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Forward-Backward Diffusion With Continuous Spectrum

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Abstract

We prove existence and uniqueness of solutions for a class of forward-backward diffusion equations via a representative example, where the second-order part has continuous spectrum, and the initial and boundary data are suitably chosen.

1. Introduction

Consider the following problem: given f and g, find $u = u(x, y)$ such that

$$xu_y - yu_x - u_{xx} = f, \tag{1}$$

$$u(x, 1) = g(x),\ x < 0, \quad \text{and} \quad u(x, -1) = g(x),\ x > 0, \tag{2}$$

where $x \in \mathbb{R}$ and $y \in [-1, 1]$. We prove existence and uniqueness of solutions for suitable choices of functional spaces for u, f, and g. In particular we impose some restrictions of the frequencies of f; see Theorems 4 and 5 for precise statements.

The methods we employ here can be used to deal with equations of the form $a(x)u_y + b(y)u_x + Au = f$ (where A is a positive operator), with variations according to each case. In [1], for example, we considered the case when A was of Sturm-Liouville type; here A has continuous spectrum.

Equation (1) is of mixed type. It behaves like a parabolic equation when $x \neq 0$, except that the direction of diffusion changes with the sign of x. This is reflected in the placement of the boundary conditions, and justifies that placement. Such equations are of great interest in physics, and have been studied at least since the 1930s (see for instance [5], [6]). The first rigorous mathematical treatment of this type of problem can be found in [2], but there does not seem to be a general theory of existence and uniqueness for this class of equations, and the literature mostly treats special cases ([4] is an exception).

2. Definitions and Results

All our functions are real-valued. We denote by $\mathcal{S}$ the Schwartz space in one variable, that is, smooth functions whose derivatives of any order, when multiplied by any polynomial, are bounded. In what follows we let I denote the interval $[-1, 1]$, $\Omega = \mathbb{R} \times I$, and $\mathcal{F} = C^\infty(I, \mathcal{S})$, which is canonically identified with the space of smooth functions u in the variables $(x, y) \in \Omega$ for which $u(\cdot, y) \in \mathcal{S}$ for every $y \in I$. Given $u \in \mathcal{F}$ we define the boundary operators B and $\overline{B}$ by setting

$$Bu(x) = \begin{cases} u(x,1), & \text{if } x < 0; \\ u(x,-1), & \text{if } x > 0; \end{cases} \qquad \overline{B}u(x) = \begin{cases} u(x,1), & \text{if } x > 0; \\ u(x,-1), & \text{if } x < 0. \end{cases} \tag{3}$$

Definition 1 *We say that $v \in \mathcal{F}_0$ if $v \in \mathcal{F}$ and $\overline{B}v = 0$.*

If $u, v \in \mathcal{F}$ we denote the standard inner product of u and v by

$$(u, v) = \int_\Omega u(x, y)\, v(x, y)\, dx\, dy. \tag{4}$$

When we fix $y \in I$ and integrate only in the variable x we will write

$$(u, v)_y = \int_{\mathbb{R}} u(x, y)\, v(x, y)\, dx, \tag{5}$$

so that $(u, v) = \int_I (u, v)_y\, dy$.

Definition 2 *The space F is defined to be the completion of $\mathcal{F}$ under the inner product*

$$\langle\, u, v\,\rangle = (u, v) + (u_x, v_x), \tag{6}$$

where u_x and u_y are the partial derivatives of u with respect to x and y, respectively. We set $\|u\|_F^2 = \langle\, u, u\,\rangle$.

The space F', dual to F, is the completion of $\mathcal{F}$ under the norm given by

$$\|f\|_{F'} = \sup_{\|u\|_F=1} |(f, u)|. \tag{7}$$

The space H is the completion of $\mathcal{F}_0$ under the inner product

$$[u, v] = \langle\, u, v\,\rangle + \frac{1}{2}\int_{\mathbb{R}} Bu(x)\, Bv(x)\, |x|\, dx. \tag{8}$$

Notice that we have the natural inclusions $H \subset F \subset L^2(\Omega) \subset F'$.

We can be even more explicit in our description of F and F', as follows. Let $\widehat{h}(\xi) = \int_{\mathbb{R}} h(x)\, e^{-i\xi x}\, dx$ be the usual Fourier transform in one variable. The Sobolev space H^s is defined to be the completion of Schwartz space $\mathcal{S}$ with respect to the norm

$$\|h\|_s^2 = \int_{\mathbb{R}} (1 + \xi^2)^s\, |\widehat{h}(\xi)|^2\, d\xi. \tag{9}$$

It is well-known that h is in H^1 if and only if both h and h' are in L^2. Moreover, H^{-1} is the dual of H^1. With that in mind we have the alternative descriptions

$$F = L^2(I, H^1), \qquad F' = L^2(I, H^{-1}). \tag{10}$$

Moreover, the norm in F can be written as

$$\|u\|_F^2 = \int_{-1}^{1}\int_{\mathbb{R}_\xi} (1 + \xi^2)\, |\widehat{u}(\xi, y)|^2\, d\xi\, dy, \tag{11}$$

where the Fourier transform is taken over the first coordinate only. The symbol $\mathbb{R}_\xi$ is just a reminder that we are integrating over the frequency side.

The weak formulation of our problem is obtained in the following way. Multiply equation (1) by $v \in \mathcal{F}$, and integrate over Ω; we obtain (formally)

$$(u_x, v_x) - (u, xv_y) + (u, yv_x) = (f, v) + \int_{\mathbb{R}} Bu\, Bv\, |x|\, dx - \int_{\mathbb{R}} \overline{B}u\, \overline{B}v\, |x|\, dx.$$

This suggests that f and g should be taken in F' and $G = L^2(|x|\, dx)$, respectively, and u should be taken in H.

Statement of the Problem *Given $f \in F'$ and $g \in G$, find $u \in H$ such that $Lu = f$ and $Bu = g$.*

Definition 3 *Given $f \in F'$ and $g \in G$, we say that $u \in H$ is a weak solution to the problem $Lu = f$, $Bu = g$ if, for all $v \in \mathcal{F}_0$,*

$$(u_x, v_x) - (u, xv_y) + (u, yv_x) = (f, v) + \int_{\mathbb{R}} g\, Bv\, |x|\, dx. \tag{12}$$

We will show in Theorem 2 that if u is a weak solution, then $Lu = f$ in the sense of distributions, and that $Bu = g$. Moreover, in Theorem 4 we obtain our main result, that if $f \in F'$ only contains high frequencies (in the variable x), then we can find $u \in H$ solving our problem.

3. First Existence Result

Theorem 1 *Let $0 < \varepsilon < 1$. If $f \in F'$ and $g \in G$, then there is a weak solution $u \in H$ to the problem $Lu + \varepsilon u = f$, $Bu = g$.*

Proof. We want to find $u \in H$ for which the equation

$$(u_x, v_x) - (u, xv_y) + (u, yv_x) + \varepsilon(u, v) = (f, v) + \int_{\mathbb{R}} g\, Bv\, |x|\, dx \tag{13}$$

holds for all $v \in \mathcal{F}_0$.

The map $v \mapsto (f,v) + \int g\, Bv\, |x|\, dx$ is a bounded linear functional on $\mathcal{F}_0$, extending uniquely to H, so there is a unique $w \in H$, depending only on f and g, such that

$$[w,v] = (f,v) + \int_{\mathbb{R}} g\, Bv\, |x|\, dx. \tag{14}$$

If we now fix $v \in \mathcal{F}_0$, then the left-hand-side of (13) is a bounded linear functional for $u \in H$, and there is a unique element $Tv \in H$ such that for all $u \in H$ we have

$$[u, Tv] = (u_x, v_x) - (u, xv_y) + (u, yv_x) + \varepsilon(u,v). \tag{15}$$

Setting $u = v$ in the above equation we obtain

$$[v, Tv] = (v_x, v_x) + \varepsilon(v,v) + \frac{1}{2}\int_{\mathbb{R}} |Bv|^2\, |x|\, dx, \tag{16}$$

from which we conclude that $[v, Tv] \geq \varepsilon[v,v]$. Thus $T : \mathcal{F}_0 \to H$ is injective, with a bounded inverse $V : H \to H$. Let V^* be the adjoint of V, and define $u = V^* w$. Then for all $v \in \mathcal{F}_0$ we have

$$[u, Tv] = [V^* w, Tv] = [w, VTv] = [w, v], \tag{17}$$

proving our claim. □

4. Technical Lemmas

Lemma 1 *Suppose $u \in H$ and $xu_y \in F'$. Then we can find a family $\{v^\delta\}_{\delta>0}$, such that $v^\delta \in C^\infty(I, H)$, $xv^\delta_y \in C^\infty(I, F')$, v^δ approximates u in H, and xv^δ_y approximates xu_y in F'.*

Proof. Let $\psi = \psi(y)$ be compactly supported on the line, smooth, with $\psi(y) \equiv 1$ for $y \in I$. Given $u \in F$ with $xu_y \in F'$, extend u to the real line in y by making even extensions around the points $y = 1$ and $y = -1$, etc. Call this extension w, and define $v(x,y) = \psi(y)w(x,y)$. Then $v \in L^2(\mathbb{R}, F)$, $xv_y \in L^2(\mathbb{R}, F')$, both compactly supported in the variable y, and coinciding with u and xu_y, respectively, when $y \in I$. Let h^δ be an approximation of the identity, $\delta > 0$; then

$$v^\delta(x,y) = \int_{\mathbb{R}} v(x, y-s)\, h^\delta(s)\, ds; \tag{18}$$

$$xv_y^\delta(x,y) = \int_{\mathbb{R}} xv_y(x, y-s)\, h^\delta(s)\, ds \tag{19}$$

are such that $v^\delta \in C^\infty(\mathbb{R}, F)$, and for $y \in I$ it approximates u in F; $xv_y^\delta \in C^\infty(\mathbb{R}, F')$, and for $y \in I$ it approximates xu_y in F'. $\square$

Lemma 2 *If $u \in H$ and $xu_y \in F'$, then $u \in C(I, G)$, and there is a constant c independent of u such that*

$$\sup_y (|x|\, u, u)_y \leq c\, \|xu_y\|_{F'}\, \|u\|_F + \int_{\mathbb{R}} |Bu|^2\, |x|\, dx. \tag{20}$$

Proof. We start with $u \in C^\infty(I, H)$, let u^+ be the restriction of u to $x > 0$, and $v = v(x,y)$ be the even extension of u^+ to the whole line. Then

$$\int_0^\infty x\, u^2(x,y)\, dx = \frac{1}{2}\int_{-\infty}^\infty |x|\, v^2(x,y)\, dx \tag{21}$$

$$= \frac{1}{4}\int_{-\infty}^\infty \int_{-1}^y |x|\, v_y(x,s)\, v(x,s)\, ds\, dx + \frac{1}{2}\int_{-\infty}^\infty |x|\, v^2(x,-1)\, dx \tag{22}$$

$$= \frac{1}{4}\int_{-1}^y (|x| v_y, v)_s\, ds + \int_0^\infty |Bu|^2\, |x|\, dx \tag{23}$$

$$\leq c\|xu_y\|_{F'}\, \|u\|_F + \int_0^\infty |Bu|^2\, |x|\, dx, \tag{24}$$

where we have used the fact that $|(|x|v_y, v)_s| \leq c'|(xu_y, u)_s|$ for some constant c' independent of u.

Similarly we can obtain the inequality

$$\int_{-\infty}^0 |x|\, u^2(x,y)\, dx \leq c\|xu_y\|_{F'}\, \|u\|_F + \int_{-\infty}^0 |Bu|^2\, |x|\, dx, \tag{25}$$

and we put these together to obtain our inequality over the whole line in the case when u is smooth in y. Now use Lemma 1 to approximate the general u by smooth functions v^δ. □

The next lemma is a standard result in vector-valued distributions; see [8] for a reference.

Lemma 3 *Let $u \in H$, $xu_y \in F'$, and $v \in \mathcal{F}$. Then the function $y \mapsto (xu, v)_y$ is absolutely continuous, with derivative given by $(xu_y, v)_y + (xu, v_y)_y$.*

Our final lemmas decompose F, F', and G into high and low frequency subspaces.

Let $D : F \to L^2(\Omega)$ denote the derivative operator in the x-variable; D is a bounded operator. Let $Mv(\xi) = i\xi v(\xi)$ denote multiplication by $i\xi$, and let $\wedge$ denote the Fourier transform in the x-variable. The following diagram is commutative:

$$\begin{array}{ccc} \widehat{F} & \xrightarrow{\;M\;} & L^2(\Omega_\xi) \\ \uparrow{\scriptstyle\wedge} & & \uparrow{\scriptstyle\wedge} \\ F & \xrightarrow{\;D\;} & L^2(\Omega). \end{array}$$

Given $\alpha > 0$, define the subspaces K_α and K^α of F by

$$h \in K_\alpha \iff \widehat{h}(\xi, y) \equiv 0 \text{ for } |\xi| > \alpha \tag{26}$$

$$h \in K^\alpha \iff \widehat{h}(\xi, y) \equiv 0 \text{ for } |\xi| \leq \alpha. \tag{27}$$

These are the low- and high-frequency subspaces of F, and $F = K_\alpha \oplus K^\alpha$. We denote by P_α and P^α the projections onto K_α and K^α, respectively. We denote by χ_α the function that is one if $|\xi| \leq \alpha$, and zero otherwise.

Lemma 4 *Given $u \in F$ we have*

$$u = P_\alpha u + P^\alpha u = P_\alpha u + (M^{-1}(1-\chi_\alpha)\widehat{Du})^\vee, \tag{28}$$

where $\vee$ denotes the inverse Fourier transform. Moreover, $\|P^\alpha u\|_F \le c\,\|Du\|$ for some constant c depending on α.

Proof. The norm of $m \in \widehat{F}$ is $\|m\|_{\widehat{F}}^2 = \int_{\Omega_\xi}(1+\xi^2)\,|m(\xi,y)|^2\,d\xi\,dy$. Let $g \in L^2(\Omega_\xi)$ be such that $g(\xi) \equiv 0$ if $|\xi| \le \alpha$. Then we can define m by setting $g(\xi) = i\xi\, m(\xi)$. We claim that m is in $\widehat{F}$:

$$\int_{\mathbb{R}_\xi}(1+\xi^2)\,|m|^2\,d\xi = \int_{|\xi|>\alpha}(1+\xi^2)\,\frac{|g|^2}{\xi^2}\,d\xi \le \left(1+\frac{1}{\alpha^2}\right)\int_{|\xi|>\alpha}|g|^2\,d\xi. \tag{29}$$

Therefore $\|m\|_{\widehat{F}} \le c\,\|g\|_{L^2(\Omega_\xi)}$, where $c = \alpha^{-1}\sqrt{1+\alpha^2}$ depends only on α. Because the Fourier transforms are isometries between the given spaces, we obtain our claim. □

The constant c from the last lemma is of interest to us. Note that it behaves asymptotically like $1/\alpha$ as α becomes small.

The decomposition $F = K_\alpha \oplus K^\alpha$ induces a decomposition of F' into $(F')_\alpha$ and $(F')^\alpha$, where $f \in (F')^\alpha$ if and only if $(f,u) = 0$ for all $u \in K_\alpha$.

Likewise we have $G = G_\alpha \oplus G^\alpha$, where $B(K^\alpha) = G^\alpha$.

5. Weak Solutions

Theorem 2 *Let $0 \le \varepsilon < 1$. If $u \in H$ is a weak solution to $Lu + \varepsilon u = f$, $Bu = g$, with $f \in F'$ and $g \in G$, then*

1. *$Lu + \varepsilon u = f$ in the sense of distributions;*
2. *$Bu = g$;*
3. *$\lim_{y\to -1}\int_0^\infty |u(x,y) - g(x)|^2\,|x|\,dx = 0$;*
4. *$\lim_{y\to 1}\int_{-\infty}^0 |u(x,y) - g(x)|^2\,|x|\,dx = 0$.*

Proof. Let v be smooth and compactly supported in the interior of Ω. Since v is zero at the boundary of Ω we have

$$(Lu + \varepsilon u, v) = (u, yv_x) - (u, xv_y) + (u_x, v_x) + \varepsilon(u, v) = (f, v), \tag{30}$$

and so $Lu + \varepsilon u = f$ in the sense of distributions. But then $Lu \in F'$, implying $xu_y \in F'$. Lemma 2 implies that $Bu \in G$. Taking $v \in \mathcal{F}_0$ we can use Lemma 3 to obtain

$$\begin{aligned} -\int_{\mathbb{R}} Bu\, Bv\, |x|\, dx &= (xu, v)_1 - (xu, v)_{-1} \qquad (31) \\ &= \int_{-1}^{1} (xu_y, v)_y + (xu, v_y)_y\, dy = (xu_y, v) + (xu, v_y). \end{aligned}$$

Since u is a weak solution we obtain

$$(xu_y, v) - (yu_x, v) - (u_{xx}, v) + \varepsilon(u, v) = (f, v) + \int_{\mathbb{R}} (g - Bu)\, Bv\, |x|\, dx, \tag{32}$$

for all $v \in \mathcal{F}_0$. Since the left hand side of (32) is just $(Lu + \varepsilon u, v)$, we conclude that

$$\int_{\mathbb{R}} (g - Bu)\, h\, |x|\, dx = 0 \tag{33}$$

for all $h \in G$, and so $Bu = g$. The last two statements are a consequence of this fact, and that $u \in C(I, G)$. □

We stress that in general $Lu + \varepsilon u = f$ in the sense of distributions, but since equation (1) is hypoelliptic (it satisfies Hörmander's criterion; see [7]), improved regularity in f automatically gives improved regularity for u, and if u has two continuous derivatives, then $Lu + \varepsilon u = f$ in the strong sense.

6. Uniqueness

Uniqueness is a consequence of the following estimate.

Theorem 3 (*A priori* **estimate**) *Let $u \in H$, and $Lu \in F'$. Then*

$$(u_x, u_x) \le (Lu, u) + \frac{1}{2} \int_{\mathbb{R}} |Bu|^2 \, |x| \, dx. \tag{34}$$

Proof. Note that if $u \in F$, then $Lu \in F'$ if and only if $xu_y \in F'$. Under our hypothesis

$$(Lu, u) = (xu_y, u) - (yu_x, u) + (u_x, u_x) = (xu_y, u) + (u_x, u_x), \tag{35}$$

since for almost all y, $(u_x, u)_y = 0$. So we need only to prove

$$(xu_y, u) \ge -\frac{1}{2} \int_{\mathbb{R}} |Bu|^2 \, |x| \, dx. \tag{36}$$

This is true if $u \in C^\infty(I, H)$. Using now Lemma 1, we approximate u by smooth functions v^δ in such a way that xu_y is approximated in F' by xv^δ_y. □

Corollary 1 *Let $0 \le \varepsilon < 1$. If $u \in H$ is a weak solution to $Lu + \varepsilon u = f$, $Bu = g$, then u is unique.*

Proof. It is enough to consider the case $f = 0$, $g = 0$. From the previous theorem we obtain $\varepsilon(u, u) + (u_x, u_x) \equiv 0$, which implies that $u_x \equiv 0$, making u a function of y only. As u is in $L^2(\Omega)$, this implies $u \equiv 0$. □

Note: In fact our proof of Theorem 3 establishes the identity

$$(u_x, u_x) + \frac{1}{2} \int_{\mathbb{R}} |\overline{B}u|^2 \, |x| \, dx = (Lu, u) + \frac{1}{2} \int_{\mathbb{R}} |Bu|^2 \, |x| \, dx. \tag{37}$$

7. Existence

From our previous results we know that there is a unique weak solution in F to the problem $Lu + \varepsilon u = f$, $Bu = g$, where $0 < \varepsilon < 1$. For the moment let's call this solution u^ε. If we prove that $\|u^\varepsilon\|_F$ is bounded as $\varepsilon \to 0$, then there is $u^0 \in F$, a weak limit of a subsequence u^{ε_n}. Such u^0 is a weak solution to $Lu = f$, $Bu = g$, since

$$(u^0, yv_x) - (u^0, xv_y) + (u^0_x, v_x) \tag{38}$$

$$= \lim_{\varepsilon_n \to 0} \left\{ (u^{\varepsilon_n}, yv_x) - (u^{\varepsilon_n}, xv_y) + (u^{\varepsilon_n}_x, v_x) + (\varepsilon_n u^{\varepsilon_n}, v) \right\} \tag{39}$$

$$= (f, v) + \int g\, Bv\, |x|\, dx. \tag{40}$$

From now on, we write $u = u^\varepsilon$, dropping the superscript. Our *a priori* estimate reads

$$\varepsilon(u, u) + (u_x, u_x) \leq (f, u) + \frac{1}{2} \int g^2\, |x|\, dx. \tag{41}$$

Our main theorem is the following.

Theorem 4 *If $g \in G$ and $f \in (F')^\alpha$ for some $\alpha > 0$, then there is a unique weak solution $u \in H$ to the problem $Lu = f$, $Bu = g$.*

Proof As mentioned above, our strategy is to show that $\|u\|_F$ is bounded as ε tends to zero. We divide the proof of Theorem 4 into steps.

Step 1 *As ε goes to zero, $\|u_x\|$ and $\varepsilon\|u\|^2$ remain bounded.*

Proof. Recall the notation from Lemma 4. Since $f \in (F')^\alpha$, we have that $(f, u) = (f, P^\alpha u)$, and Lemma 4 implies that

$$|(f, u)| \leq c\, \|f\|_{F'}\, \|u_x\|, \tag{42}$$

with c as in Lemma 4.

Taken together with (41) this yields

$$\|u_x\|^2 + \varepsilon\|u\|^2 \leq c_1\, \|u_x\| + c_2, \tag{43}$$

where $2c_2 = \int g^2\, |x|\, dx$, and c_1 is a constant depending only on α and $\|f\|_{F'}$. In particular we find that $\|u_x\| \leq \frac{1}{2}(c_1 + \sqrt{c_1^2 + 4c_2})$, proving our claim. □

Step 2 *The quantity $\|xu_y\|_{F'}$ remains bounded as ε goes to zero.*

Proof. If $v \in F$, we have

$$|(xu_y, v)| \leq |(yu_x, v)| + |(u_x, v_x)| + \varepsilon|(u, v)| + |(f, v)|. \tag{44}$$

From Step 1 we conclude that there is some constant C independent of ε such that $|(xu_y, v)| \leq C\|v\|_F$, proving our claim. □

Step 3 *To prove the theorem it is enough to show that the solution to the problem $Lu + \varepsilon u = 0$, $Bu = g$ is bounded in $L^2(\Omega)$ as $\varepsilon \to 0$.*

Proof. Let $g^\alpha = B(P^\alpha u) = B(u^\alpha) \in G^\alpha$, and $g_\alpha = B(P_\alpha u) = B(u_\alpha) \in G_\alpha$. Our problem splits into $Lu^\alpha + \varepsilon u^\alpha = f$, $B(u^\alpha) = g^\alpha$, and $Lu_\alpha + \varepsilon u_\alpha = 0$, $B(u_\alpha) = g_\alpha$. The boundedness of u_x ensures the boundedness of u_x^α, u^α, and $(u_\alpha)_x$ in L^2, so all we need to do is to bound u_α in $L^2(\Omega)$. □

From now on we concentrate on the problem $Lu + \varepsilon u = 0$, $Bu = g$.

For each $a > 0$ we break the L^2 norm of u into two pieces:

$$\|u\|_{>a}^2 = \int_{|x|>a} \int_{-1}^{1} u^2(x, y)\, dy\, dx, \qquad \|u\|_{<a}^2 = \int_{|x|<a} \int_{-1}^{1} u^2(x, y)\, dy\, dx.$$

Step 4 *To bound $\|u\|$ it is enough to bound the quantity $\|u\|_{<a}$.*

Proof. As a consequence of (20) and steps 1 and 2 we have that

$$\int_{-1}^{1} \int_{\mathbb{R}} u^2(x, y)\, |x|\, dx\, dy \leq d_1\|u\| + d_2, \tag{45}$$

where the constants d_1 and d_2 do not depend on ε. Therefore, for any fixed positive a we have

$$a\|u\|_{>a}^2 \leq \int_{-1}^{1} \int_{\mathbb{R}} u^2(x, y)\, |x|\, dx\, dy \leq d_3(\|u\|_{<a} + \|u\|_{>a}) + d_2, \tag{46}$$

with d_3 also independent of ε, from which we see that if $\|u\|_{<a}$ is bounded, then $\|u\|_{>a}$ will be bounded as well. $\square$

Step 5 *If for some $a > 0$ the quantity $\|u\|_{>a}$ is bounded in ε, then so is $\|u\|_{<a}$.*

Proof. Fix $z > 0$, and let $x \in [-z, z]$. We have

$$u(x, y) = u(z, y) + \int_z^x u_x(t, y)\, dt. \tag{47}$$

Squaring this identity we obtain

$$u^2(x, y) \le 2u^2(z, y) + 2\left(\int_z^x u_x(t, y)\, dt\right)^2, \tag{48}$$

and so

$$u^2(x, y) \le 2u^2(z, y) + 4z \int_{-z}^{z} u_x^2(t, y)\, dt. \tag{49}$$

Integrating over the rectangle $[-z, z] \times [-1, 1]$ gives us

$$\|u\|_{<z}^2 \le 4z \int_{-1}^{1} u^2(z, y)\, dy + 8z^2 \|u_x\|^2. \tag{50}$$

For $a > 0$, integrate the last inequality from $z = a$ to $z = a + 1$, and use the fact that $\|u\|_{<z}$ is a non-decreasing function of z to obtain

$$\begin{aligned} \|u\|_{<a}^2 &\le 4(a+1) \int_a^{a+1} \int_{-1}^{1} u^2(z, y)\, dy\, dz + 8(a+1)^2 \|u_x\|^2 \\ &\le 4(a+1)\|u\|_{>a}^2 + 8(a+1)^2 \|u_x\|^2. \end{aligned} \tag{51}$$

As a consequence we have

$$\|u\|_{<a} \le C_1 \|u\|_{>a} + C_2 \|u_x\|, \tag{52}$$

where $C_1 = 2\sqrt{a+1}$ and $C_2 = \sqrt{8}(a+1)$. This proves our claim. $\square$

The next step finishes the proof of Theorem 4.

Step 6 *For any $a > 0$ the quantity $\|u\|_{>a}$ is bounded in ε.*

Proof. Inequalities (46) and (52) give us

$$a\|u\|_{>a}^2 \leq C\|u\|_{>a} + E, \tag{53}$$

where C and E are constants that do not depend on ε, and consequently $\|u\|_{>a}$ is bounded as ε goes to zero. □

This finishes the proof of Theorem 4. □

From the proof of Theorem 4 we see that if $f \in (F')^\alpha$, then the solution u to the problem $Lu = f$, $Bu = 0$ satisfies the estimate $\|u\|_F \leq C\alpha^{-1}\|f\|_{F'}$, for some constant C independent of α. This suggests an improvement in Theorem 4, as follows.

Theorem 5 *If $f \in F'$ is such that*

$$\int_{-1}^{1} \int_{\mathbb{R}_\xi} \frac{|\widehat{f}(\xi, y)|^2}{\xi^2} \, d\xi \, dy < \infty, \tag{54}$$

then we can find $u \in H$ solving $Lu = f$, $Bu = 0$.

Proof. Let $\alpha_i = 2^{-i}$, and $f_i \in F'$ be the part of f with frequencies supported in the set $I_i = [-\alpha_{i-1}, -\alpha_i) \cup (\alpha_i, \alpha_{i-1}]$; alternatively, $\widehat{f}_i(\xi, y) = \widehat{f}(\xi, y)$ if $\xi \in I_i$, and zero otherwise. Let $u_i \in H$ be the solution to the problem $Lu_i = f_i$, $Bu_i = 0$. Then we know that $\|u_i\|_F^2 \leq C\alpha_i^2\|f_i\|_{F'}^2$, and that the u_i are mutually orthogonal (since their frequencies are supported on disjoint sets). Consequently, setting $u = \sum_i u_i$, we find

$$\|u\|_F^2 = \sum_i \|u_i\|_F^2 \leq C \sum_i \frac{\|f_i\|_{F'}^2}{\alpha_i^2} < \infty, \tag{55}$$

because the last sum is comparable to $\int\int |\widehat{f}(\xi, y)|^2/\xi^2 \, d\xi \, dy$. □

While the hypotheses of Theorem 5 may not be optimal, they should essentially be optimal; after all, if we want some regularity of u when solving $u_{xx} = f$ (in one variable), we need to impose the same type of restrictions on f.

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