see that δ_n is additive with respect to both arguments. Moreover, if $\delta_n = 0$ then $F = (f_i)_{i\in I\!N}$ is a GHD of U into R.

We need the following Lemma.

2.1 Lemma *Let R be a 2-torsion-free ring, U a square closed Lie ideal of R and $F = (f_i)_{i\in I\!N}$ a JGHD of U into R. Then for each fixed $n \in I\!N$ and for every $x, y, z \in U$, the following statements hold:*

(i). $f_n(xy + yx) = \sum_{i+j=n} (f_i(x)d_j(y) + f_i(y)d_j(x))$;

(ii). $f_n(xyx) = \sum_{i+j+k=n} f_i(x)d_j(y)d_k(x)$;

(iii). $f_n(xyz + zyx) = \sum_{i+j+k=n} (f_i(x)d_j(y)d_k(z) + f_i(z)d_j(y)d_k(x))$.

Proof. (i) and (ii) follow easily as in the proof of ([6], Theorem 1.3).
(iii). In fact, replacing x by $x + z$ in (ii), we obtain for $\alpha = (x + z)y(x + z)$,

$$\begin{aligned} f_n(\alpha) &= \sum_{i+j+k=n} (f_i(x)d_j(y)d_k(z) + f_i(z)d_j(y)d_k(x)) + \\ &\quad + \sum_{i+j+k=n} (f_i(x)d_j(y)d_k(x) + f_i(z)d_j(y)d_k(z)). \end{aligned}$$

On the other side, using (ii),

$$f_n(\alpha) = f_n(xyz + zyx) + \sum_{i+j+k=n} (f_i(x)d_j(y)d_k(x) + f_i(z)d_j(y)d_k(z)).$$

Comparing the expressions above, the result holds. □

By Lemma 2.1 (iii) we obtain Theorem 1.5, which generalizes ([6], Theorem 1.3).

Now we are ready to prove our main Theorem.

Proof of Theorem 1.3. By assumption, there exist elements a and b of U such that $c[a, b] = 0$ implies $c = 0$ for every $c \in R$.

We proceed by induction on $n \in I\!N$. Assume that F is a JGHD of U into R and take $a, b, x, y \in U$.

If $n = 1$: define $\beta = xy(xy) + (xy)yx$. Thus $2^3\beta = 2^3(xy(xy) + (xy)yx) = 2(2xy)(2xy) + 2(2xy)(2yx)$, where $2xy$, $2yx$, $2(2xy)(2xy)$ and $2(2xy)(2yx)$ are in U, because U is square closed. Now we are ready by ([1], Theorem), since R is 2-torsion-free and ([1], Lemma 2.1 (iv)) is also true in this case;

If $n = 2$: our aim is to show that $\delta_2(x, y) = 0$, for every $x, y \in U$.

We follow out in a similar way as in the proof of the main theorem of ([1]). First we will prove that

$$\delta_2(x, y)[x, y] = 0, \ \forall x, y \in U. \tag{1}$$

In fact, replacing z by $4xy$ in Lemma 2.1 (iii), we get for $\beta = 4xy(xy) + 4(xy)yx$,

$$\begin{aligned} f_2(\beta) \quad = \quad & 4(f_2(x)yxy + xd_2(y)xy + xyd_2(xy) + \\ & + f_1(x)d_1(y)xy + f_1(x)yd_1(xy) + xd_1(y)d_1(xy) + \\ & + f_2(xy)yx + xyd_2(y)x + xyyd_2(x) + \\ & + f_1(xy)d_1(y)x + f_1(xy)yd_1(x) + xyd_1(y)d_1(x)). \end{aligned}$$

On the other hand, since $F = (f_i)_{i\in I\!N}$ is a JGHD and $D = (d_i)_{i\in I\!N}$ is a HD, and by Lemma 2.1 (ii), we get

$$\begin{aligned} f_2(\beta) \quad = \quad & 4(f_2(xy)xy + f_1(xy)d_1(xy) + xyd_2(xy) + f_2(x)y^2x + xd_2(y^2)x + \\ & + xy^2d_2(x) + f_1(x)d_1(y^2)x + f_1(x)y^2d_1(x) + xd_1(y^2)d_1(x)) = \\ = \quad & 4(f_2(xy)xy + f_1(xy)d_1(xy) + f_2(x)y^2x + xd_2(y)yx + xyd_2(xy) + \\ & + xd_1(y)d_1(y)x + xyd_2(y)x + xy^2d_2(x) + f_1(x)d_1(y)yx + \\ & + f_1(x)yd_1(y)x + f_1(x)y^2d_1(x) + xd_1(y)yd_1(x) + xyd_1(y)d_1(x)). \end{aligned}$$

Comparing the two expressions of $f_2(\beta)$, we obtain

$$\delta_2(x, y)[x, y] + \delta_1(x, y)d_1([x, y]) = 0, \ \text{since } R \text{ is 2-torsion-free.}$$

By the $n = 1$ case, $\delta_1(x, y) = 0$. Therefore $\delta_2(x, y)[x, y] = 0$.

In particular, $\delta_2(a, b)[a, b] = 0$. Thus

$$\delta_2(a, b) = 0. \tag{2}$$

Replacing x by $x + a$ in (1), we get

$$\delta_2(x, y)[a, y] + \delta_2(a, y)[x, y] = 0, \ \forall x, y \in U. \tag{3}$$

Now replace y by $y+b$ in (3). We get

$$\delta_2(x,y)[a,b]+\delta_2(x,b)[a,y]+\delta_2(x,b)[a,b]+\delta_2(a,y)[x,b]=0,\ \forall x,y\in U. \tag{4}$$

Replacing x by a in (4), using (2) and since R is a 2-torsion-free ring, we obtain $\delta_2(a,y)[a,b]=0$ for every $y\in U$. Hence we have

$$\delta_2(a,y)=0,\ \forall y\in U. \tag{5}$$

Again replace y by b in (3) and use (2), to get $\delta_2(x,b)[a,b]=0$, for every $x\in U$. So we find that

$$\delta_2(x,b)=0,\ \forall x\in U. \tag{6}$$

Combining (4), (5) and (6) we have that $\delta_2(x,y)[a,b]=0$ and hence $\delta_2(x,y)=0$, for every $x,y\in U$.

Suppose now that $\delta_s(x,y)=0$, for every $x,y\in U$ and for all $s<n$.

Using Lemma 2.1 (iii) we have, for $\beta=4(xy(xy)+(xy)yx)$:

$$f_n(\beta)=4\left(\sum_{i+j+k=n} f_i(x)d_j(y)d_k(xy)+\sum_{i+j+k=n} f_i(xy)d_j(y)d_k(x)\right).$$

On the other side, since $F=(f_i)_{i\in\mathbb{N}}$ is a JGHD, using Lemma 2.1 (ii), we obtain

$$f_n(\beta)=4\left(\sum_{l+t=n} f_l(xy)d_t(xy)+\sum_{i+j+k=n} f_i(x)d_j(y^2)d_k(x)\right).$$

Now compare the right hand side of these two expressions of $f_n(\beta)$ to get

$$\sum_{i=1}^{n}\delta_i(x,y)\cdot d_{n-i}([x,y])=0,\ \forall x,y\in U,\ \text{since } R \text{ is 2-torsion-free.} \tag{7}$$

Since $\delta_s(x,y)=0$, for $s<n$, it follows that $\delta_n(x,y)[x,y]=0$, for every $x,y\in U$.

The proof now proceeds by using similar arguments as used in the case $n=2$. This completes the proof. □

Remark: We could have considered an alternative definition for a JGHD in the Definition 1.2: instead of supposing D as a HD, we could have taking D just as a JHD of R.

In fact, with this assumption, we can prove Lemma 2.1 and Theorem 1.3 in the same way as in our paper, just observing that the rule $d_n(ab+ba) = \sum_{i+j=n} d_i(a)d_j(b)+d_i(b)d_j(a)$ holds in this case ([6], proof of Theorem 1.3). Note that for obtaining (7) in the proof of Theorem 1.3 we only used that D was a JHD of R.

The following example shows that the assumptions of our Corollary 1.4 and ([6], Corollary 1.4) are independent each other. This example is due to M. Brešar who kindly allowed us to include it here.

Example: A semiprime ring may not contain a commutator nonzero divisor (after all, take commutative semiprime rings, or more generally, semiprime rings R containing a nonzero central idempotent element $e \in R$ such that eR is commutative). Conversely, a ring may contain a commutator nonzero divisor, but is not semiprime. For example, let $R = T_2(A_1)$ be the ring of the 2×2 upper triangular matrices whose entries are elements from the Weyl algebra A_1 (polynomials in x, y such that $xy - yx = 1$). Then R is not semiprime, but the commutator of scalar matrices generated by x and y is the identity matrix.

Finally, we give some well known examples of rings that have commutators nonzero divisors.

Example: 1. One can consider any noncommutative ring without zero divisors, the matrix algebra over a division ring, and also every noncommutative prime Goldie ring.

2. Consider the 2×2 matrix algebra $\mathcal{M}_2(D)$ over a domain D with 1. Let E_{ij} be the usual matrix units. Then the commutator $[E_{12}, E_{21}] = E_{11} - E_{22}$ is invertible.

When $char(D) \neq 2$, then in $\mathcal{M}_3(D)$ we have that $[E_{12} + E_{23}, E_{21} - E_{32}] = E_{11} - 2E_{22} + E_{33}$ is a nonzero divisor.

Clearly variations of this will work for $\mathcal{M}_n(D)$ where D is a noncommutative domain (just consider 2×2 block diagonal matrices and a 3×3 block at the bottom if n is odd). This block matrix idea will also work for the ring of all (countably infinite) row and column finite matrices over a domain.

In rings like $R = F\langle x, y\rangle/(x^2)$, the zero divisors must lie in xR or Rx, so $[x, y]$ is regular.

Of course once one has suitable examples of prime rings, then direct sums give examples for semiprime rings.

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Wagner CORTES
Departamento de Matemática
Universidade Estadual de Maringá
87020-900, Maringá-PR, BRAZIL
e-mail: wocortes@uem.br
Claus HAETINGER
Departamento III
Centro Universitário UNIVATES
95900-000, Lajeado-RS, BRAZIL
e-mail: chaet@univates.br

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