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SU(2) Representations of The Groups of Integer Tangles

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Abstract

In this work we classify the irreducible $SU(2)$ representations of $\Pi_1\left(S^3\backslash k_n\right)$ where k_n is an integer n tangle and as a result we have proved the following theorem: Let n be an odd integer then $\mathcal{R}^*\left(\Pi_1\left(S^3\backslash k_n\right)\right)/SO\left(3\right)$ is the disjoint union of n open arcs where $\mathcal{R}^*\left(\Pi_1\left(S^3\backslash k_n\right)\right)$ is the space of irreducible representations.

Key words and phrases: Representation space,knot group, quaternions

1. Introduction

As it is known a knot in S^3 is an embedding of S^1 into S^3 and the fundamental group of its complement is one of the most important invariants of the knot. Especially after 1970, following the works of Riley [5], Casson [1], Burde [3], the representation of $\Pi_1\left(S^3\backslash k_n\right)$ has gained ever increasing importance, but the $SU(2)$ representations of the knot groups still defy classification. In this context we will characterize $SU(2)$ representations of a special class of knots and the result is given in the main theorem of the paper. To begin, we give the main threads of the representation theory:

Let G be a group. We mean by an $SU(2)$ representation of G is a homomorphism from G into $SU(2)$. Two important isomorphic Lie groups will be our main devices. One is $SU(2)$ which is

$$SU(2) = \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \in M_2\left(\mathbb{C}\right) \quad | \quad z\overline{z} + w\overline{w} = 1 \right\}.$$

The other is the group of unit quaternions defined as

$$\mathbb{H} = \left\{ z + wj \quad | \quad z, w \in \mathbb{C}\ , |z|^2 + |w|^2 = 1,\ wj = j\overline{w},\ j^2 = -1 \right\}.$$

The isomorphism between these groups is given by the obvious map:

$$\begin{array}{ccc} \mathbb{H} & \longrightarrow & SU(2) \\ z+wj & \longrightarrow & \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix}. \end{array}$$

If one defines $i \cdot j = k$ then it is possible to pass from the complex presentation to real one, i.e. a quaternion can also be defined as:

$$\mathbb{H} = \left\{ q_0 + q_1 i + q_2 j + q_3 k \;\middle|\; q_0,\ q_1,\ q_2,\ q_3 \in \mathbb{R},\ q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1 \right\}$$

which implies that S^3, the unit sphere of $\mathbb{R}^4$, can be equipped with quaternions. From now on we will be taking S^3 with its quaternionic structure rather than $SU(2)$ since the geometric structure of S^3 fits in an excellent way to this algebraic structure of quaternions.

One can obtain the polar form of a unit quaternion as follows:

$$\begin{aligned} Q &= q_0 + q_1 i + q_2 j + q_3 k \\ &= q_0 + \sqrt{q_1^2 + q_2^2 + q_3^2} \left[\frac{1}{\sqrt{q_1^2+q_2^2+q_3^2}} (q_1 i + q_2 j + q_3 k) \right] \end{aligned}$$

so $\exists \alpha \in [0, \pi]$ such that $\cos\alpha = q_0$ and $\sin\alpha = \sqrt{q_1^2 + q_2^2 + q_3^2}$. If we identify the pure imaginary part of a quaternion with an element of S^2 as

$$q = \frac{1}{\sqrt{q_1^2 + q_2^2 + q_3^2}} (q_1, q_2, q_3) \longleftrightarrow \frac{1}{\sqrt{q_1^2 + q_2^2 + q_3^2}} (q_1 i + q_2 j + q_3 k),$$

then Q can be written as $Q = \cos\alpha + q \sin\alpha$ where $\alpha \in [0, \pi]$ and $q \in S^2$. In this expression we call $\alpha \in [0, \pi]$ as the argument of Q and $q \in S^2$ as the pure imaginary pure unit part of Q. It is quite clear that this polar expression is unique for each quaternions but ∓ 1. Again as in the complex numbers we denote $Q = \cos\alpha + q\sin\alpha = e^{\alpha q}$. This construction gives a geometric decomposition of S^3 into 2-spheres parametrized by the argument since the space of quaternions of a given argument is homeomorphic to S^2. That is why we employ the notation S^2_α for the quaternions whose arguments is α.What is nice is that these 2-spheres are precisely conjugacy classes of quaternions: i.e. two quaternion Q_1, Q_2 are conjugate if and only if their real parts (so their arguments) coincide. That is for two unit quaternions Q_1 and Q_2, $\exists Q \in S^3$ such that $Q_1 = Q^{-1} Q_2 Q$ if and only if Q_1 and Q_2 has the same real part. Moreover, the geometry of the conjugation can be exploited by using the Riemannian structure of S^3. Let $Q_1 = e^{\alpha q_1}$ and $Q_2 = e^{\beta q_2}$ then the quaternion $Q_1^{-1} Q_2 Q_1 = e^{-\alpha q_1} e^{\beta q_2} e^{\alpha q_1}$ has the argument β and its pure imaginary unit part is obtained as the image of q_2 by 2α right hand rotation about the axis $[1, q_1]$ (the geodesic connecting 1 and q_1).

Now let $\mathcal{R}(G)$ denote the set of S^3 (or $SU(2)$) representations of G i.e.

$$\mathcal{R}(G) = \left\{ \phi \;\middle|\; \phi : G \overset{\text{homomorphism}}{\longrightarrow} S^3 \right\}$$

S^3 has various subgroups. An important class of subgroups can be defined as for $q \in S^2_{\frac{\pi}{2}}$,

$$S^1_q = \{\cos\alpha + q\sin\alpha \mid \alpha \in [0, 2\pi]\},$$

which are called the Cartan subgroups of S^3 and clearly are abelian and isomorphic to S^1. There are S^3 representations of G such that $\operatorname{Im}(G) \leq S^1_q$ for a suitable $q \in S^2_{\frac{\pi}{2}}$; we call these representations as the reducible representations and denote the set by $\mathcal{S}(G)$ i.e.

$$\mathcal{S}(G) = \left\{ \phi \in \mathcal{R}(G) \mid \operatorname{Im}(\phi) \leq S^1_q \leq S^3 \textit{ for some } q \in S^2_{\frac{\pi}{2}} \right\}.$$

Then we can define our prime object as

$$\mathcal{R}^*(G) = \mathcal{R}(G) - \mathcal{S}(G),$$

the set of irreducible representations. Behind the set structure, $\mathcal{R}(G)$ can also be turned into a topological space provided that G is a topological group. Here we equip G with the discrete topology then $\mathcal{R}(G)$ can be made a topological space by the compact-open topology whose sub-base are the following subset of $\mathcal{R}(G)$:

$$H_{K,U} = \{\phi \in \mathcal{R}(G) \mid \phi(K) \subseteq U\},$$

where K is a compact subset of G (in this case, a finite subset of G since G is a discrete group) and U is an open subset of S^3. So we can take $\mathcal{R}(G)$ as a topological space, $\mathcal{R}^*(G)$ also a topological space with subspace topology. There is a natural action of $SO(3)$ on $\mathcal{R}(G)$. Remember that $SO(3)$, the group of special orthogonal transformations of $\mathbb{R}^3$, is homomorphic to the projective space i.e.

$$SO(3) = S^3/\{\mp 1\} = \{[Q] \mid [Q] = \{Q, -Q\}\}$$

then the action of $SO(3)$ on $\mathcal{R}(G)$ is

$$\begin{array}{llll} SO(3) & \times \quad \mathcal{R}(G) & \longrightarrow & \mathcal{R}(G) \\ [Q] & , \quad \phi(g) & \longrightarrow & Q\phi(g)Q^{-1}, \end{array}$$

where $g \in G$. Obviously this action is free and restricts on $\mathcal{R}^*(G)$, which means that $\widehat{\mathcal{R}}(G) = \mathcal{R}^*(G)/SO(3)$ is a manifold, a property we investigate throughout this paper.

1.1. The Space of SU(2) Representations of $\Pi_1\left(S^3 \backslash k_n\right)$

Let n be an odd integer and k_n be an integer tangle whose plane projection is depicted in Figure 1. Then the Wirtinger presentation of the fundamental group of its complement in S^3, G, can be read from the figure as follows:

$$G = \langle X_1, X_2, ..., X_n \mid R_1, R_2, ..., R_n \rangle,$$

where

$$\begin{array}{c} R_1 = X_n X_{n-1} X_n^{-1} X_1^{-1} \\ R_2 = X_1 X_n X_1^{-1} X_2^{-1} \\ R_i = X_{i-1} X_{i-2} X_{i-1}^{-1} X_i^{-1} \text{ for } i > 2 \end{array} \quad (1)$$

It is quite clear from the relations that any two generators of G are conjugate.

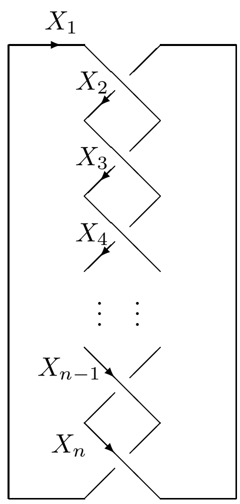

Figure 1. Plane projection of integer n-tangle

Let us assume that we have a homomorphism

$$\phi : G \longrightarrow S^3.$$

Since ϕ takes conjugate elements to conjugate elements then the images of the generators are conjugate therefore for a fixed $\alpha \in (0, \pi)$ we can define

$$\phi(X_i) = Y_i = e^{\alpha y_i} = \cos\alpha + y_i \sin\alpha,$$

i.e. images of the generators are on S^2_α. Now we can study a configuration of the generators on S^2_α such that the images of at least two generators are distinct. Let us suppose that there is a configuration of $Y_1, Y_2, ..., Y_n$ such that the conjugation relations in S^3 are satisfied. If one looks at the imaginary parts of $Y_1, Y_2, ..., Y_n,$ since they are conjugation relations, they must form a regular spherical $n-$gon as given in Figure 2.

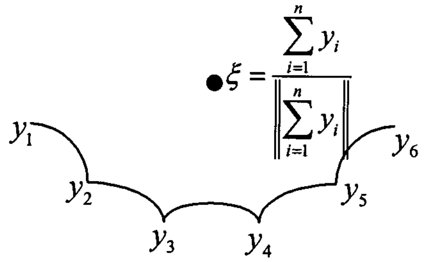

Figure 2. Configuration of the pure imaginary unit parts of the generators on $S^2_{\frac{\pi}{2}}$.

Since the conjugation is rotation by 2α then $d(y_i, y_{i+1}) = d(y_{i+1}, y_{i+2})$ for all $i \in \{1, ..., n-2\}$. Let $\xi = \frac{y_1+y_2+...+y_n}{\|y_1+y_2+...+y_n\|}$. Then for all $i, j \in \{1, ..., n\}$,

$$d(\xi, y_i) = d(\xi, y_j),$$

hence the triangles with vertices ξ, y_i, y_{i+1} are isosceles and they all are congruent to each other. We go on with this $S^2_{\frac{\pi}{2}}$ picture given in Figure 3 by considering the polar line of ξ. The lines connecting ξ and y_i cut the polar circle of ξ at two points and we call the intersection point closer to y_i as $\underline{y_i}$. Obviously with a simple calculation we can see that

$$\underline{y_i} = \frac{1}{\sqrt{1 - \langle y_i, \xi \rangle^2}} \left(y_i - \langle y_i, \xi \rangle \xi \right).$$

Because of the relations $d(\xi, y_i) = d(\xi, y_{i+1})$, we have $d\left(\underline{y_i}, \underline{y_{i+1}}\right) = d\left(\underline{y_{i+1}}, \underline{y_{i+2}}\right)$ for all i's.

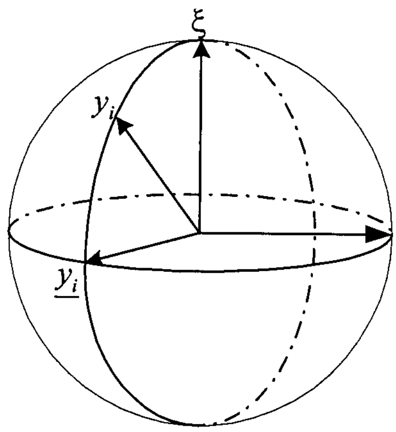

Figure 3. Lifting from $S^2_{\frac{\pi}{2}}$ configuration to S^2_α configuration.

Now a brief digression. Let us consider a particular representation called circle representation or cyclic representation. We prefer to call it by circle representation in this study. By a circle representation we mean a representation which takes all generators of the knot group G into one of the line of $S^2_{\frac{\pi}{2}}$. Before going further we would like to give an example of circle representation.

Let us consider the trefoil whose plane projection is depicted in Figure 4.

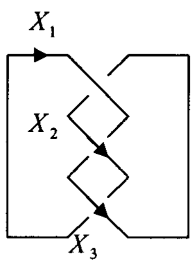

Figure 4. Plane projection of trefoil.

The group of the trefoil is

$$G = \left\langle X_1, X_2, X_3 \ \middle|\ X_1 = X_3X_2X_3^{-1}, X_2 = X_1X_3X_1^{-1},\ X_3 = X_2X_1X_2^{-1} \right\rangle$$

It is trivial that the placement, depicted in Figure 5, on a line of $S^2_{\frac{\pi}{2}}$ is a circle representation of the trefoil. For instance, we could take

$$\begin{array}{lll} \underline{y_1} & = & i \\ \underline{y_2} & = & i\cos\frac{2\pi}{3} + j\sin\frac{2\pi}{3} \\ \underline{y_3} & = & i\cos\frac{4\pi}{3} + j\sin\frac{4\pi}{3} \end{array} \tag{2}$$

where $\underline{y_i}$ is the image of X_i for $i \in \{1, 2, 3\}$.

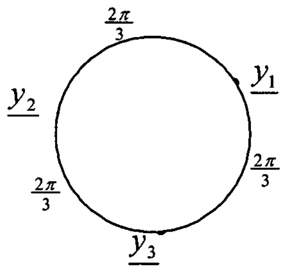

Figure 5. Circle representation of the trefoil.

If we go back to our construction, $\underline{y_1}, \underline{y_2}, ..., \underline{y_n}$ points are on the polar circle of ξ and the relations $d\left(\underline{y_i}, \underline{y_{i+1}}\right) = d\left(\underline{y_{i+1}}, \underline{y_{i+2}}\right)$ are satisfied for all i's. Therefore the points $\left\{\underline{y_1}, \underline{y_2}, ..., \underline{y_n}\right\}$ form a circle representation of G, the group of n-tangle knot. Hence we have proved the following lemma.

Lemma 1 *Let k_n be an integer n tangle where n is an positive odd integer and G denote its fundamental group. If there exist an irreducible $SU(2)$ representation of G then this representation rises to give a circle representation.*

Conversely, let us have a circle representation of G. Then we construct an $SU(2)$ representation from this circle representation. Assume that we have homomorphism

$$\begin{array}{llll} \phi: & G & \longrightarrow & S^3 \\ & x_i & \longrightarrow & \underline{y_i}, \end{array}$$

where the images of the generators are placed on the same line of $S^2_{\frac{\pi}{2}}$. We call ξ one of the poles of this line. Choose $\theta \in \left(0, \frac{\pi}{2}\right)$ and we denote the line connecting ξ and $\underline{y_i}$ by

Γ_i. If we consider the point

$$y_i = \underline{y_i} \cos\theta + \xi \sin\theta$$

then $d\left(\underline{y_i}, y_i\right) = \theta$ and $d\left(y_i, \xi\right) = \frac{\pi}{2} - \theta = \beta$. Since ϕ is a circle representation then $d\left(\underline{y_i}, \underline{y_{i+1}}\right) = d\left(\underline{y_{i+1}}, \underline{y_{i+2}}\right)$ for all i's. Hence the angle between the lines $\left[\xi, \underline{y_i}\right]$ and $\left[\xi, \underline{y_{i+1}}\right]$ equals to the angle between the lines $\left[\xi, \underline{y_{i+1}}\right]$ and $\left[\xi, \underline{y_{i+2}}\right]$, we denote this angle φ. Obviously for all i's

$$d\left(y_i, y_{i+1}\right) = d\left(y_{i+1}, y_{i+2}\right)$$

and the isosceles triangles with vertices ξ, y_i, y_{i+1} and ξ, y_{i+1}, y_{i+2} are congruent. Let α be the vertex angle of this isosceles triangle, then $\alpha > \frac{\pi-\varphi}{2}$ since the sum of interior angles of a spherical triangle exceeds π. By cosine rule we get

$$\cos\alpha = \frac{\langle \xi, y_{i+1}\rangle - \langle y_i, y_{i+1}\rangle \langle y_i \xi\rangle}{\sqrt{1 - \langle y_i, y_{i+1}\rangle^2}\sqrt{1 - \langle y_i, \xi\rangle^2}}, \tag{3}$$

and by considering

$$y_i = \underline{y_i} \cos\theta + \xi \sin\theta$$

and

$$\left\langle \underline{y_i}, \underline{y_{i+1}} \right\rangle = \cos\varphi$$

then Eqn. (3) transforms into

$$\cos\alpha = \frac{\sin\theta\sqrt{1-\cos\varphi}}{\sqrt{2+\cos^2\theta\left(\cos\varphi - 1\right)}}$$

and

$$\sin\alpha = \frac{\sqrt{1+\cos\varphi}}{\sqrt{2+\cos^2\theta\left(\cos\varphi - 1\right)}}.$$

Therefore the points $e^{\alpha y_i}$, $e^{\alpha y_2}$,...,$e^{\alpha y_n}$ give an irreducible $SU\left(2\right)$ representation for $\alpha > \frac{\pi-\varphi}{2}$. If we substitute θ with $-\theta$ then we get representations for $\alpha < \frac{\pi+\varphi}{2}$. Hence a circle representation lifts into an $SU\left(2\right)$ representation for $\frac{\pi-\varphi}{2} < \alpha < \frac{\pi+\varphi}{2}$.

Lets go back the trefoil example. If we consider the placement which is given in Eqn. (2) and an angle θ such that $0 < \theta < \frac{\pi}{2}$, we have

$$\cos\alpha = \frac{\sqrt{3}\sin\theta}{\sqrt{4-3\cos^2\theta}}, \sin\alpha = \frac{1}{\sqrt{4-3\cos^2\theta}}$$

and

$$\begin{array}{rcl} y_1 & = & i\cos\theta + k\sin\theta \\ y_2 & = & \frac{1}{2}\cos\theta\left(-i+\sqrt{3}j\right) + k\sin\theta \\ \underline{y_3} & = & \frac{1}{2}\cos\theta\left(-i-\sqrt{3}j\right) + k\sin\theta. \end{array}$$

Then the points $e^{\alpha y_1}$, $e^{\alpha y_2}$, $e^{\alpha y_3}$ give an irreducible $SU(2)$ representation of the group of trefoil.

Now to classify the irreducible $SU(2)$ representations of G, the group of n-tangle, we need to know the number of distinct irreducible circle representations of the knot group and under which conditions the knot group admits an irreducible representation.

Lemma 2 *Let k be a knot in S^3 with group G, then G admits an irreducible circle representation if and only if the determinant of the knot is not 1 [2].*

Notice that the determinant of a knot is defined as $|\Delta(-1)|$ where $\Delta(-1)$ is the Alexander polynomial of the knot evaluated at -1 [6].

On the other hand, $\sum_2$, the 2-fold branched covering of S^3 branched over k, has first homology $H_1(\sum_2)$ which is a finite abelian group and of order $|\Delta(-1)|$ [6]. Consequently, if $|\Delta(-1)| = 1$ then this double cover will be a homology 3-sphere.

Theorem 3 *Let l be a link in S^3 with group $\dot{G}$. Then G admits an irreducible circle representation if and only if the double branched cover of S^3 branched over l is not a homology 3-sphere. Furthermore, the number of inequivalent circle representatins is $\prod_{a_{i,i\neq\mp1}} \left[\frac{a_{i,i}}{2}\right]$ where $\left[\frac{a_{i,i}}{2}\right]$ denotes the biggest integer less than or equal to $\frac{a_{i,i}}{2}$ and $a_{i,i}$ is the i'th elementary divisor of the Alexander matrix. However, the space of representation is infinite if $\Delta(-1) = 0$ [2].*

Now a simple calculation for the number of distinct irreducible circle representations of the n-tangle knot group. Let $\phi(X_i) = \underline{y_i}$ where X_i is generator of the n-tangle knot group ($i = 1, 2, ..., n$) and ϕ is a circle representation. Then we know $d\left(\underline{y_i}, \underline{y_{i+1}}\right) =$

$d\left(\underline{y_{i+1}}, \underline{y_{i+2}}\right) = \varphi$, therefore $n\varphi \equiv 0 \pmod{2\pi}$. Obviously for each solution of the congruence, one can obtain a circle representation of n-tangle knot group and the solution set is

$$\left\{\varphi_t = \frac{2t\pi}{n} \mid t = 1, 2, ..., n-1\right\}$$

Because circle representations corresponding to t and $n-t$ are congruent, the order of the space of the irreducible circle representations modulo $SO(3)$ is $\frac{n-1}{2}$. Note that the determinant of the n-tangle, $|\Delta(-1)|$, is n, therefore the order of the space of the irreducible circle representations modulo $SO(3)$ is $\frac{|\Delta(-1)|-1}{2}$. Therefore we have proved the following main theorem of the paper:

Theorem 4 *Let k_n be an integer tangle in S^3 where n is an positive odd integer and G denote the Wirtinger presentation of its fundamental group. Then G has an irreducible $SU(2)$ representation such that the arguments of the images of the generators α if and only if $\frac{\pi-\varphi}{2} < \alpha < \frac{\pi+\varphi}{2}$ where φ as above. The number of distinct such representations for a given α is*

$$card\left\{\frac{2k\pi}{n} \;\middle|\; k \in \left\{1, 2, ..., \frac{n-1}{2}\right\}\right\}$$

Hence the space of irreducible representations of G modulo $SO(3)$, $\widehat{\mathcal{R}}(G) = \mathcal{R}^*(G)/SO(3)$, is a disjoint union of $\frac{n-1}{2}$ open arcs.

By considering that a n-tangle knot is a torus knot of type $(2, n)$, we realize that the results, about irreducible representations space of n-tangle, in the above theorem and in the Klassen's theorem which is following are coincide.

Theorem 5 *Let (r, s) be any pair of positive, relatively prime integers and $K_{r,s}$ denote the (r, s)-torus knot in S^3. $\widehat{R}(K_{r,s})$ is thedisjoint union of $\frac{(r-1)(s-1)}{2}$ open arcs [4].*

For instance, let go back trefoil. We know that the order of the space of the irreducible circle representations of the group of trefoil modulo $SO(3)$ is 1. For trefoil, φ is $\frac{2\pi}{3}$, therefore there exist an irreducible representation in S^2_α if and only if $\frac{\pi}{6} < \alpha < \frac{5\pi}{6}$. Hence the space of irreducible representations of the group of trefoil modulo $SO(3)$ is an open arc.

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