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Quotient f-Modules

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Abstract

Let L be an f-module over f-algebra A. Then $L^{\sim}$ is a cf-module over the f-algebra $(A^{\sim})_n^{\sim}$. Quotient f-modules are studied and subsequently a connection between $Z(L^{\sim})$ and $[A^{\sim})_n^{\sim}]_{\hat{e}}$ is investigated.

Key Words: Vector lattices, f-modules, quotient Riesz spaces.

1. Introduction

In this note Riesz spaces are asumed to have separating order duals. Let A be a Riesz algebra, i.e., A is a Riesz space which is simultaneously an associative algebra with the additional property that $a, b \in A_+$ implies that $ab \in A_+$. An f-algebra A is a Riesz algebra which satisfies the extra requirement that $a \wedge b = 0$ implies $ac \wedge b = ca \wedge b = 0$ for all $c \in A_+$. If A is an Archimedean f-algebra, then A is necessarily commutative. It is well-known that for any Archimedean f-algebra A with point separating order dual, $(A^{\sim})_n^{\sim}$ is an Archimedean f-algebra with respect to the Arens multiplication [4]. We denote by $L_b(L)$, the class of all order bounded operators from L into itself. Recall that $\pi \in L_b(L)$ is called an orthomorphism of L if $x \perp y$ in L imply that $\pi(x) \perp y$. Orthomorphisms of L will be denoted by $\mathrm{Orth}(L)$. $\mathrm{Orth}(L)$ is an f-algebra under pointwise order and composition. The principal order ideal generated by the identity operator I in $\mathrm{Orth}(L)$ is called the ideal center of L and is denoted by $Z(L)$. If L is an Dedekind complete Riesz space the $Z(L)$ is the ideal generated by I in $L_b(L)$ and $\mathrm{Orth}(L)$ is the band generated by

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I in $L_b(L)$. We refer to [1] and [7] for terminology and further information about Riesz spaces.

2. Quotient f-Modules

Definition 2.1 *Let A be an f-algebra with unit e and L be a Riesz space. L is said to be a left f-module over A if there exists a map $A \times L \to L : (a, x) \to ax$ satisfying*

(i) L is a left module over A and $ex = x$ for each $x \in L$,

(ii) for each $a \in A_+$ and $x \in L_+$ we have $ax \in L_+$

(iii) if $x \perp y$ in L, then for each $a \in A$ we have $ax \perp y$.

A right f-module over A is defined similarly. We shall only consider the left f-modules from now on and these will simply be referred to as f-modules. A f-module over A is called an cf-module if it has the following property:

(iv) If $(a_\alpha) \subseteq A$ and $a_\alpha \uparrow a$ for some $a \in A$, then $a_\alpha x \uparrow ax$ for each $x \in L_+$

If L is an f-module over A, then for each $a \in A$, the mapping p_a of L into L defined by $p_a(x) = ax, x \in L$, is an orthomorphism of L. We refer to [2] and [6] for further information about f-modules.

If A is an Archimedean f-algebra then any uniformly closed ideal in A is an r-ideal (i.e., a linear subspace of A which is a two-sided ring ideal) [3]. Let A be an f-algebra and N be a uniformly closed ideal in A. It is easy to see that the quotient Riesz space A/N is an Archimedean f-algebra with multiplication given by $(x+N)(y+N) = xy+N$. If A is an f-algebra with unit e then $\dot{e}$ is a unit of A/N.

Definition 2.2 *Let A be an f-algebra with unit e , N be a uniformly closed ideal in A and L be an f-module over A. $L_0(N) = \{x \in L : Nx = \{0\}\}$ is said to be a null ideal of L with respect to N.*

Note that $L_0(N)$ is a band in L since $L_0(N) = \bigcap_{a \in N} N_{p_a}$, where N_{p_a} is null ideal of p_a. Furthermore, $NL = \{0\}$ if and only if $L_0(N) = L$.

Example 2.3 *Let $A = C[0, 1]$ and $N = \{f \in C[0, 1] : f(x) = 0$ for $0 \leq x \leq 1/2\}$. If we take $L = N$ then $L_0(N) = \{0\}$. On the other hand, if we take $L = A$ then $L_0(N) = \{f \in C[0, 1] : f(x) = 0$ for $1/2 \leq x \leq 1\}$.*

Proposition 2.4 *Let A be an f-algebra with unit e and N be a uniformly closed ideal in A. If L is an f-module over A then $L_0(N)$ is an f-module over A/N with multiplication given by*

$$(a+N)x = ax.$$

Proof. Once we have shown that multiplication is well defined, the proof that $L_0(N)$ is an f-module over A/N is routine as $L_0(N)$ is a band in L. $ax \in L_0(N)$ as A commutative. Suppose $a+N = b+N$. Since $a \in a+N = b+N, a = b+n$ for some $n \in N$. Consequently $ax = (b+n)x = bx + nx$ for each x in $L_0(N)$. As $nx = 0, ax = bx$.

In an Archimedean Riesz space, any relatively unifomly convergent sequence is order convergent. Thus, any band in an Archimedean Riesz space is uniformly closed. Let A be a Dedekind complete Riesz space and N be a band in A. Since $A/N \cong N^d, A/N$ is a Dedekind complete.

Let L be an f-module over A. Let $x \in L$ be arbitrary and $0 \leq y \leq x$. L is said to be discrete with respect to $Z(A)$ (topologically full with respect to A) if there exists $0 \leq a \leq e$ such that $ax = y$ (if there exists a net $0 \leq a_\alpha \leq e$ such that $a_\alpha x \to y$ in $\sigma(L, L^\sim)$) [2]. □

Proposition 2.5 *Let L be an f-module over A and N be a uniformly closed ideal in A. Then the following statements hold.*

i) If L is discrete with respect to $Z(A)$ then $L_0(N)$ is discrete with respect to $Z(A/N)$.

ii) If $NL = \{0\}$ and L is a topologically full with respect to A then $L_0(N)$ is topologically full with respect to A/N.

iii) If L is an cf-module over A and N is a projection band in A then $L_0(N)$ is an cf-module over A/N.

Proof. i) Suppose $x, y \in L_0(N)$ be such that $0 \leq y \leq x$. By hypothesis, there exists $0 \leq a \leq e$ such that $ax = y$. Therefore, there exists $0 \leq \dot{a} \leq \dot{e}$ such that $\dot{a}x = ax = y$.

ii) This statement can be proven similarly.

iii) Let P be the band projection of A onto N^d. $\overline{P} : A/N \to N^d; \dot{a} \to \overline{P}(\dot{a}) = P(a)$ is a Riesz isomorphism. Suppose that $(\dot{a}_\alpha) \subseteq A/N$ and $\dot{a}_\alpha \uparrow \dot{a}$ in A/N. As $\overline{P}$ is order continuous, $\overline{P}\dot{a}_\alpha \uparrow \overline{P}\dot{a}$ and so $Pa_\alpha \uparrow Pa$. On the other hand, there exists $b_\alpha \in \dot{a}_\alpha, b \in \dot{a}$ such that $Pa_\alpha = b_\alpha, Pa = b$ for each α. Since L is an cf-module over $A, b_\alpha x \uparrow bx$ for each $x \in L_0(N)_+$. By definition of the multiplication, we obtain that $\dot{a}_\alpha x \uparrow \dot{a}x$ for each $x \in L_0(N)_+$. □

Examples 2.6 *(i) Let $A = \ell_\infty$ and $L = \ell_p, (1 \leq p < \infty)$ and $N = \{x \in \ell_\infty : x = (x_1, 0, 0, ..., 0, ...)\}$. It is easy to see that $L_0(N) = \{(x_n) \in \ell_p : x_1 = 0\}$. $L_0(N)$ is an cf-module over A/N and discrete with respect to $Z(A/N)$, since L is an cf-module over A and discrete with respect to $Z(A)$.*

(ii) Let $A = \ell_\infty$ and $L = \{(x_n) \in \ell_p : x_{2n-1} = 0$, for all $n \in N\}$, $(1 \leq p < \infty)$ and $N = \{(a_n) \in \ell_\infty : a_{2n} = 0$, for all $n \in N\}$. Since $NL = \{0\}$, $L_0(N) = L$. Moreover, L is an cf-module over A/N and discrete with respect to $Z(A/N)$.

3. The Connection Between $\mathbf{Z(L^\sim)}$ and $[(A^\sim)^\sim_n]_{\hat{e}}$

Let L be an f-module over A. It is known that $L^\sim$ is an f-module over $(A^\sim)^\sim_n$. Furthermore, $L^\sim$ is topologically full with respect to $(A^\sim)^\sim_n$ when L is topologically full with respect to A. It can also be seen that $L^\sim$ is discrete with respect to $Z((A^\sim)^\sim_n)$ under the hypothesis of Proposition 3.12 in [6].

Let us consider a particular bilinear map $\phi : L \times L^\sim \to A^\sim, (x, f) \to \psi_{x,f} : \psi_{x,f}(a) = f(a.x)$ for each $a \in A$ of an f-module L over A. For each $x \in L_+$ the map $f \to \phi(x, f)$ and for each $0 \leq f \in L^\sim$ the map $x \to \phi(x, f)$ are positive and we have $|\phi(x, f)| \leq \phi(|x|, |f|)$ for each $(x, f) \in L \times L^\sim$. If L is a topologically full f-module then ϕ is a bilattice homomorphism. Let $x \in L$ be arbitrary and consider $S(x) = \{\psi_{x,f} : f \in L^\sim\}$. Then $S(x)$ is an ideal in $A^\sim$ [6]. We denote by $L \otimes L^\sim$ the union of $S(x)$ for each x in L, i.e., $L \otimes L^\sim = \{\psi_{x,f} : x \in L, f \in L^\sim\}$.

Proposition 3.1 *Let L be an f-module over A. Then $L^\sim$ is an cf-module over $(A^\sim)^\sim_n$.*

Proof. Let $(F_\alpha) \subseteq (A^\sim)^\sim_n$ and $F_\alpha \uparrow F$ in $(A^\sim)^\sim_n$. We shall show that $F_\alpha f \uparrow Ff$ for each $0 \leq f \in L^\sim$. For this, we pick $0 \leq f \in L^\sim$ and $0 \leq x \in L$. Then $0 \leq \psi_{x,f} \in A^\sim$ and $F_\alpha(\psi_{x,f}) \uparrow F(\psi_{x,f})$ holds [1]. Thus, $F_\alpha f(x) \uparrow Ff(x)$ holds for each $0 \leq x \in L$ because of module structure on $L^\sim$. So $F_\alpha f \uparrow Ff$ in $L^\sim$ for each $0 \leq f \in L^\sim$. □

Proposition 3.2 *Let L be an f-module over A. Then $L \otimes L^{\sim}$ is an ideal in $A^{\sim}$.*

Proof. Let $0 \leq |x| \leq y$ in L. For each $f \in L^{\sim}$ $|\psi_{x,f}| \leq \psi_{|x|,|f|} \leq \psi_{y,|f|}$ holds in $A^{\sim}$. Since $S(y)$ is an ideal, $S(x) \subseteq S(y)$. Let $u, v \in L \otimes L^{\sim}$. There exists $x, y \in L$ such that $u \in S(x), v \in S(y)$. Since $S(x) \subseteq S(|x| \vee |y|)$ and $S(y) \subseteq S(|x| \vee |y|), \lambda u + v \in S(|x| \vee |y|)$ for each $\lambda \in R$. Now suppose $0 \leq |u| \leq |v|; u \in A^{\sim}, v \in L \otimes L^{\sim}$. Then $v \in S(x)$ for some $x \in L$. As $S(x)$ is ideal, $u \in S(x)$ and so $u \in L \otimes L^{\sim}$ □

Proposition 3.3 *Let L be an f-module over A. Then $N = \{F \in (A^{\sim})_n^{\sim} : F\,|_{L \otimes L^{\sim}} = 0\}$ is a band in $(A^{\sim})_n^{\sim}$ and $NL^{\sim} = \{0\}$.*

Proof. N is clearly a subspace. Let $0 \leq |F| \leq |G|$ with $G \in N$. Since $|G\,|_{L \otimes L^{\sim}}| = |G|\,|_{L \otimes L^{\sim}}$holds in $(L \otimes L^{\sim})_n^{\sim}$, we see that $F\,|_{L \otimes L^{\sim}} = 0$. So N is an ideal in $(A^{\sim})_n^{\sim}$. We shall show that it is a band. Let $(F_\alpha) \subseteq N$ and $0 \leq F_\alpha \uparrow F$ in $(A^{\sim})_n^{\sim}$. Then $F_\alpha(\mu) \uparrow F(\mu)$ for each $0 \leq \mu \in L \otimes L^{\sim}$. Thus $F(\mu) = 0$ for each $0 \leq \mu \in L \otimes L^{\sim}$, that is $F \in N$. To show that $NL^{\sim} = 0$, we pick $F \in N$ and $f \in L^{\sim}$. For each $x \in L, \psi_{x,f} \in L \otimes L^{\sim}$ and so $Ff(x) = F(\psi_{x,f}) = 0$. That is, $Ff = 0$. □

The mapping $p : A \to Orth(L)$, defined by $p(a) = p_a$, $a \in A$, is an algebraic homomorphism, p is also positive linear mapping of A into Orth(L) satisfying $p(e) = I$. The principal ideal generated by unit in A will be denoted by I_e. We quote the following from [2].

Proposition 3.4. *Let A be Dedekind complete f-algebra with unit e, L be a Dedekind complete Riesz space and assume that L is an cf-module over A. Then $p : I_e \to Z(L)$ is surjective if and only if L is discrete with respect to $Z(A)$.*

Remark. Let L be an f-module over A. Since I_e is a subalgebra of A, we see that L is an f-module over I_e. Furthermore, $Z(L)$ is f-module over I_e with $I_e \times Z(L) \to Z(L)$ $(a, \pi) \to a\pi = p(a)\pi$. Under the hypothesis of Proposition 3.4, $Z(L)$ is discrete with respect to $Z(I_e)$ whenever L is discrete with respect to $Z(A)$. Indeed, $\pi, \tau \in Z(L)$ with $0 \leq \pi \leq \tau$ then Dedekind completeness of $Z(L)$ ensures that there exists $0 \leq \mu \leq I$ with $\mu\tau = \pi$. As p is surjective, there exists $0 \leq a \leq e$ with $\mu = p(a)$ and so $a\tau = \pi$. Since p is an algebra homomorphism, we obtain that p is I_e-linear. Note that p is an f-orthomorphism whenever $Z(L)$ is discrete with respect to $Z(I_e)$ [6]. □

Let L, A and N be as in Proposition 3.3. The principal ideal generated by unit in $(A^{\sim})_n^{\sim}/N$ will be denoted by $[(A^{\sim})_n^{\sim}/N]_{\dot{\hat{e}}}$. $(A^{\sim})_n^{\sim}/N$ is Dedekind complete and $L_0^{\sim}(N) = L^{\sim}$ as we discussed earlier. By Proposition 2.5 (iii) and Proposition 3.1, $L^{\sim}$ is an cf-module over $(A^{\sim})_n^{\sim}/N$. Furthermore, $L^{\sim}$ is discrete with respect to $Z((A^{\sim})_n^{\sim}/N)$ whenever L is topologically full f-module over A [6]. As an application of the quotient f- modules let us obtain special case of 3.4

Corollary 3.5 *Let L, A and N be as in Proposition 3.3 and L be a topologically full f-module over A. Then $p : [(A^{\sim})_n^{\sim}/N]_{\dot{\hat{e}}} \to Z(L^{\sim})$ is a unital algebra and a Riesz isomorphism. In addition, p is an f-orthomorphism.*

Proof. To see that p is surjective, we can take respectively $(A^{\sim})_n^{\sim}/N$ and $L^{\sim}$ instead of A, L in Proposition 3.4. p is clearly a unital algebra and a Riesz homomorphism and a f-orthomorphism. Thus, it is enough to show p is injective. For this, let $\dot{F} \in [(A^{\sim})_n^{\sim}/N]_{\dot{\hat{e}}}$ and $p(\dot{F}) = 0$. Then $\dot{F}f = Ff = 0$ for each $f \in L^{\sim}$. Therefore, $Ff(x) = F(\psi_{x,f}) = 0$ for each $x \in L, f \in L^{\sim}$. That is, $F \mid_{L\otimes L^{\sim}} = 0$ and so $F \in N$. □

Example 3.6 *Let A be an f-algebra with unit e and $L = A$. It is well known that A is topologically full with respect to itself [6]. Since $\psi_{e,f} = f$ for each $f \in A^{\sim}$, $L \otimes L^{\sim} = A^{\sim}$ and so $N = \{0\}$. Thus we obtained that $[(A^{\sim})_n^{\sim}/N]_{\dot{\hat{e}}} = [(A^{\sim})_n^{\sim}]_{\hat{e}} = Z(A^{\sim})$.*

Let $p : [(A^{\sim})_n^{\sim}]_{\hat{e}} \to Z(L^{\sim})$ where $p_F(f) = Ff$ for each $f \in L^{\sim}$. It is clear that Kerp $= N \cap [(A^{\sim})_n^{\sim}]_{\hat{e}}$. Note that $\overline{p} : [(A^{\sim})_n^{\sim}]_{\hat{e}}/\text{Kerp} \to Z(L^{\sim})$ is a Riesz isomorphism under the hypothesis of 3.5. When is p injective? Following result to give necessary and sufficient conditions for this.

Proposition 3.7 *Let L be an f-module over A. p is injective if and only if $L \otimes L^{\sim}$ is an order dense ideal in $A^{\sim}$, i.e., $(L \otimes L^{\sim})^d = \{0\}$.*

Proof. Suppose Kerp $= \{0\}$. Since $A^{\sim}$ is Dedekind complete, there exists a band projection P of $A^{\sim}$ onto $(L \otimes L^{\sim})^d$. By Theorem 5.2 in [4] , $(A^{\sim})_n^{\sim}$ is an f-algebra isomorphic to Orth$(A^{\sim})$. Since $P \in Z(A^{\sim})$, there exists $F \in [(A^{\sim})_n^{\sim}]_{\hat{e}}$ such that $P(g) = Fg$ for each $g \in A^{\sim}$. Thus $P(\psi_{x,f}) = F\psi_{x,f} = 0$ for each $x \in L$ and $f \in L^{\sim}$ from the definition of $L \otimes L^{\sim}$. Therefore, $F\psi_{x,f}(e) = F(\psi_{x,f}e) = F(\psi_{x,f}) = 0$ for each $x \in L$ and $f \in L^{\sim}$. That is, $F \in N$ and so $F \in$ Kerp. By our hypothesis, $F = 0$. So $P = 0$,i.e.,

$(L \otimes L^{\sim})^d = \{0\}$.

Conversely, suppose that $L \otimes L^{\sim}$ is order dense in $A^{\sim}$. Let $F \in$ Kerp. Since p is an Riesz homomorphism, Kerp is an ideal. So, we can assume that F is positive. Let $0 \leq g \in A^{\sim}$. There exists $(g_\alpha) \subseteq L \otimes L^{\sim}$ such that $0 \leq g_\alpha \uparrow g$. Since F is order continuous, $0 = F(g_\alpha) \uparrow F(g)$ and so $F = 0$. □

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