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On Fuzzy Cosets of Gamma Nearrings

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Abstract

In this paper, we consider fuzzy notion of a Γ-near ring, introduce the notion of a fuzzy coset and obtained some related important fundamental isomorphism theorems.

Key Words: Gamma nearring, ideal, fuzzy ideal, fuzzy coset, Gamma nearring homomorphism.

Introduction

A non-empty set N with two binary operations $+$ and $\cdot$ is called a *nearring* if it satisfies the following axioms.

(i) $(N, +)$ is a group (not necessarily Abelian);

(ii) $(N, \cdot)$ is a semi-group;

(iii) $(a + b)c = ac + bc$ for all $a, b, c \in N$.

Precisely speaking, it is a right near ring. Moreover, a near ring N is said to be a *zero-symmetric* nearring if $n0 = 0$ for all $n \in N$, where 0 is the additive identity in N. The concept of Γ-nearring, a generalization of both the concepts nearring and Γ-ring was introduced by Satyanarayana [10]. Later, several authors such as Satyanarayana [11, 12], Booth [1-3] and Booth and Groenewald [4] studied the ideal theory of Γ-nearrings.

Let (M, +) be a group (not necessarily Abelian) and Γ a non-empty set. Then M is said to be a Γ-nearring if there exists a mapping $M \times \Gamma \times M \rightarrow M$ (the image of (a, α, b) is denoted by aαb), satisfying the following conditions:

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(i) $(a+b)\alpha c = a\alpha c + b\alpha c$;

(ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and $\alpha,\ \beta \in \Gamma$. Moreover, M is said to be *zero-symmetric* if aα0 = 0 for all a $\in$ M and $\alpha \in \Gamma$, where 0 is the additive identity in M. A normal subgroup (I, +) of (M, +) is called

(i) a *left ideal,* if aα(b + i) − aαb $\in$ I for all a, b $\in$ M, $\alpha \in \Gamma$, i $\in$ I;

(ii) a *right ideal,* if iαa $\in$ I for all a $\in$ M, $\alpha \in \Gamma$, i $\in$ I;

(iii) an *ideal,* if it is both a left and a right ideal.

It is clear that if M is a Γ-nearring, then the elements of Γ act as binary operations on M such that the system (M, +, γ) is a nearring for all $\gamma \in \Gamma$. The relations between the concepts Γ-nearring and nearring were studied in Section 1 of Satyanarayana [12]. Throughout this paper, M stands for a zero-symmetric Γ-nearring. The ideal generated by an element a $\in$ M is denoted by $< a >$. For other definitions and preliminary results on Γ-nearrings we refer to [7, 11, 12].

The concept of fuzzy subset was introduced by Zadeh [14]. A fuzzy set in a set A is a function μ: A→ [0, 1]. For any t $\in$ [0, 1], the set μ_t defined by $\mu_t = \{x \in A | \mu(x) \geq t\}$ is called as a *level subset* of μ. For any two fuzzy sets μ, σ in A, we write $\mu \subseteq \sigma$ if

$\mu(x) \leq \sigma(x)$ for all $x \in A$. (In this case, we also say that μ is a subset of σ). Let X and Y be two non empty sets, f: X → Y, μ and σ be fuzzy subsets of X and Y respectively. Then f(μ), the *image* of μ under f is a fuzzy subset of Y defined by

$$(f(\mu))(y) = \begin{cases} \sup\limits_{f(x)=y} \mu(x) \text{ if } f^{-1}(y) \neq \phi \\ 0 \text{ if } f^{-1}(y) = \phi. \end{cases}$$

$f^{-1}(\sigma)$, the *preimage* of σ under f is a fuzzy subset of X defined by $(f^{-1}(\sigma))(x) = \sigma(f(x))$ for all $x \in X$.

Jun, Sapanci and Ozturk [7] introduced the concept "fuzzy ideal" in Γ-nearrings and studied some fundamental properties. It is clear that a fuzzy set μ in a Γ-nearring M is a mapping f: M → [0, 1].

A fuzzy set μ in a Γ-nearring M is called a *fuzzy left (resp., right) ideal* of (or in) M if

(i) μ is a fuzzy normal subgroup with respect to addition (that is, μ(x - y) $\geq$ min $\{\mu$(x), μ(y)$\}$, and μ(y + x - y) $\geq$ μ(x));

(ii) μ(xα(y+z)–xαy) $\geq$ μ(z) (resp., μ(xαy) $\geq$ μ(x)) for all x, y, z $\in$ M and $\alpha \in \Gamma$.

If μ is both left and right ideal, then μ is said to be a *fuzzy ideal* in M.

It is easy to verify that if μ is a fuzzy ideal of M, then the following three conditions hold: (i) $\mu(0) \geq \mu(x)$ (in other words, $\mu(0) = \max \{\mu(x) / x \in M\}$);

(ii) $\mu(x + y) = \mu(y + x)$; and

(iii) $\mu(x-y) = \mu(0)$ implies $\mu(x) = \mu(y)$, for all x, y $\in$ M.

Jun, Sapanci and Ozturk [7] proved that for a fuzzy set μ in M, μ is a fuzzy left (resp., right) ideal of M if and only if each level subset μ_t, t $\in$ im (μ), of μ is a left (resp. right) ideal of M.

A fuzzy (left, right) ideal μ of M with $\mu(0) = 1$ is called a *normal* (left, right) ideal of M. Jun, Kim and Ozturk [5] introduced the fuzzy maximal ideals and some related properties were studied. A fuzzy ideal μ of M is said to be a *fuzzy maximal* ideal if it satisfies the two conditions: (i) μ is non-constant; and (ii) μ^* is a maximal element among all the normal fuzzy ideals of M where $\mu * (x) = \mu(x) + 1 - \mu(0)$, for all $x \in M$.

For other preliminary definitions and results related to fuzzyness, see [7].

This paper is divided into three sections. In Section 1, we prove a result on fuzzy ideals.

In Section 2, we introduce the concept fuzzy coset in Γ-nearrings and prove that the set of all fuzzy cosets forms a Γ-nearring (theorem 2.4). In the last section, that is, in Section 3, we prove the following important fundamental results related to isomorphism theorems on Γ-nearrings;

(i). M/μ is isomorphic to the quotient Γ-nearring M/M_μ, where M/μ is the Γ-nearring of all the cosets of M with respect to the fuzzy ideal μ, and

$$M_\mu = \{x \in M/\mu(x) = \mu(0)\}.$$

(ii) There exists an order preserving bijection between the set P of all fuzzy ideals σ of M such that $\sigma \supseteq \mu$ and $\sigma(0) = \mu(0)$ and the set Q of all fuzzy ideals θ of M/μ such that $\theta \supseteq \theta_\mu$, where θ_μ is a fuzzy ideal of M/μ defined by $\theta_\mu(x+\mu) = \mu(x)$ for all x $\in$ M.

(iii) Let h: M $\rightarrow$ M^1 be a Γ-nearring epimorphism and let σ be a fuzzy ideal of M^1 and $\mu = h^{-1}(\sigma)$. Then the map ψ: M/μ $\rightarrow$ M^1/σ defined by $\psi(x+\mu) = h(x) + \sigma$ is a Γ-near ring isomorphism.

As a consequence of (iii), we obtain the following result: If μ and σ are two fuzzy ideals of M such that $\mu \subseteq \sigma$ and $\sigma(0) = \mu(0)$, then $M/\sigma \cong (M/\mu)/(\sigma/\mu)$.

1. Fuzzy ideals

Theorem 1.1 *If μ is a fuzzy ideal of M, and $a \in M$ then $\mu(x) \geq \mu(a)$ for all $x \in< a >$.*

Proof. By straightforward verification, we conclude that for $a \in M, < a >= \bigcup_{i=0}^{\infty} A_i$, where $A_{k+1} = A_k^* \cup A_k^+ \cup A_k^0 \cup A_k^{++}, A_0 = \{a\}$ and

$A_k^* = \{n+x-n \ / \ n \in N, x \in A_k\}$;

$A_k^+ = \{n_1\alpha(n_2+a)-n_1\alpha n_2 \ / \ n_1, n_2 \in M, a \in A_k, \alpha \in M\}$;

$A_k^0 = \{x - y \ / \ x, y \in A_k\}$;

$A_k^{++} = \{x\alpha m \ / \ x \in A_k, \alpha \in \Gamma \text{ and } m \in M\}$.

We prove that $\mu(u) \geq \mu(a)$ for all $u \in A_m$ for $m \geq 1$. For this, we use induction on m. It is obvious if m = 0. Suppose the induction hypothesis for k. That is., $\mu(x) \geq \mu(a)$ for all $x \in A_k$. Now let $v \in A_k^* \cup A_k^+ \cup A_k^0 \cup A_k^{++}$. Suppose $v \in A_k^*$. Then $v = n+x-n$ for some $x \in A_k$. Now $\mu(v) = \mu(n+x-n) \geq \mu(x)$ (since μ is a fuzzy ideal of N) $\geq \mu(a)$ (by induction hypothesis). Let $v \in A_k^0$. Then $v = x_1-x_2$ for some $x_1, x_2 \in A_k$. Now $\mu(v) = \mu(x_1-x_2) \geq \min(\mu(x_1), \mu(x_2)) \geq \mu(a)$, by induction hypothesis.

Suppose that $v \in A_k^+$. Then $v = n_1\alpha(n_2+x)-n_1\alpha n_2$ for some $n_1, n_2 \in M$, $x \in A_k$ and $\alpha \in \Gamma$. Now $\mu(v) = \mu(n_1\alpha(n_2+x)-n_1\alpha n_2) \geq \mu(x)$ (since μ is a fuzzy ideal of M) $\geq \mu(a)$ (by induction hypothesis on k).

Suppose $v \in A_k^{++}$. Then $v = x\alpha m$ for some $x \in A_k$, $\alpha \in \Gamma$ and $m \in M$. Now $\mu(v) = \mu(x\alpha m) \geq \mu(x)$ (since μ is a fuzzy ideal of M) $\geq \mu(a)$ (by induction hypothesis on k).

Thus in all cases we proved that $\mu(v) \geq \mu(a)$ for all $v \in A_{k+1}$. Hence by the principle of mathematical induction, we conclude that $\mu(v) \geq \mu(a)$ for all $v \in A_m$ and for all positive integers m. Hence $\mu(x) \geq \mu(a)$ for all $x \in< a >$. □

Corollary 1.2 *If I is an ideal of N with $I =< a >=< b >$, then $\mu(a) = \mu(b)$.*

Proof. Since $a \in< b >$ and $b \in< a >$, we have $\mu(a) \geq \mu(b)$ and $\mu(b) \geq \mu(a)$, so $\mu(a) = \mu(b)$. □

2. Fuzzy Cosets

Definition 2.1 *Let μ be a fuzzy ideal of M and $m \in M$. Then the fuzzy subset $m+\mu$ defined by $(m+\mu)(m^1) = \mu(m^1-m)$ for all $m^1 \in M$ is called a fuzzy coset of the fuzzy ideal μ.*

Lemma 2.2 *Let μ be a fuzzy ideal of M. Then for x, y, $z \in M$ we have the following:*

(i) $x+\mu = y+\mu$ if and only if $\mu(x-y) = \mu(0)$;

(ii) If $x + \mu = y + \mu$, then $\mu(x) = \mu(y)$;

(iii) $\mu(x+y) = \mu(y+x)$;

(iv) $M_\mu = \{x \in M / \mu(x) = \mu(0)\}$ is an ideal of M;

(v) Every fuzzy coset of a fuzzy ideal μ of M is constant on M_μ;

(vi) If $z \in M_\mu$, then $(x+\mu)(z) = \mu(x)$.

Proof. (i), (ii), (iii) have straightforward verifications.

(iv) Proved in Jun, Sapanci and Ozturk [7].

(v) Let y, $z \in M_\mu$. We show that $(x+\mu)(y) = (x+\mu)(z)$. Since y, $z \in M_\mu$, we have that $\mu(y) = \mu(0)$ and $\mu(z) = \mu(0)$. Since M_μ is an ideal, we have that $y-z \in M_\mu$. So $\mu(y-z) = \mu(0)$. Now $(x+\mu)(y) = \mu(y-x)$ (by the definition of fuzzy coset)

$= \mu(-(x-y))$ (since μ is a fuzzy ideal of M)

$= \mu(y-x) = \mu(-z+y-x+z)$ (since μ is a fuzzy normal subgroup)

$\geq \min\{\mu(-z+y), \mu(x-z)\}$ (since μ is a fuzzy ideal of M)

$= \min\{\mu(y-z), \mu(x-z)\} = \min\{\mu(0), \mu(x-z)\}$ (since $\mu(y-z) = \mu(0)$)

$= \mu(x-z)$ (since $\mu(0) \geq \mu(x-z)$)

$= (x+\mu)(z)$ (by definition of fuzzy coset).

Therefore $(x+\mu)(y) \geq (x+\mu)(z)$.

Similarly by interchanging y and z in above part, we can show that $(x+\mu)(z) \geq (x+\mu)(y)$. Hence $(x+\mu)(y) = (x+\mu)(z)$ for all y, $z \in M_\mu$.

(vi) Let $z \in M_\mu$. Then $\mu(z) = \mu(0)$. Since z, $0 \in M_\mu$, we have $(x+\mu)(z) = (x+\mu)(0)$ (by (v)) $\Rightarrow \mu(z-x)=\mu(0-x) = \mu(-x) = \mu(x)$ (since μ is a fuzzy ideal of M). Therefore $\mu(z-x) = \mu(x)$. Hence $(x+\mu)(z) = \mu(x)$. □

Notation 2.3 *We write $M/\mu = \{m + \mu\}\ m \in M\}$, the set of all fuzzy cosets of μ.*

Theorem 2.4 *Let μ be a fuzzy ideal of M. Then the set M/μ of all fuzzy cosets of μ is a Γ-nearring with respect to the operations defined by*

$(x+\mu)+(y+\mu) = (x+y)+\mu$; and $(x+\mu)\alpha(y+\mu) = x\alpha y+\mu$ for all x, $y \in M$ and $\alpha \in \Gamma$.

Proof. First we verify that "+" is well defined. Suppose $x+\mu = u+\mu$, $y+\mu = v+\mu$. Then by Lemma 2.2 (i), $\mu(x-u) = \mu(y-v) = \mu(0)$. Now $\mu\{(x+y)-(u+v)\} = \mu\{(x+y-v-u)\} = \mu\{(x+y-v)-u\} = \mu\{-u+(x+y-v)\} = \mu\{(-u+x)+(y-v)\} \geq \min\{\mu(-u+x), \mu(y-v)\}$ (since μ is a fuzzy ideal of M) $= \min\{\mu(x-u), \mu(y-v)\} = \mu(0)$. Also it is clear that $\mu(0) \geq \mu\{(x+y)-(u+v)\}$. Therefore $\mu\{(x+y)-(u+v)\} = \mu(0)$. Hence by Lemma 2.2 (i), $(x+y)+\mu = (u+v)+\mu$. This shows that "+" is well defined.

Next we verify that "·" is well defined. Now $\mu(x\alpha y-u\alpha v) = \mu(u\alpha v-x\alpha y) = \mu(u\alpha v-x\alpha v+x\alpha v-x\alpha y) = \mu((u-x)\alpha v+x\alpha(y+(-y+v))-x\alpha y) \geq \min\{\mu(u-x), \mu(-y+v)\}$ (since μ is a fuzzy ideal of M) $= \min\{\mu(0), \mu(0)\} = \mu(0) \geq \mu(x\alpha y-u\alpha v)$. This shows that $\mu(x\alpha y-u\alpha v)= \mu(0)$. By Lemma 2.2 (i), $x\alpha y+\mu = u\alpha v+\mu$.

Now we verify that $M/\mu = \{x+\mu \,/\, x \in M\}$ is a Γ-nearring with respect to the above operations defined. A direct verification shows that $(M/\mu, +)$ is a group.

Let x, y, $z \in M$ and $\alpha, \beta \in \Gamma$. Now $((x+\mu)+(y+\mu))\alpha(z+\mu)$

$= ((x+y)+\mu)\alpha(z+\mu)$

$= ((x+y)\alpha z)+\mu$ (by definition of multiplication)

$= ((x\alpha z)+(y\alpha z))+\mu$ (by right distributive law in M)

$= ((x\alpha z)+\mu)+((y\alpha z)+\mu)$

$= (x+\mu)\alpha(z+\mu)+(y+\mu)\alpha(z+\mu)$.

Also $((x+\mu)\alpha(y+\mu))\beta(z+\mu) = ((x\alpha y)+\mu)\beta(z+\mu)$ (by definition of addition)

$= (x\alpha y)\beta(z+\mu) = x\alpha(y\beta z)+\mu = (x+\mu)\alpha(y\beta z+\mu) = (x+\mu)\alpha((y+\mu)\beta(z+\mu))$.

Hence M/μ is a Γ-nearring. □

Notation 2.5 *Let M be a fuzzy ideal. We define θ_μ: $M/\mu \to [0, 1]$ by $\theta_\mu(x + \mu) = \mu(x)$ for all $x \in M$.*

Lemma 2.6 *If μ is a fuzzy ideal, then θ_μ is a fuzzy ideal of M/μ.*

Proof. Given that $\theta_\mu(x+\mu) = \mu(x)$. Suppose $x+\mu = y+\mu$. Then $\mu(x-y) = \mu(0)$. This implies $\mu(x) = \mu(y)$. That is, $\theta_\mu(x+\mu) = \theta_\mu(y+\mu)$. Therefore θ_μ is well defined.

We verify that θ_μ is a fuzzy ideal of M/μ.

(i) $\theta_\mu((x+\mu)+(y+\mu)) = \theta_\mu(x+y+\mu) = \mu(x+y)$ (by definition of θ_μ)
$\geq \min\{\mu(x), \mu(y)\}$(since μ is a fuzzy ideal of M)
$= \min\{\theta_\mu(x+\mu), \theta_\mu(y+\mu)\}$.

Therefore $\theta_\mu((x+\mu)+(y+\mu)) \geq \min\{\theta_\mu(x+\mu), \theta_\mu(y+\mu)\}$.

(ii) $\theta_\mu(x+\mu) = \mu(x) = \mu(-x)$ (since μ is a fuzzy ideal of M) $= \theta_\mu(-x+\mu)$, by definition of θ_μ.

(iii) $\theta_\mu((y+\mu)+(x+\mu)-(y+\mu)) = \theta_\mu((y+x-y)+\mu) = \mu(y+x-y) = \mu(x)= \theta_\mu(x+\mu)$

(iv) $\theta_\mu((x+\mu)\alpha(y+\mu)) = \theta_\mu(x\alpha y+\mu) = \mu(x\alpha y) \geq \mu(x) = \theta_\mu(x+\mu)$, by definition of θ_μ.

(v) $\theta_\mu\{(x+\mu)\alpha((y+\mu)+(z+\mu))-(x+\mu)\alpha(y+\mu)\} = \theta_\mu\{(x+\mu)\alpha((y+z)+\mu)-$
$(x+\mu)\alpha(y+\mu)\} = \theta_\mu\{(x\alpha(y+z)+\mu))-(x\alpha y+\mu)\}= \theta_\mu\{(x\alpha(y+z)-(x\alpha y)+\mu)\}$
$= \mu\{x\alpha(y+z)-(x\alpha y)\} \geq \mu(z) = \theta_\mu(z+\mu)$.

Hence θ_μ is a fuzzy ideal of M/μ. □

3. Some Isomorphism Theorems

Theorem 3.1 *(Jun, Sapanci and Ozturk [7]): If μ is a fuzzy (left, right) ideal of M then the set $M_\mu = \{x \in M \,/\, \mu(x) = \mu(0)\}$ is a fuzzy (left, right) ideal of M.*

Definition 3.2 *Let M and N be Γ-nearrings. A map θ: $M \rightarrow N$ is called a Γ-nearring homomorphism if $\theta(x+y) = \theta(x) + \theta(y)$ and $\theta(x\alpha y) = \theta(x)\alpha\theta(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$. Moreover if θ is one-one (onto, bijection, respectively) then θ is called as monomorphism (epimorphism, isomorphism, respectively).*

Now we prove the following theorem.

Theorem 3.3 *If μ is a fuzzy ideal of M then the map θ: $M \rightarrow M/\mu$, defined by $\theta(x) = x+\mu$, $x \in M$, is a Γ-near-ring epimorphism with kernel M_μ where*

$M_\mu = \{x \in M/ \mu(x) = \mu(0)\}$. Moreover M/M_μ is isomorphic to M/μ (under the mapping $x+M_\mu \rightarrow x+\mu$).

Proof. $\theta(x+y) = \theta(x) + \theta(y)$ is clear.

Now $\theta(x\alpha y) = (x\alpha y)+\mu$ (by definition of θ) $= (x+\mu)\alpha(y+\mu)$
$= \theta(x)\alpha\theta(y)$. Therefore θ is a homomorphism.

Now x $\in$ kernel $\theta \Leftrightarrow \theta(x) = 0 = 0+\mu \Leftrightarrow x+\mu = 0+\mu \Leftrightarrow \mu(x-0) = \mu(0)$, (by Lemma 2.2(i)) $\Leftrightarrow \mu(x) = \mu(0) \Leftrightarrow x \in M_\mu$. This shows that kernel $\theta = M_\mu$. □

Notation 3.4 *Let μ and σ be two fuzzy ideals of M such that $\mu \subseteq \sigma$ and $\sigma(0) = \mu(0)$. Then we define a fuzzy set θ_σ: $M/\mu \to [0, 1]$ by $\theta_\sigma(x+\mu) = \sigma(x)$ for all $x+\mu \in M/\mu$.*

Lemma 3.5 *θ_σ is a fuzzy ideal of M/μ such that $\theta_\mu. \subseteq \theta_\sigma$ where θ_σ and θ_μ are given by the above notation. Also $\theta_\mu(0) = \theta_\sigma(0)$.*

Proof. A direct verification shows that θ_σ is well-defined and is a fuzzy normal subgroup of M/μ. Now we verify that θ_σ is a fuzzy ideal of M/μ.

$\theta_\sigma\ ((x+\mu)\alpha(y+\mu)) = \theta_\sigma(x\alpha y+\mu) = \sigma(x\alpha y)$ (by definition of θ_σ) $\geq \sigma(x) = \theta_\sigma(x+\mu)$.

$\theta_\sigma\ \{(x+\mu)\alpha((y+\mu)+(z+\mu))-(x+\mu)\alpha(y+\mu)\}$

$= \theta_\sigma\{(x+\mu)\alpha((y+z)+\mu)-(x\alpha y+\mu)\}$

$=\theta_\sigma\{(x\alpha(y+z)+\mu)-(x\alpha y+\mu)\}$

$= \theta_\sigma\{(x\alpha(y+z)-x\alpha y)+\mu\}$

$= \sigma(x\alpha(y+z)-x\alpha y)$ (by definition of θ_σ)

$\geq \sigma\ (z)$ (since σ is a fuzzy ideal of M)

$= \theta_\sigma(z+\mu)$ (by definition of θ_σ).

Also $\theta_\sigma(x+\mu) = \sigma(x) \geq \mu(x) = \theta_\mu(x+\mu)$. Hence $\theta_\mu \subseteq \theta_\sigma$. □

Notation 3.6 *(i) The fuzzy ideal θ_σ of M/μ is denoted by σ/μ. Note that $\mu \subseteq \sigma$ with $\sigma(0) = \mu(0)$.*

(ii) Let μ be a fuzzy ideal of M and θ a fuzzy ideal of M/μ such that $\theta_\mu \subseteq \theta$ and $\theta_\mu(0) = \theta(0)$. Then we define σ_θ: $M \to [0, 1]$ by $\sigma_\theta(x) = \theta(x+\mu)$ for all $x \in M$.

Lemma 3.7 *σ_θ (defined above in notation 3.6), is a fuzzy ideal of M such that $\mu \subseteq \sigma_\theta$and $\mu(0) = \sigma_\theta(0)$.*

Proof. It is easy to verify that σ_θ is a fuzzy normal subgroup of M. Now

$\sigma_\theta\ (x\alpha y) = \theta(x\alpha y+\mu) = \theta((x+\mu)\alpha(y+\mu)) \geq \theta(x+\mu)$ (since θ is a fuzzy right ideal of M/μ)

$= \sigma_\theta(x)$ (by definition of σ_θ).

Therefore σ_θ is a fuzzy right ideal of M.

Also $\sigma_\theta\{x\alpha(y+z)-x\alpha y\} = \theta\{(x\alpha(y+z)-x\alpha y)+\mu\}$

$= \theta\{(x+\mu)\alpha((y+\mu)+(z+\mu))-(x+\mu)\alpha(y+\mu)\}$

$\geq \theta\ (z+\mu)$ (since θ is a fuzzy left ideal of M/μ)

$= \sigma_\theta(z)$ (by definition of σ_θ).

Therefore σ_θ is a fuzzy left ideal of M.

This shows that σ_θ is a fuzzy ideal of M.

Now we have $\sigma_\theta(x) = \theta(x+\mu) \geq \theta_\mu(x+\mu)$ (since $\theta_\mu \subseteq \theta$) $= \mu(x)$and so $\mu \subseteq \sigma_\theta$.

Also $\sigma_\theta(0) = \theta(0+\mu)= \theta(0) = \theta_\mu(0) = \theta_\mu(0+\mu) = \mu(0)$. □

Notation 3.8 *Let μ be a fuzzy ideal of M. We write $P = \{\sigma/\sigma$ is a fuzzy ideal of M, $\mu \subseteq \sigma$, $\sigma(0) = \mu(0)\}$ and $Q = \{\theta\ /\ \theta$ is a fuzzy ideal of M/μ, $\theta_\mu \subseteq \theta$ and $\theta(0) = \theta_\mu(0)\}$.*

Theorem 3.9 *Let μ be a fuzzy ideal of M. There exists an order preserving bijective mapping between the sets P and Q.*

Proof. Define η: P $\to$ Q by $\eta(\sigma) = \theta_\sigma$. By the lemma 3.5, $\eta(\sigma) = \theta_\sigma$ is a fuzzy ideal of M/μ such that $\theta_\mu \subseteq \theta_\sigma$and $\theta_\mu(0) = \theta_\sigma(0)$. By the definition of θ_σ, the mapping η is well defined. Suppose $\eta(\sigma) = \eta\ (\beta) \Rightarrow \theta_\sigma= \theta_\beta$.

Now $\sigma(x) = \theta_\sigma(x+\mu) = \theta_\beta(x+\mu) = \beta(x)$ for all $x \in M$. We have proved that $\eta(\sigma) = \eta(\beta) \Rightarrow \sigma = \beta$. Therefore η is one-one.

Let $\theta \in$ Q. Consider σ_θ: M $\to$ [0, 1] defined in notation 3.6 (ii). By Lemma 3.7, $\sigma_\theta \in$ P. Now we have to show that $\eta(\sigma_\theta) = \theta$.

$(\eta(\sigma_\theta))\ (x+\mu) = \sigma_\theta(x)$ (by definition of η and the definition of θ_σ in notation 3.4) $= \theta(x+\mu)$ (by the definition of σ_θ in notation 3.6 (ii)).

This is true for all $x+\mu \in M/\mu$. Hence $\eta(\sigma_\theta) = \theta$ and so η is onto.

Thus η: P $\to$ Q is a bijection.

Let σ, $\beta \in$ P such that $\sigma \subseteq \beta$. Now $(\eta(\sigma))\ (x+\mu) = \theta_\sigma(x+\mu)$ (by the definition of η) $\sigma(x) \leq \beta(x)$ (since $\sigma \subseteq \beta$) $= \theta_\beta(x+\mu) = (\eta(\beta))\ (x+\mu)$.

Since this is true for all $x+\mu \in M/\mu$, we have that $\eta(\sigma) \subseteq \eta(\beta)$.

Thus η: P $\to$ Q is an order preserving bijection. □

Theorem 3.10 *(Jun, Sapanci and Ozturk [7]) A Γ-nearing homomorphic pre-image of a fuzzy (left, right) ideal is a fuzzy (left, right) ideal.*

Theorem 3.11 *Let h: $M \to M^1$ be an epimorphism, σ is a fuzzy ideal of M^1 and $\mu = h^{-1}(\sigma)$. Then the map ψ: $M/\mu \to M^1/\sigma$ defined by $\psi(x+\mu) = h(x) + \sigma$ is a Γ-near- ring isomorphism.*

Proof. First we show that the mapping ψ is well defined.

Let $z^1 \in M^1$. Since h is an epimorphism, $h(z) = z^1$ for some $z \in M$.

Now $x+\mu = y+\mu \Rightarrow (x+\mu)(z) = (y+\mu)(z) \Rightarrow \mu(x–z) = \mu(y–z) \Rightarrow (h^{-1}(\sigma))(x–z) = (h^{-1}(\sigma))(y–z) \Rightarrow \sigma\ (h(x–z)) = \sigma(h(y–z)) \Rightarrow \sigma(h(x)–z^1)) = \sigma(h(y)–z^1))$

$\Rightarrow (h(x)+\sigma)(z^1) = (h(y)+\sigma)(z^1)$

This is true for all $z^1 \in M^1$. Hence $h(x) + \sigma = h(y) + \sigma$.

Now we proved that $x+\mu = y+\mu \Rightarrow \psi(x+\mu) = \psi(y+\mu)$. Thus ψ is well defined.

It is easy to verify that $\psi((x+\mu)+(y+\mu)) = \psi(x+\mu) + \psi(y+\mu)$.

Now $\psi((x+\mu)\alpha(y+\mu)) = \psi(x\alpha y+\mu) = h(x\alpha y)+\sigma$, by definition of ψ. Since h is a homomorphism, we have $h(x\alpha y)+\sigma = (h(x)\alpha h(y))+\sigma = (h(x)+\sigma)\alpha(h(y)+\sigma)$ $= \psi(x+\mu)\alpha\psi(y+\mu)$, by definition of ψ. Therefore ψ is a homomorphism.

Now we verify that ψ is one-one. Suppose $h(x)+\sigma = h(y)+\sigma$. Then $\sigma[h(x)-h(y)] = \sigma[h(0)]$, by definition. Since h is a homomorphism, $\sigma[h(x-y)] = \sigma(h(0))$. This implies $(h^{-1}(\sigma))(x-y) = (h^{-1}(\sigma))(0)$, which implies $\mu(x-y) = \mu(0)$. By Lemma 2.2 (i), $x+\mu = y+\mu$. This shows that ψ is one-one. Let $y \in M^1/\mu$. Then $y = h(x)+\sigma$ for some $x \in M$. Now $\psi(x + \mu) = h(x) + \sigma = y$. Therefore ψ is onto. Hence ψ is an isomorphism. □

As a consequence of Theorem 3.11, we obtain the following corollary.

Corollary 3.12 *Let μ and σ be two fuzzy ideals of M such that $\mu \subseteq \sigma$ and $\sigma(0) = \mu(0)$. Then $M/\sigma \cong (M/\mu)/(\sigma/\mu)$.*

Proof. Define ψ: $M \to M/\mu$ by $\psi(x) = x + \mu$ for all $x \in M$.

By theorem 3.3, ψ is Γ-nearring epimorphism. From the notation 3.4 and 3.6 we have $\theta_\sigma = \sigma/\mu$ and by Lemma 3.5, σ/μ is a fuzzy ideal of M/μ such that $\theta_\mu \subseteq \theta_\sigma = \sigma/\mu$ and $\theta_\mu(0) = \theta_\sigma(0)$. Now $\psi^{-1}(\sigma/\mu)$ is a fuzzy set in M and for any $x \in M$ we have

$(\psi^{-1}(\sigma/\mu))(\text{x}) = \psi^{-1}(\theta_\sigma)(\text{x}) = \theta_\sigma(\psi(\text{x})) = \theta_\sigma(\text{x}+\mu) = \sigma(\text{x})$. Therefore $\psi^{-1}(\sigma/\mu) = \sigma$ is a fuzzy ideal of M. Define ψ^*: M/ $\sigma \rightarrow (\text{M}/\mu)/(\sigma/\mu)$ by $\psi^*(\text{x}+\sigma) = \psi(\text{x}) + (\sigma/\mu)$.

By theorem 3.11, ψ^* is a Γ- nearring isomorphism. This completes the proof. □

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