Turk J Math
29 (2005) , 75 – 97.

Weighted Boundedness for a Rough Homogeneous Singular Integral

Hussain Al-Qassem

Abstract

A weighted norm inequality for a homogeneous singular integral with a kernel belonging to a certain block space is proved. Also, some applications of this inequality are obtained. Our results are essential improvements as well as extensions of some known results on the weighted boundedness of singular integrals.

Key Words: Singular integral, rough kernel, block space, weighted norm inequality.

1. Introduction

Let $\mathbf{S}^{n-1}$ denote the unit sphere in $\mathbf{R}^n$ ($n \geq 2$) equipped with the normalized Lebesgue measure $d\sigma = d\sigma(\cdot)$. Throughout this paper, p' will denote the dual exponent to p, that is $1/p + 1/p' = 1$. Also, we shall let Ω be a homogeneous function of zero which satisfies $\Omega \in L^1(\mathbf{S}^{n-1})$ and

$$\int_{\mathbf{S}^{n-1}} \Omega(u)\, d\sigma(u) = 0. \tag{1.1}$$

For $\gamma > 1$, let $\Delta_\gamma(\mathbf{R}^+)$ denote the set of all measurable functions h on $\mathbf{R}^+$ such that

$$\sup_{R>0} \frac{1}{R} \int_0^R |h(t)|^\gamma \, dt < \infty.$$

2000 *Mathematics Subject Classification:* 42B20; 42B25; 42B30.

Let $\Gamma(t)$ be a C^1 function on the interval $(0,\infty)$. We define the singular integral operator $T_{\Gamma,h,\Omega}$ and its truncated maximal operator $T^*_{\Gamma,h,\Omega}$ by

$$T_{\Gamma,h,\Omega}f(x)=\text{p.v.}\int_{\mathbf{R}^n} f\left(x-\Gamma(|y|)y'\right)|y|^{-n}\,\Omega(y)h(|y|)dy \tag{1.2}$$

and

$$T^*_{\Gamma,h,\Omega}f(x)=\sup_{\varepsilon>0}\left|\int_{|y|>\varepsilon} f\left(x-\Gamma(|y|)y'\right)|y|^{-n}\,\Omega(y)h(|y|)dy\right|, \tag{1.3}$$

where $y'=y/|y|\in\mathbf{S}^{n-1}$, and $f\in\mathcal{S}(\mathbf{R}^n)$, the space of Schwartz functions.

If $\Gamma(t)=t$, we shall denote $T_{\Gamma,h,\Omega}$ by $T_{h,\Omega}$ and $T^*_{\Gamma,h,\Omega}$ by $T^*_{h,\Omega}$. Also, we denote $T_{\Gamma,h,\Omega}$ by T_Ω and $T^*_{\Gamma,h,\Omega}$ by T^*_Ω when $h=1$ and $\Gamma(t)=t$.

The study of the $L^p(1<p<\infty)$ boundedness of the operators T_Ω and T^*_Ω began with Calderón-Zygmund in [6] under the condition $\Omega\in L\log^+L\left(\mathbf{S}^{n-1}\right)$. Some years later, Connett [10] and Coifman-Weiss [9] obtained an improvement over the result of Calderón and Zygmund by considering the Hardy space $H^1\left(\mathbf{S}^{n-1}\right)$. The study of the $L^p(1<p<\infty)$ boundedness of the more general class of operators $T_{h,\Omega}$ and $T^*_{h,\Omega}$ began in R. Fefferman in [16] if $h\in L^\infty(\mathbf{R}^+)$ and Ω satisfies some Lipschitz condition of positive order on S^{n-1} and subsequently by many authors under various conditions on Ω and h (see for example, [25], [7], [11], [13], [14], [22], [4]). In the meantime, the study of the weighted L^p boundedness of $T_{h,\Omega}$ and $T^*_{h,\Omega}$ has also attracted the attention of many authors ([5], [11], [12], [18], [20], [24], [28], [15]).

In 1993, J. Duoandikoetxea [12] proved the following two results:

Theorem A. *Suppose that $h\in L^\infty(\mathbf{R}^+)$ and $\Omega\in L^q(\mathbf{S}^{n-1})$ for some $q>1$. Then $T_{h,\Omega}$ is bounded on $L^p(\omega)$ if $q'\le p<\infty$, $p\ne 1$ and $\omega\in A_{p/q'}$,where $A_p(\mathbf{R}^n)$ is the Muckenhoupt weight class (see [17] for the definition) and $L^p(\omega)=L^p(\mathbf{R}^n,\omega(x)dx)$,* $\omega\ge 0$, is defined by

$$L^p(\mathbf{R}^n,\omega(x)dx)=\left\{f:\|f\|_{L^p(\omega)}=\left(\int_{\mathbf{R}^n}|f(x)|^p\,\omega(x)dx\right)^{1/p}<\infty\right\}.$$

For a special class of radial weights $\tilde{A}_p(\mathbf{R}_+)$, Duoandikoetxea proved the following sharper result:

Theorem B. *If $\omega \in \tilde{A}_p(\mathbf{R}_+)$ for $1 < p < \infty$, then T_Ω is bounded on $L^p(\omega)$ provided that $\Omega \in L\log^+ L\left(\mathbf{S}^{n-1}\right)$.*

We point out the class of weights $\tilde{A}_p(\mathbf{R}_+)$ was introduced by Duoandikoetxea [12] and its definition will be reviewed in Section 2.

In 1999, Fan-Pan-Yang in [15] improved the result in Theorem B and obtained the following:

Theorem C. *If $h \in \Delta_\gamma(\mathbf{R}^+)$ for some $\gamma \geq 2$ and $\Omega \in H^1\left(\mathbf{S}^{n-1}\right)$, then*

(i) $T_{h,\Omega}$ is bounded on $L^p(\omega)$ for $\gamma' \leq p < \infty$ and $\omega \in \tilde{A}^I_{p/\gamma'}(\mathbf{R}_+)$;

(ii) $T^*_{h,\Omega}$ is bounded on $L^p(\omega)$ for $\gamma' < p < \infty$ and $\omega \in \tilde{A}^I_{p/\gamma'}(\mathbf{R}_+)$,

where $\tilde{A}^I_{p/\gamma'}(\mathbf{R}_+)$ is a subclass of $\tilde{A}_{p/\gamma'}(\mathbf{R}_+)$ (which will be recalled in Section 2).

On the other hand, Jiang and Lu introduced a special class of block spaces $B_q^{(\kappa,\upsilon)}\left(\mathbf{S}^{n-1}\right)$ with respect to the study of the mapping properties of singular integral operators $T_{h,\Omega}$ (see [22]). In fact, they obtained the following L^2 boundedness result.

Theorem D ([22]). *Let $T_{h,\Omega}$ and $T^*_{h,\Omega}$ be given as above. Then we have*

(i) if $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1})$ and $h \in L^\infty$, $T_{h,\Omega}$ is a bounded operator on $L^2(\mathbf{R}^n)$;

(ii) if $\Omega \in B_q^{(0,1)}(\mathbf{S}^{n-1})$ and $h \in L^\infty$, $T^*_{h,\Omega}$ is a bounded operator on $L^2(\mathbf{R}^n)$.

Some years later, the L^p boundedness of the operators $T_{h,\Omega}$ and $T^*_{h,\Omega}$ were proved for all $p \in (1,\infty)$ under the conditions $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1})$ and $h \in \Delta_\gamma(\mathbf{R}^+)$ for some $\gamma > 1$ (see for example, [1], [2], [4]). Also, it was proved in [3] that the condition $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1})$ is the best possible for the L^p boundedness of T_Ω to hold. Namely, the L^p boundedness of T_Ω may fail for any p if it is replaced by a weaker condition $\Omega \in B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$ for any $-1 < \upsilon < 0$ and $q > 1$. The definition of the block space $B_q^{(\kappa,\upsilon)}\left(\mathbf{S}^{n-1}\right)$ will be recalled in Section 2.

The primary concern of this paper is studying the $L^p(\omega)$ boundedness of the operators $T_{h,\Omega}$ and $T^*_{h,\Omega}$ for $\omega \in \tilde{A}^I_{p/\gamma'}(\mathbf{R}_+)$, $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1})$ for some $q > 1$ and $h \in \Delta_\gamma(\mathbf{R}^+)$ for some $\gamma > 1$. The main results of this paper are the following:

Theorem 1.1. *Let $h \in \Delta_\gamma(\mathbf{R}^+)$ with $\gamma > 1$. Let Γ be in $C^2([0,\infty))$, convex, and an* *increasing function with* $\Gamma(0)=0$. *If* $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1})$ *for some* $q>1$, *then* $T_{\Gamma,h,\Omega}$ *is bounded on* $L^p(\mathbf{R}^n)$ *for* $1<p<\infty$.

Theorem 1.2. *Let* $h \in \Delta_\gamma(\mathbf{R}^+)$ *with* $\gamma \geq 2$ and $1<p<\infty$. *Let* Γ *be in* $C^2([0,\infty))$, *convex, and an increasing function with* $\Gamma(0)=0$. *If* $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1})$ *for some* $q>1$, *and* $\omega \in \tilde{A}^I_{p/\gamma'}(\mathbf{R}_+)$ *with* $p \geq \gamma'$, *then* $T_{\Gamma,h,\Omega}$ *is bounded on* $L^p(\omega)$.

As for the maximal truncated singular integral $T^*_{\Gamma,h,\Omega}$ we have the following results.

Theorem 1.3. *Let* Γ *be in* $C^2([0,\infty))$, *convex, and an increasing function with* $\Gamma(0)=0$. *If* $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1})$ *for some* $q>1$ and $h \in \Delta_\gamma(\mathbf{R}^+)$ *with* $\gamma>1$, *then*

(i) $T^*_{\Gamma,h,\Omega}$ *is bounded on* $L^p(\mathbf{R}^n)$ *for* $1<p<\infty$;

(ii) $T^*_{\Gamma,h,\Omega}$ *is bounded on* $L^p(\omega)$ *for* $\gamma'<p<\infty, \gamma \geq 2$ *and* $\omega \in \tilde{A}^I_{p/\gamma'}(\mathbf{R}_+)$.

Remark. Obviously, Theorems 1.1–1.3 extend and improve (in the weighted case) the results in [1], [2] and [4]. Also, one observes that Theorems 1.2 and 1.3 (b) represent an improvement and extension over Theorem A in the case $\omega \in \tilde{A}^I_p(\mathbf{R}_+)$ because Ω is allowed to be in the space $B_q^{(0,0)}(\mathbf{S}^{n-1})$; and bearing in mind the relation, for any fixed $q>1$, $L^d(\mathbf{S}^{n-1}) \subset B_q^{(0,0)}(\mathbf{S}^{n-1})$ for all $d>1$. With regard to the relation between $B_q^{(0,0)}(\mathbf{S}^{n-1})$ and $L\log^+ L\left(\mathbf{S}^{n-1}\right)$ remains open, as pointed out in [21].

In order to prove our results, we use the machinery developed by Duoandikoetxea and Rubio de Francia in [11] and we follow some ideas employed in [15] and [4]. We shall now point out some of the main differences between the proofs in this paper and the ones in [15]: (i) To treat the operators under consideration with kernels given by Ω's in the space $B_q^{(0,0)}(\mathbf{S}^{n-1})$ for some $q>1$, we shall first make an appropriate decomposition $\Omega = \sum_{\mu=1}^{\infty} c_\mu \tilde{b}_\mu$ and then we make further appropriate decomposition to each $T_{\Gamma,h,\tilde{b}_\mu}$ and $T^*_{\Gamma,h,\tilde{b}_\mu}$. These decompositions, together with a specially constructed partition of unity on $(0,\infty)$ and keeping track of certain constants allow us to obtain our results. (ii) We notice that the results in [15] are proved under the condition $h \in \Delta_\gamma(\mathbf{R}^+)$ with $\gamma \geq 2$, but the case $1<\gamma<2$ was left open, whereas in this paper we are able to obtain (in the unweighted case) some results under the condition $h \in \Delta_\gamma(\mathbf{R}^+)$ with $\gamma>1$. (For more details, see Theorem 1.1 and Theorem 1.3 (i) in this paper).

Throughout this paper, the letter C will stand for a positive constant that may vary at each occurrence. However, C does not depend on any of the essential variables.

2. Some Definitions and Lemmas

We start this section recalling the definition of some special classes of weights and some of their important properties relevant to our current study.

Definition 2.1. *Let $\omega(t) \geq 0$ and $\omega \in L^1_{loc}(\mathbf{R}_+)$. For $1 < p < \infty$, we say that $\omega \in A_p(\mathbf{R}_+)$ if there is a positive constant C such that for any interval $I \subset \mathbf{R}_+$,*

$$\left(|I|^{-1}\int_I \omega(t)dt\right)\left(|I|^{-1}\int_I \omega(t)^{-1/(p-1)}dt\right)^{p-1} \leq C < \infty.$$

$A_1(\mathbf{R}_+)$ is the class of weights ω for which M_{HL} satisfies a weak-type estimate in $L^1(\omega)$, where $M_{HL}(f)$ is the Hardy-Littlewood maximal function of f.

It is well-known that the class $A_1(\mathbf{R}_+)$ is also characterized by all weights ω for which $M_{HL}\omega(t) \leq C\omega(t)$ for a.e. $t \in \mathbf{R}_+$ and for some positive constant C.

Definition 2.2. *Let $1 \leq p < \infty$. We say that $\omega \in \tilde{A}_p(\mathbf{R}_+)$ if $\omega(x) = \nu_1(|x|)\nu_2(|x|)^{1-p}$, where either $\nu_i \in A_1(\mathbf{R}_+)$ is decreasing or $\nu_i^2 \in A_1(\mathbf{R}_+)$, $i = 1, 2$.*

Let $A_p^I(\mathbf{R}^n)$ be the weight class defined by exchanging the cubes in the definitions of A_p for all n-dimensional intervals with sides parallel to coordinate axes (see [19]). Let $\tilde{A}_p^I = \tilde{A}_p \cap A_p^I$. If $\omega \in \tilde{A}_p$, it follows from [12] that the classical Hardy-Littlewood maximal function $M_{HL}f$ is bounded on $L^p(\mathbf{R}^n, \omega(|x|)dx)$. Therefore, if $\omega(t) \in \tilde{A}_p(\mathbf{R}_+)$, then $\omega(|x|) \in A_p(\mathbf{R}^n)$.

By following the same argument as in the proof of the elementary properties of A_p weight class (see for example [17]) we get the following lemma.

Lemma 2.3. *If $1 \leq p < \infty$, then the weight class $\tilde{A}_p^I(\mathbf{R}_+)$ has the following properties:*

(i) $\tilde{A}_{p_1}^I \subset \tilde{A}_{p_1}^I$, if $1 \leq p_1 < p_2 < \infty$;

(ii) For any $\omega \in \tilde{A}_p^I$, there exists an $\varepsilon > 0$ such that $\omega^{1+\varepsilon} \in \tilde{A}_p^I$;

(iii) For any $\omega \in \tilde{A}_p^I$ and $p > 1$, there exists an $\varepsilon > 0$ such that $p - \varepsilon > 1$ and $\omega \in \tilde{A}_{p-\varepsilon}^I$.

Definition 2.4. (1) *For* $x_0' \in \mathbf{S}^{n-1}$ *and* $0 < \theta_0 \leq 2$, *the set* $B(x_0', \theta_0) = \{x' \in \mathbf{S}^{n-1} : |x' - x_0'| < \theta_0\}$ *is called a cap on* $\mathbf{S}^{n-1}$.

(2) For $1 < q \leq \infty$, *a measurable function* b *is called a q-block on* $\mathbf{S}^{n-1}$ *if* b *is a function supported on some cap* $I = B(x_0', \theta_0)$ *with* $\|b\|_{L^q} \leq |I|^{-1/q'}$, *where* $|I| = \sigma(I)$.

(3) $B_q^{(\kappa,\upsilon)}(\mathbf{S}^{n-1}) = \{\Omega \in L^1(\mathbf{S}^{n-1})$*:* $\Omega = \sum_{\mu=1}^{\infty} c_\mu b_\mu$ *where each* c_μ *is a complex number; each* b_μ *is a q-block supported on a cap* I_μ *on* $\mathbf{S}^{n-1}$*; and* $M_q^{(\kappa,\upsilon)}\left(\{c_\mu\}, \{I_\mu\}\right) = \sum_{\mu=1}^{\infty} |c_\mu| \left(1 + \phi_{\kappa,\upsilon}(|I_\mu|)\right) < \infty\}$, *where* $\phi_{\kappa,\upsilon}(t) = \chi_{(0,1)}(t) \int_t^1 u^{-1-\kappa} \log^{\upsilon}\left(u^{-1}\right) du$.

One observes that

$$\begin{aligned} \phi_{\kappa,\upsilon}(t) &\sim t^{-\kappa} \log^{\upsilon}(t^{-1}) \text{ as } t \to 0 \text{ for } \kappa > 0, \upsilon \in \mathbf{R}, \\ \phi_{0,\upsilon}(t) &\sim \log^{\upsilon+1}\left(t^{-1}\right) \text{ as } t \to 0 \text{ for } \upsilon > -1. \end{aligned}$$

The following properties of $B_q^{(\kappa,\upsilon)}$ can be found in [21]:

$$\begin{aligned} (i)\ B_q^{(\kappa,\upsilon_2)} &\subset B_q^{(\kappa,\upsilon_1)} \text{ if } \upsilon_2 > \upsilon_1 > -1 \text{ and } \kappa \geq 0; \\ (ii)\ B_{q_2}^{(\kappa,\upsilon)} &\subset B_{q_1}^{(\kappa,\upsilon)} \text{ if } 1 < q_1 < q_2; \\ (iii)\ L^q(\mathbf{S}^{n-1}) &\subset B_q^{(\kappa,\upsilon)}(\mathbf{S}^{n-1}) \text{ for } \upsilon > -1 \text{ and } \kappa \geq 0. \end{aligned}$$

In their investigations of block spaces, Keitoku and Sato showed in [21] that these spaces enjoy the following properties.

Lemma 2.5.

(i) If $1 < p \leq q \leq \infty$, *then for* $\kappa > \frac{1}{p'}$ *we have* $B_q^{(\kappa,\upsilon)}(\mathbf{S}^{n-1}) \subseteq L^p(\mathbf{S}^{n-1})$ for any $\upsilon > -1$;

(ii) $B_q^{(\kappa,\upsilon)}(\mathbf{S}^{n-1}) = L^q(\mathbf{S}^{n-1})$ if and only if $\kappa \geq \frac{1}{q'}$ and $\upsilon \geq 0$;

(iii) for any $\upsilon > -1$, *we have* $\bigcup_{q>1} B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}) \nsubseteq \bigcup_{q>1} L^q(\mathbf{S}^{n-1})$.

Definition 2.6. Let $\Gamma(t)$ be a C^1 function on the interval $(0, \infty)$ and for $\mu \in \mathbf{N}$, *let* $\tilde{b}_\mu$ *be a function on* $\mathbf{S}^{n-1}$ *satisfying the conditions:*

$$\left\|\tilde{b}_\mu\right\|_q \leq \left|I_\mu\right|^{-1/q'} \text{ for some } q > 1 \text{ and for some cap } I_\mu \text{ on } \mathbf{S}^{n-1}; \tag{2.1}$$

$$\left\|\tilde{b}_\mu\right\|_1 \leq 1. \tag{2.2}$$

Define the sequence of measures $\{\sigma_{\mu,\Gamma,k} : k \in \mathbf{Z}\}$ *and their corresponding maximal operator* $\sigma^*_{\mu,\Gamma}$ *on* $\mathbf{R}^n$ *by*

$$\int_{\mathbf{R}^n} f d\sigma_{\mu,\Gamma,k} = \int_{\rho_\mu^k \le |y| < \rho_\mu^{k+1}} f(\Gamma(|y|)y')h(|y|)\frac{\tilde{b}_\mu(y')}{|y|^n}dy,$$

$$\sigma^*_{\mu,\Gamma} f(x) = \sup_{k\in\mathbf{Z}} \left||\sigma_{\mu,\Gamma,k}| * f(x)\right|,$$

where $|\sigma_{\mu,\Gamma,k}|$ *is defined in the same way as* $\sigma_{\mu,\Gamma,k}$, *but with* $h\tilde{b}_\mu$ *replaced by* $\left|h\tilde{b}_\mu\right|$, $\rho_\mu = 2^{\beta_\mu}$ *and*

$$\beta_\mu = \begin{cases} 1 & , \text{ if } \left|I_\mu\right| \ge e^{-1}, \\ \log(\left|I_\mu\right|^{-1}) & \text{if } \left|I_\mu\right| < e^{-1}. \end{cases} \tag{2.3}$$

Lemma 2.7. *Let* $\mu \in \mathbf{N}$ *and* $h \in \Delta_\gamma(\mathbf{R}^+)$ *for some* $\gamma, 1 < \gamma \le 2$. *Let* $\tilde{b}_\mu$ *be a function on* $\mathbf{S}^{n-1}$ *satisfying (2.1)–(2.2) and (1.1) with* Ω *replaced by* $\tilde{b}_\mu$. *Suppose that* Γ *is in* $C^2([0,\infty))$, *convex, and an increasing function with* $\Gamma(0) = 0$. *Then there exist constants* C *and* $0 < \alpha < 1/q'$ *such that for all* $k \in \mathbf{Z}$ *and* $\xi \in \mathbf{R}^n$ *we have*

$$\|\lambda_{\mu,k}\| \le C\beta_\mu \tag{2.4}$$

$$|\hat{\sigma}_{\mu,\Gamma,k}(\xi)| \le C\beta_\mu \left|\Gamma(\rho_\mu^k)\xi\right|^{-\frac{\alpha}{\gamma'\beta_\mu}}; \tag{2.5}$$

$$|\hat{\sigma}_{\mu,\Gamma,k}(\xi)| \le C\beta_\mu \left|\Gamma(\rho_\mu^{k+1})\xi\right|^{\frac{\alpha}{\gamma'\beta_\mu}}. \tag{2.6}$$

The constant C *is independent of* k, I_μ, ξ *and* $\Gamma(\cdot)$.

Proof. It is easy to verify that (2.4) holds with a constant C independent of I_μ. We prove the inequalities (2.5)–(2.6) only for the case $\left|I_\mu\right| < e^{-1}$, because the proof of these inequalities for the case $\left|I_\mu\right| \ge e^{-1}$ can be dealt with quite similarly and more easily.

$$|\hat{\sigma}_{\mu,\Gamma,k}(\xi)| \le \left(\int_{\rho_\mu^k}^{\rho_\mu^{k+1}} |h(t)|^\gamma \frac{dt}{t}\right)^{1/\gamma} \left(\int_1^{\rho_\mu} |H_{k,\mu}(t)|^{\gamma'} \frac{dt}{t}\right)^{1/\gamma'},$$

where

$$H_{k,\mu}(t)=\int_{\mathbf{S}^{n-1}}e^{-i\Gamma(\rho_\mu^k t)\xi\cdot x}\tilde{b}_\mu(x)d\sigma(x)\,.$$

Since $|H_{k,\mu}(t)|\le 1$ we immediately get

$$\begin{aligned}|\hat{\sigma}_{\mu,\Gamma,k}(\xi)| &\le C\left(\sum_{s=1}^{[\log\left|I_\mu\right|^{-1}]+1}\int_{\rho_\mu^k 2^{s-1}}^{\rho_\mu^k 2^s}|h(t)|^\gamma\frac{dt}{t}\right)^{1/\gamma}\left(\int_1^{\rho_\mu}|H_{k,\mu}(t)|^2\frac{dt}{t}\right)^{1/\gamma'}\\ &\le\left(\log\left(\left|I_\mu\right|^{-1}\right)\right)^{1/\gamma}\left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}}\tilde{b}_\mu(x)\overline{\tilde{b}_\mu(y)}J_{\mu,k}(\xi,x,y)d\sigma(x)\,d\sigma(y)\right)^{1/\gamma'},\end{aligned}$$

where

$$J_{\mu,k}(\xi,x,y)=\int_1^{\rho_\mu}e^{-i\Gamma(\rho_\mu^k t)\xi\cdot(x-y)}\frac{dt}{t}.$$

We now show that

$$|J_{\mu,k}(\xi,x,y)|\le C(\log\left|I_\mu\right|^{-1})\left|\Gamma(\rho_\mu^k)\xi\right|^{-\alpha}|\xi'\cdot(x-y)|^{-\alpha}\,; \tag{2.7}$$

for some $0<\alpha q'<1$. To this end, we notice that

$$J_{\mu,k}(\xi,x,y)=\int_1^{\rho_\mu}G'(t)\frac{dt}{t},\ \text{ where } G(t)=\int_1^t e^{-i\Gamma(\rho_\mu^k w)\xi\cdot(x-y)}dw,\ 1\le t\le\rho_\mu.$$

By the assumptions on Γ and the mean value theorem we have

$$\frac{d}{dw}\left(\Gamma(\rho_\mu^k w)\right)=\rho_\mu^k\Gamma'(\rho_\mu^k w)\ge\frac{\Gamma(\rho_\mu^k w)}{w}\ge\frac{\Gamma(\rho_\mu^k)}{t}\text{ for }1\le w\le t\le\rho_\mu.$$

Thus by van der Corput's lemma, $|G(t)|\le\left|\Gamma(\rho_\mu^k)\xi\right|^{-1}|\xi'\cdot(x-y)|^{-1}t$, for $1\le t\le\rho_\mu$. Hence by integration by parts,

$$|J_{\mu,k}(\xi,x,y)|\le C\log\left(\left|I_\mu\right|^{-1}\right)\left|\Gamma(\rho_\mu^k)\xi\right|^{-1}|\xi'\cdot(x-y)|^{-1}.$$

Now combining this bound with the trivial bound $|J_{\mu,k}(\xi,x,y)|\le(\log 2)\log(\left|I_\mu\right|^{-1})$ and choosing α so that $0<\alpha q'<1$, yields the assertion of (2.7). Therefore, by Hölder's inequality

$$|\hat{\sigma}_{\mu,\Gamma,k}(\xi)| \leq$$

$$C\log\left(\left|I_{\mu}\right|^{-1}\right)\left|\Gamma(\rho_{\mu}^{k})\xi\right|^{-\alpha/\gamma'}\left|\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}}\left|\tilde{b}_{\mu}(x)\tilde{b}_{\mu}(y)\right|\left|\xi'\cdot(x-y)\right|^{-\alpha}d\sigma(x)\,d\sigma(y)\right|^{1/\gamma'}$$

$$\leq C\log\left(\left|I_{\mu}\right|^{-1}\right)\left|\Gamma(\rho_{\mu}^{k})\xi\right|^{-\alpha/\gamma'}\left\|\tilde{b}_{\mu}\right\|_{L^{q}(\mathbf{S}^{n-1})}^{2/\gamma'}\left\{\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}}|x_1-y_1|^{-\alpha q'}d\sigma(x)\,d\sigma(y)\right\}^{1/\gamma' q'},$$

where $x=(x_{1,\dots,}x_n)$ and $y=(y_{1,\dots,}y_n)$. Since the last integral is finite, by (2.1) we obtain

$$|\hat{\sigma}_{\mu,\Gamma,k}(\xi)| \leq C\log\left(\left|I_{\mu}\right|^{-1}\right)\left|I_{\mu}\right|^{-2/\gamma' q'}\left|\Gamma(\rho_{\mu}^{k})\xi\right|^{-\alpha/\gamma'}.$$

By interpolation between this estimate and the trivial estimate

$$|\hat{\sigma}_{\mu,\Gamma,k}(\xi)| \leq C\log\left(\left|I_{\mu}\right|^{-1}\right)$$

we get the estimate in (2.5). To get the estimate (2.6), we use the mean zero property (1.1) of $\tilde{b}_{\mu}$ to get

$$|\hat{\sigma}_{\mu,\Gamma,k}(\xi)| \leq \int_{\mathbf{S}^{n-1}}\int_{1}^{\rho_{\mu}}\left|e^{-i\Gamma(\rho_{\mu}^{k}t)\xi\cdot x}-1\right|\left|h(\rho_{\mu}^{k}t)\tilde{b}_{\mu}(x)\right|\frac{dt}{t}d\sigma(x).$$

Hence,

$$|\hat{\sigma}_{\mu,\Gamma,k}(\xi)| \leq C\log\left(\left|I_{\mu}\right|^{-1}\right)\left|\Gamma(\rho_{\mu}^{k+1})\xi\right|$$

which, when combined with the trivial estimate $|\hat{\sigma}_{\mu,\Gamma,k}(\xi)| \leq C\log\left(\left|I_{\mu}\right|^{-1}\right)$, yields the estimate in (2.6). This completes the proof of the lemma. □

By the same argument as in [27, p. 57] we get

Lemma 2.8. *Let φ be a nonnegative, decreasing function on $[0,\infty)$ with $\int_{[0,\infty)}\varphi(t)dt=1$. Then*

$$\left|\int_{[0,\infty)}f(x-ty')\varphi(t)dt\right| \leq M_{y'}f(x),$$

where

$$M_{y'}f(x)=\sup_{R\in\mathbf{R}}\frac{1}{R}\int_0^R|f(x-sy')|\,ds$$

is the Hardy-Littlewood maximal function of f in the direction of y'.

Lemma 2.9. *Let $\mu\in\mathbf{N}$, $h\in\Delta_{\gamma}(\mathbf{R}^+)$ for some $\gamma>1$ and $\omega\in\tilde{A}_{p/\gamma'}(\mathbf{R}^+)$. Let $\tilde{b}_{\mu}$ be a function on $\mathbf{S}^{n-1}$ satisfying (2.1)–(2.2) and let Γ be in $C^2([0,\infty))$, convex, and an increasing function with $\Gamma(0)=0$. Then*

$$\left\|\sigma^*_{\mu,\Gamma}(f)\right\|_{L^p(\omega)}\le C_p\beta_{\mu}\left\|f\right\|_{L^p(\omega)} \tag{2.8}$$

for $\gamma'<p\le\infty$, where C_p is independent of μ and f.

Proof. As above, we prove (2.8) only for the case that $\left|I_{\mu}\right|<e^{-1}$. By Hölder's inequality and (2.2), we have

$$\begin{aligned}\left|\sigma_{\mu,\Gamma,k}*f(x)\right| &\le \left(\int_{\rho_\mu^k}^{\rho_\mu^{k+1}}|h(t)|^{\gamma}\frac{dt}{t}\right)^{1/\gamma}\left(\int_{\rho_\mu^k}^{\rho_\mu^{k+1}}\left|\int_{\mathbf{S}^{n-1}}\tilde{b}_{\mu}(y')f(x-\Gamma(t)y')d\sigma(y')\right|^{\gamma'}\frac{dt}{t}\right)^{1/\gamma'}\\ &\le C\left(\log\left(\left|I_{\mu}\right|^{-1}\right)\right)^{1/\gamma}\left(\int_{\rho_\mu^k}^{\rho_\mu^{k+1}}\int_{\mathbf{S}^{n-1}}\left|\tilde{b}_{\mu}(y')\right|\left|f(x-\Gamma(t)y')\right|^{\gamma'}d\sigma(y')\frac{dt}{t}\right)^{1/\gamma'}\\ &\le C\left(\log\left(\left|I_{\mu}\right|^{-1}\right)\right)^{1/\gamma}\left(\int_{\mathbf{S}^{n-1}}\left|\tilde{b}_{\mu}(y')\right|\mathcal{M}_{\Gamma,\mu,y'}(|f|^{\gamma'})(x)d\sigma(y')\right)^{1/\gamma'},\end{aligned} \tag{2.9}$$

where

$$\mathcal{M}_{\Gamma,\mu,y'}f(x)=\sup_{k\in\mathbf{Z}}\left|\int_{\rho_\mu^k}^{\rho_\mu^{k+1}}f(x-\Gamma(t)y')\frac{dt}{t}\right|.$$

By a change of variable we have

$$\mathcal{M}_{\Gamma,\mu,y'}f(x)\le\sup_{k\in\mathbf{Z}}\left(\int_{\Gamma(\rho_\mu^k)}^{\Gamma(\rho_\mu^{k+1})}|f(x-ty')|\frac{dt}{\Gamma^{-1}(t)\Gamma'(\Gamma^{-1}(t))}\right).$$

Without loss of generality, we may assume that $\Gamma(t)>0$ for all $t>0$. Since the function $\frac{1}{\Gamma^{-1}(t)\Gamma'(\Gamma^{-1}(t))}$ is nonnegative, decreasing and its integral over $[\Gamma(\rho_\mu^k),\,\Gamma(\rho_\mu^{k+1})]$ is equal to $(\log 2)\log(\left|I_{\mu}\right|^{-1})$, by Lemma 2.8 we have

$$\mathcal{M}_{\Gamma,\mu,y'}f(x) \leq C\log\left(\left|I_{\mu}\right|^{-1}\right) M_{y'}f(x). \tag{2.10}$$

By (2.9)–(2.10) and Minkowski's inequality for integrals we get

$$\left\|\sigma^*_{\mu,\Gamma}f\right\|_{L^p(\omega)} \leq C\log\left(\left|I_{\mu}\right|^{-1}\right)\left(\int_{\mathbf{S}^{n-1}}\left|\tilde{b}_{\mu}(y')\right|\left\|M_{y'}(|f|^{\gamma'})\right\|_{L^{p/\gamma'}(\omega)}d\sigma(y')\right)^{1/\gamma'}. \tag{2.11}$$

By (8) in [12] and since $\omega \in \tilde{A}_{p/\gamma'}(\mathbf{R}^+)$ we have

$$\|M_{y'}f\|_{L^{p/\gamma'}(\omega)} \leq C\,\|f\|_{L^{p/\gamma'}(\omega)} \tag{2.12}$$

with C independent of y'. By (2.2) and (2.11)–(2.12) we get (2.8) which finishes the proof of the lemma. □

Lemma 2.10. *Let* $\mu \in \mathbf{N}$ and $h \in \Delta_{\gamma}(\mathbf{R}^+)$ *for some* $\gamma > 1$. *Let* $\tilde{b}_{\mu}$ *be a function on* $\mathbf{S}^{n-1}$ *satisfying (2.1)–(2.2) and let* Γ *be in* $C^2([0,\infty))$, *convex, and an increasing function with* $\Gamma(0) = 0$. *Then*

$$\left\|\sigma^*_{\mu,\Gamma}(f)\right\|_{L^p(\mathbf{R}^n)} \leq C_p\beta_{\mu}\,\|f\|_{L^p(\mathbf{R}^n)} \tag{2.13}$$

for $1 < p < \infty$, *where* C_p *is independent of* μ *and* f.

Before presenting a proof of Lemma 2.10, it is worth pointing out that Lemma 2.10 gives the boundedness of $\sigma^*_{\mu,\Gamma}$ on $L^p(\mathbf{R}^n)$ for the full range $1 < p \leq \infty$, which is much better than the range $\gamma' < p \leq \infty$ (when $\gamma \to 1$) if we apply Lemma 2.9 for $\omega = 1$.

Our proof of Lemma 2.10 will rely on the following result (see also Theorem B in [11]):

Lemma 2.11. *Let* $\{\lambda_{\mu,k} : k \in \mathbf{Z},\ \mu \in \mathbf{N}\}$ *be a sequence of Borel measures on* $\mathbf{R}^n$. *Let* Γ *be in* $C^2([0,\infty))$, *convex, and increasing function with* $\Gamma(0) = 0$. *Suppose that for all* $k \in \mathbf{Z}$, $\xi \in \mathbf{R}^n$, *for some* $\alpha, C > 0$ *and* $p_0 > 2$ *we have*

$$\|\lambda_{\mu,k}\| \leq C\beta_{\mu}; \tag{2.14}$$

$$\left|\hat{\lambda}_{\mu,k}(\xi)\right| \leq C\beta_{\mu}\left|\Gamma(\rho^k_{\mu})\xi\right|^{-\frac{\alpha}{\beta_{\mu}}}; \tag{2.15}$$

$$\left|\hat{\lambda}_{\mu,k}(\xi)\right| \leq C\beta_{\mu}\left|\Gamma(\rho^{k+1}_{\mu})\xi\right|^{\frac{\alpha}{\beta_{\mu}}}; \tag{2.16}$$

$$\left\|(\sum_{k\in\mathbf{Z}}|\lambda_{\mu,k}*g_k|^2)^{\frac{1}{2}}\right\|_{p_0} \leq C\beta_{\mu}\left\|(\sum_{k\in\mathbf{Z}}|g_k|^2)^{\frac{1}{2}}\right\|_{p_0} \tag{2.17}$$

for arbitrary functions $\{g_k\}_{k\in\mathbf{Z}}$ *on* $\mathbf{R}^n$. *Then for any* $p \in (p_0', p_0)$, *there exists a positive constant* C_p *such that*

$$\left\| \sum_{k\in\mathbf{Z}} \lambda_{\mu,k} * f \right\|_p \leq C_p \beta_\mu \|f\|_p ; \tag{2.18}$$

$$\left\| (\sum_{k\in\mathbf{Z}} |\lambda_{\mu,k} * f|^2)^{\frac{1}{2}} \right\|_p \leq C_p \beta_\mu \|f\|_p \tag{2.19}$$

hold for all f *in* $L^p(\mathbf{R}^n)$. *The constant* C_p *is independent of* β_μ.

A proof of Lemma 2.11 can be obtained by the same proof (with only minor modifications) as that of Lemma 3.2 in [4]. We omit the details.

Proof of Lemma 2.10. Choose and fix a $\varphi \in \mathcal{S}(\mathbf{R}^n)$ such that $\hat{\varphi}(\xi) = 1$ for $|\xi| \leq 1$ and $\hat{\varphi}(\xi) = 0$ for $|\xi| \geq 2$. Let $(\varphi_k)\hat{}(\xi) = \hat{\varphi}(\Gamma(\rho_\mu^k)\xi)$. Define the sequence of measures $\{\lambda_{\mu,k}\}$ by

$$\hat{\lambda}_{\mu,k}(\xi) = (|\hat{\sigma}_{\mu,\Gamma,k}|)(\xi) - (|\hat{\sigma}_{\mu,\Gamma,k}|)(0)(\varphi_k)\hat{}(\xi). \tag{2.20}$$

By Lemma 2.7 and the choice of φ we find that $\lambda_{\mu,k}$ satisfies (2.4)–(2.6) for some constants C and α.

Now let

$$S_\mu(f) = \left(\sum_{k\in\mathbf{Z}} |\lambda_{\mu,k} * f|^2 \right)^{1/2},$$

$$\lambda_\mu^* f(x) = \sup_{k\in\mathbf{Z}} ||\lambda_{\mu,k}| * f(x)| .$$

Thus we have

$$\lambda_\mu^*(f) \leq S_\mu(f) + C\beta_\mu M_{HL}(f); \tag{2.21}$$

$$\sigma_{\mu,\Gamma}^*(f) \leq S_\mu(f) + 2C\beta_\mu M_{HL}(f). \tag{2.22}$$

By (2.5)–(2.6) and using Plancherel's theorem we get

$$\|S_\mu(f)\|_2 \leq C\beta_\mu \|f\|_2 . \tag{2.23}$$

By the boundedness of M_{HL} on L^p $(1 < p < \infty)$,(2.21) and (2.23) we get

$$\left\|\lambda_\mu^*(f)\right\|_2 \leq C\beta_\mu \|f\|_2 . \tag{2.24}$$

Now, by (2.4), (2.24) and using the lemma and its proof in ([11], p. 544) with $p_0 = 4$ and $q = 2$, we find that

$$\left\|\left(\sum_{k\in\mathbf{Z}} |\lambda_{\mu,k} * g_k|^2\right)^{\frac{1}{2}}\right\|_4 \leq C\beta_\mu \left\|\left(\sum_{k\in\mathbf{Z}} |g_k|^2\right)^{\frac{1}{2}}\right\|_4 \tag{2.25}$$

for arbitrary functions $\{g_k\}_{k\in\mathbf{Z}}$ on $\mathbf{R}^n$. By (2.4)–(2.6), (2.25) and applying Lemma 2.11 we get

$$\|S_\mu(f)\|_p \leq C_p\beta_\mu \|f\|_p \tag{2.26}$$

for all p satisfying $p \in (4/3, 4)$ and $L^p(\mathbf{R}^n)$.

By replacing $p = 2$ with $p = 4/3 + \varepsilon$ $(\varepsilon \to 0^+)$ in (2.23) and repeating the preceding arguments, we get (2.26) for every p satisfying $p \in (8/7, 8)$ and $L^p(\mathbf{R}^n)$. By continuing this process we ultimately get

$$\|S_\mu(f)\|_p \leq C_p\beta_\mu \|f\|_p \tag{2.27}$$

for all $p \in (1,\infty)$ and $L^p(\mathbf{R}^n)$. Therefore, by (2.22) and (2.27), we obtain (2.13) to complete the proof of the lemma. □

Lemma 2.12. *Let $h \in \Delta_\gamma(\mathbf{R}^+)$ for some $\gamma \geq 2$ and let $\tilde{b}_\mu$ be a function on $\mathbf{S}^{n-1}$ satisfying (2.1)–(2.2). Let Γ be in $C^2([0,\infty))$, convex, and an increasing function with $\Gamma(0) = 0$. Then for $\gamma' < p < \infty$ and $\omega \in \tilde{A}_{p/\gamma'}(\mathbf{R}^+)$, there exists a positive constant C_p which is independent of $\tilde{b}_\mu$ such that*

$$\left\|\left(\sum_{k\in\mathbf{Z}} |\sigma_{\mu,\Gamma,k} * g_k|^2\right)^{1/2}\right\|_{L^p(\omega)} \leq C_p\beta_\mu \left\|\left(\sum_{k\in\mathbf{Z}} |g_k|^2\right)^{1/2}\right\|_{L^p(\omega)} \tag{2.28}$$

holds for arbitrary functions $\{g_k\}_{k\in\mathbf{Z}}$ on $\mathbf{R}^n$.

Proof. Let $\gamma' < p < \infty$. By Hölder's inequality and following a similar argument as in the proof of (2.9) we get

$$|\sigma_{\mu,\Gamma,k} * g_k(x)|^{\gamma'} \leq C(\beta_\mu)^{\gamma'-1} \int_{\rho_\mu^k}^{\rho_\mu^{k+1}} \int_{\mathbf{S}^{n-1}} \left|\tilde{b}_\mu(y')\right| |g_k(x-\Gamma(t)y')|^{\gamma'} d\sigma(y') \frac{dt}{t}. \tag{2.29}$$

Let $d = p/\gamma'$. By duality, there is a nonnegative function $f \in L^{d'}(\omega^{1-d'}, \mathbf{R}^n)$ satisfying $\|f\|_{L^{d'}(\omega^{1-d'})} \le 1$ such that

$$\left\| \left(\sum_{k\in\mathbf{Z}} |\sigma_{\mu,\Gamma,k} * g_k|^{\gamma'} \right)^{1/\gamma'} \right\|_{L^p(\omega)}^{\gamma'} = \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} |\sigma_{\mu,\Gamma,k} * g_k|^{\gamma'} f(x)dx. \tag{2.30}$$

Therefore, by (2.29) and a change of variable we get

$$\left\| \left(\sum_{k\in\mathbf{Z}} |\sigma_{\mu,\Gamma,k} * g_k|^{\gamma'} \right)^{1/\gamma'} \right\|_{L^p(\omega)}^{\gamma'} \le C(\beta_\mu)^{\gamma'-1} \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} |g_k(x)|^{\gamma'} M_\mu f(x)dx,$$

where

$$M_\mu f(x) = \sup_{k\in\mathbf{Z}} \int_{\rho_\mu^k \le |y| < \rho_\mu^{k+1}} f(x + \Gamma(|y|)y') \left| \tilde{b}_\mu(y') \right| |y|^{-n} \, dy.$$

By Hölder's inequality, we obtain

$$\left\| \left(\sum_{k\in\mathbf{Z}} |\sigma_{\mu,\Gamma,k} * g_k|^{\gamma'} \right)^{1/\gamma'} \right\|_{L^p(\omega)}^{\gamma'} \le C(\beta_\mu)^{\gamma'-1} \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^{\gamma'} \right)^{1/\gamma'} \right\|_{L^p(\omega)}^{\gamma'} \left\| M_\mu f \right\|_{L^{d'}(\omega^{1-d'})}.$$

It is easy to verify that $\omega \in \tilde{A}_d(\mathbf{R}^+)$ if and only if $\omega^{1-d'} \in \tilde{A}_{d'}(\mathbf{R}^+)$. By the same argument as in the proof of Lemma 2.9, we have

$$\left\| M_\mu f \right\|_{L^{d'}(\omega^{1-d'})} \le C_p \beta_\mu$$

which in turn implies

$$\left\| \left(\sum_{k\in\mathbf{Z}} |\sigma_{\mu,\Gamma,k} * g_k|^{\gamma'} \right)^{1/\gamma'} \right\|_{L^p(\omega)} \le C_p \beta_\mu \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^{\gamma'} \right)^{1/\gamma'} \right\|_{L^p(\omega)}. \tag{2.31}$$

Moreover, again by Lemma 2.9 we have

$$\left\| \sup_{k\in\mathbf{Z}} |\sigma_{\mu,\Gamma,k} * g_k| \right\|_{L^p(\omega)} \le \left\| \sigma^*_{\mu,\Gamma}(\sup_{k\in\mathbf{Z}} |g_k|) \right\|_{L^p(\omega)} \le C_p \beta_\mu \left\| (\sup_{k\in\mathbf{Z}} |g_k|) \right\|_{L^p(\omega)}. \tag{2.32}$$

By using the operator interpolation theorem between (2.31) and (2.32) and since $\gamma' \in [1, 2]$ we get (2.28) which concludes the proof if the lemma. □

We now have everything we need to prove our main theorems.

3. Proofs of Main Results

The proof of our results will be based on a technique developed by Duoandikoetxea and Rubio de Francia in [11]. Now, we shall introduce certain notations and prove some estimates that will be needed in our proofs. Assume that $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1})$ for some $q > 1$ and satisfies (1.1). Thus Ω can be written as $\Omega = \sum_{\mu=1}^{\infty} c_\mu b_\mu$ where $c_\mu \in \mathbf{C}$, b_μ is a q-block supported on a cap I_μ on $\mathbf{S}^{n-1}$ and

$$M_q^{(0,0)}\left(\{c_\mu\}, \{I_\mu\}\right) = \sum_{\mu=1}^{\infty} \left|c_\mu\right| \left(1 + (\log |I_\mu|^{-1})\right) < \infty. \tag{3.1}$$

To each block function $b_\mu(\cdot)$, let $\tilde{b}_\mu(\cdot)$ be a function defined by

$$\tilde{b}_\mu(x) = b_\mu(x) - \int_{\mathbf{S}^{n-1}} b_\mu(u) d\sigma(u). \tag{3.2}$$

Then we can easily see that the following inequalities hold for all μ:

$$\int_{\mathbf{S}^{n-1}} \tilde{b}_\mu(u)\, d\sigma(u) = 0, \tag{3.3}$$

$$\left\|\tilde{b}_\mu\right\|_{L^q} \leq 2\left|I_\mu\right|^{-1/q'}, \tag{3.4}$$

$$\left\|\tilde{b}_\mu\right\|_{L^1} \leq 2. \tag{3.5}$$

Using the assumption that Ω has the mean zero property (1.1), and the definition of $\tilde{b}_\mu$, we deduce that Ω can be written as $\Omega = \sum_{\mu=1}^{\infty} c_\mu \tilde{b}_\mu$, which in turn gives

$$T_{\Gamma,h,\Omega}(f) = \sum_{\mu=1}^{\infty} c_\mu T_{\Gamma,h,\tilde{b}_\mu}(f). \tag{3.6}$$

Since Γ is convex and increasing in $(0, \infty)$, we have $\Gamma(t)/t$ is also increasing for $t > 0$. Therefore, for $\mu \in \mathbf{N}$, the sequence $\left\{\Gamma(\rho_\mu^k) : k \in \mathbf{Z}\right\}$ is a lacunary sequence with $\Gamma(\rho_\mu^{k+1})/\Gamma(\rho_\mu^k) \geq \rho_\mu$. As in [4], let $\{\varphi_{k,\mu,\Gamma}\}_{-\infty}^{\infty}$ be a smooth partition of unity in $(0, \infty)$

adapted to the interval $E_{k,\mu,\Gamma} = [\left(\Gamma(\rho_{\mu}^{k+1})\right)^{-1}, \left(\Gamma(\rho_{\mu}^{k-1})]\right)^{-1}$. To be precise, we require the following:

$$\begin{aligned} \varphi_{k,\mu,\Gamma} &\in C^{\infty},\ 0 \leq \varphi_{k,\mu,\Gamma} \leq 1,\ \sum_{k} \left(\varphi_{k,\mu,\Gamma}(t)\right)^2 = 1, \\ \text{supp } \varphi_{k,\mu,\Gamma} &\subseteq E_{k,\mu,\Gamma},\ \left| \frac{d^s \varphi_{k,\mu,\Gamma}(t)}{dt^s} \right| \leq \frac{C_s}{t^s}, \end{aligned}$$

where C_s is independent of the lacunary sequence $\{\Gamma(\rho_{\mu}^{k}) : k \in \mathbf{Z}\}$. Define the multiplier operators $S_{k,\mu}$ in $\mathbf{R}^n$ by

$$(\widehat{S_{k,\mu} f})(\xi) = \varphi_{k,\mu,\Gamma}(|\xi|)\hat{f}(\xi).$$

Define

$$F_{j,\mu}(f) = \sum_{k \in \mathbf{Z}} S_{k+j,\mu}(\sigma_{\mu,\Gamma,k} * S_{k+j,\mu} f).$$

Then it is easy to see that the following identity

$$T_{\Gamma,h,\tilde{b}_{\mu}}(f) = \sum_{j \in \mathbf{Z}} F_{j,\mu}(f) \tag{3.7}$$

holds for $f \in \mathcal{S}(\mathbf{R}^n)$. By Plancherel's theorem we have

$$\|F_{j,\mu}(f)\|_{L^2}^2 \leq \sum_{k \in \mathbf{Z}} \int_{\Delta_{k+j,\mu}} \left| \hat{f}(\xi) \right|^2 |\hat{\sigma}_{\mu,\Gamma,k}(\xi)|^2 \, d\xi,$$

where

$$\Delta_{k,\mu} = \{\xi \in \mathbf{R}^n : \left(\Gamma(\rho_{\mu}^{k+1})\right)^{-1} \leq |\xi| < \left(\Gamma(\rho_{\mu}^{k-1})\right)^{-1}\}.$$

By a straightforward computations and (2.5)–(2.6) we get

$$\|F_{j,\mu}(f)\|_{L^2} \leq C\beta_{\mu} 2^{-\alpha|j|} \|f\|_{L^2}. \tag{3.8}$$

Proof of Theorem 1.1. Since $\Delta_{\gamma}(\mathbf{R}^+) \subset \Delta_2(\mathbf{R}^+)$ when $\gamma \geq 2$, we may assume that $1 < \gamma \leq 2$. Now we need to compute the L^p estimate $F_{j,\mu}(f)$.

For $1 < p < \infty$, we have

$$\begin{aligned} \|F_{j,\mu}(f)\|_p &= \left\| \sum_{k\in\mathbf{Z}} S_{k+j,\mu}(\sigma_{\mu,\Gamma,k} * S_{k+j,\mu}f) \right\|_p \\ &\le C_p \left\| \left(\sum_{k\in\mathbf{Z}} |\sigma_{\mu,\Gamma,k} * S_{k+j,\mu}f|^2 \right)^{\frac{1}{2}} \right\|_p \\ &\le C_p\beta_\mu \left\| \left(\sum_{k\in\mathbf{Z}} |S_{k+j,\mu}f|^2 \right)^{\frac{1}{2}} \right\|_p \\ &\le C_p\beta_\mu \|f\|_p , \end{aligned} \tag{3.9}$$

where the second inequality follows from (2.4), Lemma 2.11 along with the lemma and its proof in ([11], p. 544), while the first and the last inequalities follow by Littlewood-Paley theory. The constant C_p may depend on the dimension n, the constants C_s, and p, but it is independent of $\{\Gamma(\rho_\mu^k) : k \in \mathbf{Z}\}$, and β_μ (for more details see [26]).

Now by an interpolation between (3.8) and (3.9), we get

$$\|F_{j,\mu}(f)\|_{L^p} \le C\beta_\mu 2^{-\alpha_p|j|} \|f\|_{L^p} \tag{3.10}$$

for $1 < p < \infty$. Therefore, the proof of Theorem 1.1 is completed by (3.1), (3.6)–(3.7) and (3.10). □

Proof of Theorem 1.2. Let us first consider the case $p > \gamma'$. Then

$$\begin{aligned} \|F_{j,\mu}(f)\|_{L^p(\omega)} &\le C_p \left\| \left(\sum_{k\in\mathbf{Z}} |\sigma_{\mu,\Gamma,k} * S_{k+j,\mu}f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\omega)} \\ &\le C_p\beta_\mu \left\| \left(\sum_{k\in\mathbf{Z}} |S_{k+j,\mu}f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\omega)} \\ &\le C_p\beta_\mu \|f\|_{L^p(\omega)} , \end{aligned} \tag{3.11}$$

where the first and the last inequalities follow by the weighted Littlewood-Paley theory since $\omega \in \tilde{A}_{p/\gamma'}(\mathbf{R}_+) \subset A_p(\mathbf{R}_+)$, whereas the second inequality follows by Lemma 2.12.

By Lemma 2.3, for any $\omega \in \tilde{A}^I_{p/\gamma'}(\mathbf{R}_+)$, there is an $\varepsilon > 0$ such that $\omega^{1+\varepsilon} \in \tilde{A}^I_{p/\gamma'}(\mathbf{R}_+)$. Therefore by (3.11) we have

$$\|F_{j,\mu}(f)\|_{L^p(\omega^{1+\varepsilon})} \le C_p\beta_\mu \|f\|_{L^p(\omega^{1+\varepsilon})} \text{ for } p > \gamma'. \tag{3.12}$$

Thus by using Stein-Weiss's interpolation theorem with change of measures [28] between (3.10) and (3.12) we get

$$\|F_{j,\mu}(f)\|_{L^p(\omega)} \le C\beta_\mu 2^{-\tilde{\alpha}_p|j|}\,\|f\|_{L^p(\omega)} \tag{3.13}$$

for some $\tilde{\alpha}_p > 0$ which in turn implies

$$\left\|T_{\Gamma,h,\tilde{b}_\mu}(f)\right\|_{L^p(\omega)} \le \left\|\sum_{j\in\mathbf{Z}} F_{j,\mu}(f)\right\|_{L^p(\omega)} \le C_p\beta_\mu\,\|f\|_{L^p(\omega)} \text{ for } p > \gamma'. \tag{3.14}$$

Thus by (3.1), (3.6) and (3.14) we get that $T_{\Gamma,h,\Omega}$ is bounded on $L^p(\omega)$ for $p > \gamma'$ and $\omega \in \tilde{A}^I_{p/\gamma'}(\mathbf{R}_+)$. Now, let us prove the $L^p(\omega)$ boundedness of $T_{\Gamma,h,\Omega}$ whenever $p = \gamma'$ and $\omega \in \tilde{A}_1(\mathbf{R}_+)$. To this end we notice by Lemma 2.3 (i) that $T_{\Gamma,h,\Omega}$ is bounded on $L^{\upsilon}(\omega)$ for any $\omega \in \tilde{A}_1(\mathbf{R}_+)$ and $\upsilon > \gamma'$. Also, by Theorem 1.1, $T_{\Gamma,h,\Omega}$ is bounded on L^s for $1 < p \le \gamma'$. Therefore, by interpolating with change of measures we get that $T_{\Gamma,h,\Omega}$ is bounded on $L^{\gamma'}(\omega)$ for any $\omega \in \tilde{A}_1(\mathbf{R}_+)$. This concludes the proof of Theorem 1.2. □

Proof of Theorem 1.3. We shall only present the proof of part (ii) of this theorem and the proof for part (i) will be much the same. So assume that $p > \gamma'$ and $\omega \in \tilde{A}^I_{p/\gamma'}(\mathbf{R}_+)$. Since $\Omega = \sum_{\mu=1}^{\infty} c_\mu \tilde{b}_\mu$, we have

$$T^*_{\Gamma,h,\Omega}(f) \le \sum_{\mu=1} |c_\mu|\, T^*_{\Gamma,h,\tilde{b}_\mu}(f)\,. \tag{3.15}$$

Thus, as above it suffices to establish appropriate $L^p(\omega)$ bounds for $T^*_{\Gamma,h,\tilde{b}_\mu}$, $\mu \ge 1$. For any $\varepsilon > 0$ there is an integer k such that $\rho_\mu^j \le \varepsilon < \rho_\mu^{j+1}$. So we have

$$T^*_{\Gamma,h,\tilde{b}_\mu}(f) \le \sigma^*_{\mu,\Gamma}(f) + g_\mu(f), \tag{3.16}$$

where $g_\mu(f) = \sup_{k\in\mathbf{Z}} |X_k(f)|$ and $X_k(f) = \sum_{j=k}^{\infty} \sigma_{\mu,\Gamma,j} * f$. By Lemma 2.9, it suffices to show that

$$\left\|g_\mu(f)\right\|_{L^p(\omega)} \le C_p\beta_\mu\,\|f\|_{L^p(\omega)} \text{ for } p > \gamma' \text{ and } \omega \in \tilde{A}_{p/\gamma'}(\mathbf{R}_+). \tag{3.17}$$

We follow a similar argument employed in the unweighted case in the proof of Lemma 6.3 in [14]. Let $\varphi \in \mathcal{S}(\mathbf{R}^n)$ be such that $\varphi(\xi) = 1$ for $|\xi| < 1$ and $\varphi(\xi) = 0$ for $|\xi| > 2$.

Define ϕ, ϕ_k by $(\phi)\hat{} = \varphi$ and $\phi_k(\xi) = \frac{1}{\left(\Gamma(\rho_\mu^k)\right)^n}\phi_k(\frac{\xi}{\Gamma(\rho_\mu^k)})$. Define the sequence of measures $\{\nu_{k,j} : k \in \mathbf{Z},\, j \geq 0\}$ by

$$\nu_{k,j} = (\delta_{\mathbf{R}^n} - \phi_k) * \sigma_{\mu,\Gamma,k+j} - \phi_k * \sigma_{\mu,\Gamma,k-j-1}. \tag{3.18}$$

Then

$$X_k f = X_k^{(1)} f - X_k^{(2)} f, \tag{3.19}$$

where

$$X_k^{(1)} f = \sum_{j=0}^{\infty} \nu_{k,j} * f \text{ and } X_k^{(2)} f = \phi_k * T_{\Gamma,h,\tilde{b}_\mu}(f).$$

By (3.14) and since $\omega \in \tilde{A}_{p/\gamma'}(\mathbf{R}_+) \subset A_p(\mathbf{R}_+)$ we get immediately

$$\left\| \sup_{k\in\mathbf{Z}} \left| X_k^{(2)} f \right| \right\|_{L^p(\omega)} \leq C \left\| M_{HL}(T_{\Gamma,h,\tilde{b}_\mu}) \right\|_{L^p(\omega)} \leq C_p \beta_\mu \|f\|_{L^p(\omega)} \tag{3.20}$$

for $p > \gamma'$ and $\omega \in \tilde{A}_{p/\gamma'}(\mathbf{R}_+)$. It remains now to show that

$$\left\| \sup_{k\in\mathbf{Z}} \left| X_k^{(1)} f \right| \right\|_{L^p(\omega)} \leq C_p \beta_\mu \|f\|_{L^p(\omega)} \tag{3.21}$$

for $p > \gamma'$ and $\omega \in \tilde{A}_{p/\gamma'}(\mathbf{R}_+)$. By definition of $X_k^{(1)} f$ we get

$$\sup_{k\in\mathbf{Z}} \left| X_k^{(1)} f(x) \right| \leq \sum_{j=0}^{\infty} \sup_{k\in\mathbf{Z}} \left| \nu_{k,j} * f(x) \right|. \tag{3.22}$$

By definition of $\nu_{k,j}$ and Lemma 2.9 we have

$$\left\| \sup_{k\in\mathbf{Z}} \left| \nu_{k,j} * f \right| \right\|_{L^p(\omega)} \leq C_p \beta_\mu \|f\|_{L^p(\omega)} \tag{3.23}$$

for $p > \gamma'$ and $\omega \in \tilde{A}_{p/\gamma'}(\mathbf{R}_+)$. For $j \geq 0$, we let

$$R_j(f) = (\sum_{k\in\mathbf{Z}} |\nu_{k,j} * f|^2)^{\frac{1}{2}}.$$

Then by Plancherel's theorem and Lemma 2.7 we have

$$
\begin{aligned}
\|R_j(f)\|_2^2 &= \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} |\hat{\nu}_{k,j}(\xi)|^2 \, |f(\xi)|^2 \, d\xi \\
&\leq \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \left\{ \left|1-(\hat{\phi_k})(\xi)\right|^2 |\hat{\sigma}_{\mu,\Gamma,k+j}(\xi)|^2 + \left|(\hat{\phi_k})(\xi)\right|^2 |\hat{\sigma}_{\mu,\Gamma,k-j-1}(\xi)|^2 \right\} \left|\hat{f}(\xi)\right|^2 d\xi \\
&\leq \quad C\beta_\mu^2 2^{-2j\alpha} \|f\|_2^2 . \qquad (3.24)
\end{aligned}
$$

By (2.23)–(2.24) and interpolation with $\omega = 1$ we get

$$
\left\| \sup_{k\in\mathbf{Z}} |\nu_{k,j} * f| \right\|_p \leq C\beta_\mu 2^{-j\alpha(p)} \|f\|_p \qquad (3.25)
$$

for $\gamma' < p < \infty$, $f \in L^p(\mathbf{R}^n)$, and for some $\alpha(p) > 0$. By interpolating between (3.23) and (3.25) we find δ such that

$$
\left\| \sup_{k\in\mathbf{Z}} |\nu_{k,j} * f| \right\|_{L^p(\omega)} \leq C\beta_\mu 2^{-j\delta} \|f\|_{L^p(\omega)} . \qquad (3.26)
$$

Hence (3.20)–(3.21) we obtain (3.17). The theorem is proved. □

4. Power Weights $|x|^\alpha$

One of the important special classes of radial weights is the power weights $|x|^\alpha$, $\alpha \in \mathbf{R}$. It is know that $|x|^\alpha \in A_p(\mathbf{R}^n)$ if and only if $-n < \alpha < n(p-1)$. Let us recall the following result:

Theorem E. Let $1 < q \leq \infty$ and $1 < p < \infty$. Let T_Ω and T_Ω^* be the operators defined as in Section 1 with $\Omega \in L^q(\mathbf{S}^{n-1})$ satisfying (1.1). Then T_Ω and T_Ω^* are bounded on $L^p(|x|^\alpha)$ if

$$
\max(-n, -1-(n-1)p/q') < \alpha < \min(n(p-1), p-1+(n-1)p/q'). \qquad (4.1)
$$

Moreover, the range (4.1) is optimal.

This result was proved for T_Ω by Muckenhoupt and Wheeden in [24] and for both operators T_Ω and T_Ω^* using a different method by Duoandikoetxea [12]. We notice that in the limit case $q = 1$, the range in (4.1) becomes $\alpha \in (-1, p-1)$. It is well-known that the theorem fails for some $\Omega \in L^1(\mathbf{S}^{n-1})$, even in the unweighted case $\alpha = 0$. However Theorem E remains true if $\alpha \in (-1, p-1)$ and $\Omega \in L\log^+ L$ as pointed out in ([12], p. 880). In the ensuing development of this result, an improvement was obtained by Fan-Pan-Yang in [15] who proved that T_Ω is bounded on $L^p(|x|^\alpha)$ if $\alpha \in (-1, p-1)$ and $\Omega \in H^1(\mathbf{S}^{n-1})$. Our result regarding this class of weights is the following:

Theorem 4.1. *Let $h \in \Delta_\gamma(\mathbf{R}^+)$ with $\gamma \geq 2$. Let Γ be in $C^2([0,\infty))$, convex, and an increasing function with $\Gamma(0) = 0$. If $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1})$ for some $q > 1$, and $p > \gamma'$, then $T_{\Gamma,h,\Omega}$ and $T^*_{\Gamma,h,\Omega}$ are bounded on $L^p(|x|^\alpha)$ if $\alpha \in (-1, p/\gamma' - 1)$.*

A proof of this theorem can be obtained by Theorems 1.2 and 1.3 and noticing that $|x|^\alpha \in \tilde{A}_p^I(\mathbf{R}_+)$ for $\alpha \in (-1, p-1)$.

References

[1] Al-Hasan, A.: A note on a maximal singular integral, Func. Diff. Equ. 5 (1998), 309–314.

[2] Al-Hasan, A., Fan, D.: A singular integral operator related to block spaces, Hokkaido Math. J. 28 (1999), 285–299.

[3] Al-Qassem, H., Al-Salman, A., Pan, Y.: Singular integrals associated to homogeneous mappings with rough kernels. To appear in Hokkaido Math. J.

[4] Al-Qassem, H., Pan, Y.: L^p estimates for singular integrals with kernels belonging to certain block spaces, Revista Matemática Iberoamericana (18) 3 (2002), 701–730.

[5] Alvarez, J., Bagby, R., Kurtz, D., Pérez, C.: Weighted estimates for commutators of linear operators, Studia Math. 104 (1993), 195–209.

[6] Calderón, A., Zygmund, A.: On singular integrals, Amer. J. Math. 78 (1956), 289–309.

[7] Chen, L.: On a singular integral, Studia Math. 85 (1987), 61–72.

[8] Christ, M., Nagel, A., Stein, E. M., Wainger, S.: Singular and Maximal Radon Transforms, Annals of Math., Vol. 150 (1999), 489–577.

[9] Coifman, R., Weiss, G.: Extension of Hardy spaces and their use in analysis, Bull. Amer. Math. Soc. 83 (1977), 569–645.

[10] Connett, W. C.: Singular integrals near L^1, Proc. Sympos. Pure. Math. (S. Wainger and G. Weiss, eds.), Vol. 35, Amer. Math. Soc., Providence. RI, (1979), 163–165.

[11] Duoandikoetxea, J., Rubio de Francia, J. L.: Maximal functions and singular integral operators via Fourier transform estimates, Invent. Math. 84 (1986), 541–561.

[12] Duoandikoetxea, J.: Weighted norm inequalities for homogeneous singular integrals, Trans. Amer. Math. Soc. **336** (1993), 869–880.

[13] Fan, D., Pan, Y.: Singular integral with rough kernel, Proc. Amer. J. Math. Soc. 125 (1997), 3695–3703.

[14] Fan, D., Pan, Y.: Singular integral operators with rough kernels supported by subvarieties, Amer J. Math. 119 (1997), 799-839.

[15] Fan, D., Pan, Y., Yang, D.: A weighted norm inequality for rough singular integrals, Tohoku Math. J. 51 (1999), 141–161.

[16] Fefferman, R.: A note on singular integrals, Proc. Amer. Math. Soc. 74 (1979), 266–270.

[17] Garcia, J. , Rubio de Francia, J. L.: Weighted norm inequalities and related topics, North-Holland, (1985).

[18] Hofmann, S.: Weighted inequalities for commutators of rough singular integrals, Indiana Univ. Math. J. 39 (1990), 1275–1304.

[19] Kurtz, D.: Littlewood-Paley and multiplier theorems on weighted L^p spaces, Trans. Amer. Math. Soc. 259 (1980), 235–254.

[20] Kurtz, D., Wheeden, R.: A note on a singular integral with weights, Proc. Amer. Math. Soc. 81 (1993), 391–397.

[21] Keitoku, M., Sato, E.: Block spaces on the unit sphere in $\mathbf{R}^n$, Proc. Amer. Math. Soc. 119 (1993), 453–455.

[22] Lu, S., Taibleson, M., Weiss, G.: "Spaces Generated by Blocks", Beijing Normal University Press, 1989, Beijing.

[23] Lu, S., Wu, H.: Oscillatory singular integrals and commutators with rough kernels, Ann. Sci. Math. Québec 27 (2003), no 1, 47-66.

[24] Muckenhoupt, B., Wheeden, R.: Weighted norm inequalities for singular and fractional integrals, Trans. Amer. Math. Soc. 161 (1971), 249–258.

[25] Namazi, J.: A singular integral, Proc. Amer. Math. Soc. 96 (1986), 421–424.

[26] Stein, E. M.: "Singular integrals and Differentiability Properties of Functions" Princeton University Press, Princeton, NJ, 1970.

[27] Stein, E. M.: Harmonic analysis real-variable methods, orthogonality and oscillatory integrals, Princeton University Press, Princeton, NJ, 1993.

[28] Stein, E. M., Weiss, G.: Interpolation of operators with change of measures, Trans. Amer. Math. Soc. **87** (1958), 159-172.

[29] Watson, D.: Weighted estimates for singular integrals via Fourier transform estmates, Duke Math. J. 60 (1990), 389–399.

Hussain AL-QASSEM
Department of Mathematics,
Yarmouk University,
Irbid-JORDAN
e-mail: husseink@yu.edu.jo

Received 04.11.2003