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On Rough Singular Integrals Along Surfaces on Product Domains

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Abstract

In this paper, we study a class of singular integrals along surfaces on product domains with kernels in $L(\log L)^2(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$. We formulate a general theorem concerning the L^p boundedness of these operators. As a consequence of this theorem we establish L^p estimates of several classes of operators whose L^p boundedness in the one parameter setting is known. The condition $L(\log L)^2(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ is known to be an optimal size condition.

1. Introduction

Let $\mathbf{R}^d$ $(d = n, m \geq 2)$ be the d-dimensional Euclidean space and $\mathbf{S}^{d-1}$ be the unit sphere in $\mathbf{R}^d$ equipped with the normalized Lebesgue measure $d\sigma_d$. Let $\Omega \in L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ be such that

$$\Omega(tx, sy) = \Omega(x, y) \text{ for any } t, s > 0; \tag{1.1}$$

$$\int_{\mathbf{S}^{n-1}} \Omega(u, \cdot)\, d\sigma_n(u) = \int_{\mathbf{S}^{m-1}} \Omega(\cdot, v)\, d\sigma_m(v) = 0. \tag{1.2}$$

Consider the classical singular integral operator on product domains T_Ω given by

$$(T_\Omega f)(x, y) = \text{p.v.}\int_{\mathbf{R}^n\times\mathbf{R}^m} f(x-u, y-v)\, |u|^{-n}\, |v|^{-m}\, \Omega(u, v)\, dudv \tag{1.3}$$

The L^p boundedness of the operator T_Ω, under various conditions on Ω, has been investigated by many authors ([1], [8], [12], [13]). For example, R. Fefferman and E. Stein proved in ([13]) that T_Ω is bounded on $L^p(\mathbf{R}^{n+m})$ for $1 < p < \infty$ if Ω satisfies certain Lipschitz conditions. Subsequently in ([8]) Duoandikoetxea established the L^p $(1 < p < \infty)$ boundedness of T_Ω under the weaker condition that $\Omega \in L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ with $q > 1$.

Motivated by Calderón-Zygmund's result in the one parameter setting ([7]), Al-Salman, Al-Qassem, and Pan ([5]) studied the operator T_Ω under the condition that $\Omega \in L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$, i.e.,

$$\int_{\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}} |\Omega(u, v)| \left(\log 2 + |\Omega(u, v)|\right)^2 d\sigma_n(u)\, d\sigma_m(v) < \infty. \tag{1.4}$$

They proved that T_Ω is bounded on L^p $(1 < p < \infty)$ provided that Ω satisfies (1.1)-(1.2) and (1.4). Moreover, they showed that the condition $\Omega \in L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ is nearly optimal in the sense that the exponent 2 in $L(\log L)^2$ can not be replaced by any smaller numbers.

In this paper, we study the L^p boundedness of a class of singular integrals along surfaces with kernels satisfying (1.4). Namely, for suitable mappings ϕ_1, $\phi_2 : \mathbf{R}^+ \to \mathbf{R}$, consider the operator

$$(T_{\Omega,\phi_1,\phi_2} f)(x, y) = \text{p.v.} \int_{\mathbf{R}^n \times \mathbf{R}^m} f(x - \phi_1(|u|)u', y - \phi_2(|v|)v')\, |u|^{-n}\, |v|^{-m}\, \Omega(u, v)\, du dv. \tag{1.5}$$

It is clear that if $\phi_1(u) = u$ and $\phi_2(v) = v$, then $T_{\Omega,\phi_1,\phi_2} = T_\Omega$.

Also, we shall consider the corresponding truncated singular integral operator $(T_{\Omega,\phi_1,\phi_2})^*$ given by

$$(T_{\Omega,\phi_1,\phi_2})^*(f)(x, y) = \sup_{\epsilon>0,\delta>0} \left| \int_{|u|>\epsilon, |v|>\delta} f(x - \phi_1(|u|)u', y - \phi_2(|v|)v')\, |u|^{-n}\, |v|^{-m}\, \Omega(u, v)\, du dv \right|.$$

Our main purpose in this paper is presenting sufficient conditions on the functions $\phi_1$and ϕ_2 such that the corresponding operators T_{Ω,ϕ_1,ϕ_2} and $(T_{\Omega,\phi_1,\phi_2})^*$ are bounded on L^p for all $1 < p < \infty$ provided that $\Omega \in L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$.

Our main result is the following theorem.

Theorem 1.1. *Let $d_1 = n$ and $d_2 = m$. If ϕ_1 and ϕ_2 are real valued functions defined on $\mathbf{R}^+$ that satisfy the property that for each $\kappa \in \mathbf{N}$, there exists a lacunary sequence $\{a_{j,\kappa}^{(l)} : j \in \mathbf{Z}\}$ with $\inf_{j\in\mathbf{Z}} \frac{a_{j+1,\kappa}^{(l)}}{a_{j,\kappa}^{(l)}} \geq 2^{\kappa}$ such that*

$$I_l(j,\kappa,\lambda) = \left|\int_{2^{\kappa j}}^{2^{\kappa(j+1)}} e^{-i\lambda\phi_l(r)} r^{-1} dr\right| \leq \kappa C \left|a_{j,\kappa}^{(l)}\lambda\right|^{-\varepsilon}; \tag{1.6}$$

$$J_l(j,\kappa,\lambda) = \left|\int_{2^{\kappa j}}^{2^{\kappa(j+1)}} \{e^{-i\lambda\phi_l(r)} - 1\} r^{-1} dr\right| \leq \kappa C \left|a_{j+1,\kappa}^{(l)}\lambda\right|^{\varepsilon} \tag{1.7}$$

for all $j \in \mathbf{Z}$, $\lambda \in \mathbf{R}$, $\kappa \in \mathbf{N}$, and $l = 1, 2$.

Then the operators T_{Ω,ϕ_1,ϕ_2} and $(T_{\Omega,\phi_1,\phi_2})^*$ are bounded on $L^p(\mathbf{R}^n \times \mathbf{R}^m)$ for $p \in (1,\infty)$ provided that $\Omega \in L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ *and satisfies (1.1)-(1.2). Here, C is a constant independent of the essential variables and $0 < \epsilon << 1$.*

It can be easily seen that the assumptions (1.6)–(1.7) given in Theorem 1.1 are satisfied by many functions. For showing the strength and generality of Theorem 1.1, we present in Section 3 of this paper several classes of operators whose L^p boundedness follows by applying this theorem.

Throughout this paper the letter C stands for a constant that may vary at each occurrence, but it is independent of the essential variables.

2. Proof of Main Theorem

We start this section by recalling the following result in ([3]):

Lemma 2.1 ([3]). *Suppose that $d \geq 1$ and $\{\mu_k : k \in \mathbf{Z}\}$ is a family of Borel measures with $\mu_k \geq 0$ and $\|\mu_k\| = 1$ such that*
(i)$|\hat{\mu}_k(\xi)| \leq |a_k L(\xi)|^{-\beta}$,
(ii) $|\hat{\mu}_k(\xi) - 1| \leq |a_{k+1}L(\xi)|$, *where $\{a_k\}$ is a lacunary sequence that has the property that $\inf_{k\in\mathbf{Z}} a_{k+1}/a_k > 1$ and L is a linear transformation from $\mathbf{R}^n$ into $\mathbf{R}^d$.*

Then the maximal function $Mf(x) = \sup_{k\in\mathbf{Z}} |\mu_k * f(x)|$ is bounded on L^p $\forall\ 1 < p < \infty$ with bound independent of the linear transformation L.

As a consequence of this lemma, we immediately obtain the following result.

Corollary 2.2. *Let $\{a_j : j \in \mathbf{Z}\}$ be a lacunary sequence with $\inf_{j\in\mathbf{Z}} a_{j+1}/a_j > 1$. Suppose also that $\phi : \mathbf{R} \to \mathbf{R}^+$ is a function that satisfy (1.6)–(1.7) in Theorem 1.1, with $a^{(l)}_{j,\kappa}$ replaced by a_j.* For $z' \in \mathbf{S}^{n-1}$ *let $M_{\phi,z'}$ be the maximal function defined on $\mathbf{R}^n$ by*

$$M_{\phi,z'}(f)(x) = \sup_{j\in\mathbf{Z}} \int_{2^j}^{2^{j+1}} |f(x - \phi(r)z')| \, r^{-1} dr.$$

Then

$$\|M_{\phi,z'}(f)\|_p \le C \|f\|_p$$

for all $1 < p < \infty$ and $f \in L^p(\mathbf{R}^n)$, where C is a constant independent of z'.

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 We shall follow similar ideas as in [1], [5], [6]. We start by decomposing the operator T_{Ω,ϕ_1,ϕ_2}. For $\kappa \in \mathbf{N}$, let

$$\mathbf{E}_\kappa = \{(x', y') \in \mathbf{S}^{n-1} \times \mathbf{S}^{m-1} : 2^\kappa \le |\Omega\ (x', y')| < 2^{\kappa+1}\},$$

and

$$\mathbf{E}_0 = \{(x', y') \in \mathbf{S}^{n-1} \times \mathbf{S}^{m-1} : |\Omega\ (x', y')| \le 2\}.$$

We let $\mathbf{D}$ be the set of all $\kappa \in \mathbf{N}$ that satisfy

$$\iint_{\mathbf{E}_\kappa} |\Omega(u, v)| \, d\sigma_n(u) \, d\sigma_m(v) \ge 2^{-3\kappa}.$$

Define the sequence of functions $\{\Omega_\kappa : \kappa \in \mathbf{D} \cup \{0\}\}$ by

$$\begin{aligned}
\Omega_0(x, y) \quad = \quad & \Omega\chi_{\mathbf{E}_0}(x, y) + \sum_{\kappa\notin\mathbf{D}} \Omega\chi_{\mathbf{E}_\kappa}(x, y) - \int\limits_{\mathbf{S}^{n-1}} \Omega\chi_{\mathbf{E}_0}(u, y) d\sigma(u) \\
& - \int\limits_{\mathbf{S}^{m-1}} \Omega\chi_{\mathbf{E}_0}(x, v) d\sigma(v) - \sum_{\kappa\notin\mathbf{D}} [\int\limits_{\mathbf{S}^{n-1}} \Omega\chi_{\mathbf{E}_\kappa}(u, y) d\sigma(u) \\
& + \int\limits_{\mathbf{S}^{m-1}} \Omega\chi_{\mathbf{E}_\kappa}(x, v) d\sigma(v)] + \iint_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} \Omega\chi_{\mathbf{E}_0}(u, v) d\sigma(u) d\sigma(v) \\
& + \sum_{\kappa\notin\mathbf{D}} \iint_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} \Omega\chi_{\mathbf{E}_\kappa}(u, v) d\sigma(u) d\sigma(v)
\end{aligned}$$

$$\begin{aligned}\Omega_{\kappa}(x,y) &= (\left\|\Omega\chi_{\mathbf{E}_{\kappa}}\right\|_1)^{-1}\{\Omega\chi_{\mathbf{E}_{\kappa}}(x,y)-\int\limits_{\mathbf{S}^{n-1}}\Omega\chi_{\mathbf{E}_{\kappa}}(u,y)d\sigma(u)\\ &\quad -\int\limits_{\mathbf{S}^{m-1}}\Omega\chi_{\mathbf{E}_{\kappa}}(x,v)d\sigma(v)+\iint_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}\Omega\chi_{\mathbf{E}_{\kappa}}(u,v)d\sigma(u)d\sigma(v)\}.\end{aligned}$$

It is straightforward to see the following:

$$\int_{\mathbf{S}^{n-1}}\Omega_{\kappa}(u,\cdot)\,d\sigma_n(u) = \int_{\mathbf{S}^{m-1}}\Omega_{\kappa}(\cdot,v)\,d\sigma_m(v)=0, \tag{2.1}$$

$$\|\Omega_{\kappa}\|_1 \leq C,\ \|\Omega_{\kappa}\|_{\infty}\leq C2^{4(\kappa+1)},\ ||\Omega_0||_2\leq C, \tag{2.2}$$

$$\Omega(x,y) = \sum_{\kappa\in\mathbf{D}\cup\{0\}}\theta_{\kappa}\Omega_{\kappa}(x,y), \tag{2.3}$$

$$\sum_{\kappa\in\mathbf{D}\cup\{0\}}(\kappa+1)^2\theta_{\kappa} \leq C\,\|\Omega\|_{L(\log L)^2(\mathbf{S}^{n-1}\times\mathbf{S}^{\kappa-1})}\,. \tag{2.4}$$

Here, $\theta_{\kappa}=\left\|\Omega\chi_{\mathbf{E}_{\kappa}}\right\|_1$ for $\kappa\in\mathbf{D}$ and $\theta_0=1$.

Now, for $\kappa\in\mathbf{D}\cup\{0\}$, we let $T_{\Omega_{\kappa},\phi_1,\phi_2}$be the operator defined by (1.5) with Ω replaced by Ω_{κ}. Therefore, by (2.3) we have the following decomposition for the operator T_{Ω,ϕ_1,ϕ_2}:

$$T_{\Omega,\phi_1,\phi_2}f(x,y)=\sum_{\kappa\in\mathbf{D}\cup\{0\}}\theta_{\kappa}T_{\Omega_{\kappa},\phi_1,\phi_2}f(x,y). \tag{2.5}$$

By (2.4), (2.5), and Minkowski's inequality, it suffices to show that

$$\left\|T_{\Omega_{\kappa},\phi_1,\phi_2}f\right\|_p\leq(\kappa+1)^2C\,\|f\|_p \tag{2.6}$$

for all κ and $p\in(1,\infty)$.

To prove (2.6), we argue as follows:

For $j,k\in\mathbf{Z}$, let $\overset{\kappa}{\sigma}_{j,k}$ be the measure defined by

$$\int fd\overset{\kappa}{\sigma}_{j,k}=\int_{A(j,k,\kappa)}f(x-\phi_1(|u|)u',y-\phi_2(|v|)v')\,|u|^{-n}\,|v|^{-m}\,\Omega_{\kappa}(u,v)\,dudv, \tag{2.7}$$

where

$$A(j,k,\kappa)=\left\{(x,y)\in\mathbf{R}^n\times\mathbf{R}^m:2^{k(\kappa+1)}\leq|u|<2^{(k+1)(\kappa+1)}\text{ and }2^{j(\kappa+1)}\leq|v|<2^{(j+1)(\kappa+1)}\right\}.$$

Let $(\sigma^{\kappa})^{*}$ be the maximal function

$$(\sigma^{\kappa})^{*}(f)=\sup_{k,j\in\mathbf{Z}}\left|\left|\sigma_{j,k}^{\kappa}\right|*f\right|.$$

Now, we have the following:

$$\left\|\sigma_{j,k}^{\kappa}\right\|\leq C(\kappa+1)^{2};\tag{2.8}$$

$$\begin{aligned}
\left|\hat{\sigma}_{j,k}^{\kappa}(\xi,\eta)\right| &\leq C(\kappa+1)^{2}\left|a_{j,\kappa+1}^{(1)}\xi\right|^{-\frac{\delta}{\kappa+1}}\left|a_{k,\kappa+1}^{(2)}\eta\right|^{-\frac{\beta}{\kappa+1}} &(2.9)\\
\left|\hat{\sigma}_{j,k}^{\kappa}(\xi,\eta)\right| &\leq C(\kappa+1)^{2}\left|a_{j+1,\kappa+1}^{(1)}\xi\right|^{\frac{\delta}{\kappa+1}}\left|a_{k+1,\kappa+1}^{(2)}\eta\right|^{\frac{\beta}{\kappa+1}} &(2.10)\\
\left|\hat{\sigma}_{j,k}^{\kappa}(\xi,\eta)\right| &\leq C(\kappa+1)^{2}\left|a_{j+1,\kappa+1}^{(1)}\xi\right|^{\frac{\delta}{\kappa+1}}\left|a_{k,\kappa+1}^{(2)}\eta\right|^{-\frac{\beta}{\kappa+1}} &(2.11)\\
\left|\hat{\sigma}_{j,k}^{\kappa}(\xi,\eta)\right| &\leq C(\kappa+1)^{2}\left|a_{j,\kappa+1}^{(1)}\xi\right|^{-\frac{\delta}{\kappa+1}}\left|a_{k+1,\kappa+1}^{(2)}\eta\right|^{\frac{\beta}{\kappa+1}}. &(2.12)
\end{aligned}$$

Clearly (2.8) holds. To see (2.9), notice that by polar coordinates we have

$$\left|\hat{\sigma}_{j,k}^{\kappa}(\xi,\eta)\right|\leq 2^{4(\kappa+1)}\iint\limits_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}I_{1}(j,\kappa+1,\xi\cdot u')I_{2}(k,\kappa+1,\eta\cdot v')d\sigma(u)d\sigma(v),\tag{2.13}$$

where I_l is given by (1.6) in the statement of the theorem. Therefore, by assumption, we get

$$\begin{aligned}
I_{1}(j,\kappa+1,\xi\cdot u') &\leq (\kappa+1)C\left|a_{j,\kappa+1}^{(1)}\xi\cdot u'\right|^{-\varepsilon_{1}} &(2.14)\\
I_{2}(k,\kappa+1,\eta\cdot v') &\leq (\kappa+1)C\left|a_{k,\kappa+1}^{(2)}\eta\cdot v'\right|^{-\varepsilon_{2}}. &(2.15)
\end{aligned}$$

Thus by (2.13) and (2.14)–(2.15), we obtain

$$\left|\hat{\sigma}_{j,k}^{\kappa}(\xi,\eta)\right|\leq(\kappa+1)^{2}2^{4(\kappa+1)}C\left|a_{j,\kappa+1}^{(1)}\xi\right|^{-\varepsilon_{1}}\left|a_{k,\kappa+1}^{(2)}\eta\right|^{-\varepsilon_{2}}.\tag{2.16}$$

Therefore, by interpolation between (2.8) and (2.17), the inequality (2.9) follows.

Next, we show (2.11). By polar coordinates and using the cancelation property of $\Omega_{\kappa}(u,\cdot)$, we get

$$\left|\hat{\sigma}^{\kappa}_{j,k}(\xi,\eta)\right| \leq 2^{4(\kappa+1)} \iint\limits_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} J_1(j,\kappa+1,\xi\cdot u')I_2(k,\kappa+1,\eta\cdot v')d\sigma(u)d\sigma(v), \qquad (2.17)$$

where J_l is given by (1.7) in the statement of the theorem. Thus, we have

$$\left|\hat{\sigma}^{\kappa}_{j,k}(\xi,\eta)\right| \leq (\kappa+1)^2 2^{4(\kappa+1)} C \left|a^{(1)}_{j+1,\kappa+1}\xi\right|^{\varepsilon_1} \left|a^{(2)}_{k,\kappa+1}\eta\right|^{-\varepsilon_2}. \qquad (2.18)$$

Therefore, (2.11) follows by (2.18) and an interpolation argument similar to that led to (2.9). The proofs of (2.10) and (2.12) can be obtained similarly with minor modification. We omit the details.

Now, we show that

$$\left\|(\sigma^{\kappa})^*(f)\right\|_p \leq (\kappa+1)^2 C\,\|f\|_p \qquad (2.19)$$

for all $1 < p < \infty$. To see this notice that

$$(\sigma^{\kappa})^*(f)(x,y) \leq (\kappa+1)^2 \iint\limits_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |\Omega_{\kappa}(u',v')|\, M^1_{\phi_1,u'} \circ M^2_{\phi_2,v'} f(x,y)d\sigma(u)d\sigma(v), \qquad (2.20)$$

where $M^1_{\phi_1,u'}f(x,y) = M_{\phi_1,u'}f(\cdot,y)(x)$, $M^2_{\phi_2,v'}f(x,y) = M_{\phi_2,v'}f(x,\cdot)(y)$, and $\circ$ denotes the composition of operators. Thus, by (2.20), Hölder's inequality, the estimate $\|\Omega_{\kappa}\|_1 \leq C$, the estimates (1.6)–(1.7) (for $\kappa = 1$), and Corollary 2.2, we immediately obtain (2.19).

Now, by (2.8)–(2.12), (2.19), and adapting the same argument in the one parameter setting in ([6], [9]) (see also Theorem 11 in [1]), we can easily obtain (2.6). We omit the details.

Finally, the boundedness of $(T_{\Omega,\phi_1,\phi_2})^*$ follows by (2.3)–(2.4), (2.8)–(2.12), and similar argument as in the one parameter setting in ([6], [10]) (see also, [1]). □

3. Applications

As we mentioned in the introduction section, this section is devoted for presenting examples on operators whose L^p boundedness can be obtained by applying Theorem 1.1.

We should point out here that all results presented here have been extensively investigated in the one parameter setting by many authors ([2], [4], [11], among others).

We start by the following interesting result:

Corollary 3.1. *If* $\phi_1$and ϕ_2 are convex increasing functions with $\phi_1(0) = \phi_2(0) = 0$, then the corresponding operators T_{Ω,ϕ_1,ϕ_2} *and* $(T_{\Omega,\phi_1,\phi_2})^*$ *are bounded on* $L^p(\mathbf{R}^n \times \mathbf{R}^m)$ *for* $p \in (1, \infty)$ provided that $\Omega \in L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$.

Proof. We only need to verify the assumptions of Theorem 1.1, i.e., the estimates (1.6) and (1.7). For $\kappa \in \mathbf{N}$ and $j \in \mathbf{Z}$, let $a^{(1)}_{j,\kappa} = \phi_1(2^{\kappa j})$ and $a^{(2)}_{j,\kappa} = \phi_2(2^{\kappa j})$. Then since $\phi_1$and ϕ_2 are convex increasing functions with $\phi_1(0) = \phi_2(0) = 0$, it follows that $\{a^{(1)}_{j,\kappa} : j \in \mathbf{Z}\}$ and $\{a^{(2)}_{j,\kappa} : j \in \mathbf{Z}\}$ are lacunary sequences with $\inf_{j\in\mathbf{Z}} \frac{a^{(1)}_{j+1,\kappa}}{a^{(1)}_{j,\kappa}}$ and $\inf_{j\in\mathbf{Z}} \frac{a^{(2)}_{j+1,\kappa}}{a^{(2)}_{j,\kappa}} \geq 2^{\kappa}$.

Now, we show that (1.6) and (1.7) hold. To see (1.6) notice that $\frac{d}{dr}(\phi_l(2^{\kappa j}r)) = 2^{\kappa j}\phi_l'(2^{\kappa j}r) \geq \phi_l(2^{\kappa j}r) \geq a^{(l)}_{j,\kappa}$ for $r \geq 1$. Thus by integration by parts, we get

$$I_l(j, \kappa, \lambda) = \left| \int_1^{2^{\kappa}} e^{-i\lambda\phi_l(2^{\kappa j}r)} r^{-1} dr \right| \leq \left| a^{(l)}_{j,\kappa}\lambda \right|^{-1} \tag{3.1}$$

when combined with the estimate $I_l(j, \kappa, \lambda) \leq \kappa$ implies (1.6) for any $0 < \epsilon < 1$.

The proof (1.7) is clear. In fact, one only needs to observe that

$$J_l(j, \kappa, \lambda) \leq \int_{2^{\kappa j}}^{2^{\kappa(j+1)}} |\lambda\phi_l(r)|\, r^{-1} dr \leq \kappa \left| a^{(l)}_{j+1,\kappa}\lambda \right|.$$

Hence the result follows by an application of Theorem 1.1. This completes the proof.

Corollary 3.2. *If* $\phi_1$and ϕ_2 are of the form t^{α} $(\alpha \neq 0)$, then the corresponding operators T_{Ω,ϕ_1,ϕ_2} *and* $(T_{\Omega,\phi_1,\phi_2})^*$ *are bounded on* $L^p(\mathbf{R}^n \times \mathbf{R}^m)$ *for* $p \in (1, \infty)$ provided that $\Omega \in L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$.

This result has the following generalization:

Corollary 3.3. *Suppose that* ϕ_l*,* $l = 1, 2$ *satisfy*

$$|\phi_l(t)| \leq C_{1,l}t^{d_l}, \qquad |\phi_l''(t)| \leq C_{2,l}t^{d_l-2}, \tag{3.2}$$

$$C_{3,l}t^{d_l-1} \leq |\phi_l'(t)| \leq C_{4,l}t^{d_l-1} \tag{3.3}$$

for some $d_l \neq 0$ *and* $t \in (0, \infty)$, *where* $C_{1,l}$, $C_{2,l}$, $C_{3,l}$, *and* $C_{4,l}$ *are positive constants independent of* t.

Then the corresponding operators T_{Ω,ϕ_1,ϕ_2} *and* $(T_{\Omega,\phi_1,\phi_2})^*$ *are bounded on* $L^p(\mathbf{R}^n \times \mathbf{R}^m)$ *for* $p \in (1,\infty)$ *provided that* $\Omega \in L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$.

Proof. For $\kappa \in \mathbf{N}$, let $\{a^{(1)}_{j,\kappa} : j \in \mathbf{Z}\}$ and $\{a^{(2)}_{j,\kappa} : j \in \mathbf{Z}\}$ be the lacunary sequences given by $a^{(l)}_{j,\kappa} = 2^{\kappa d_l j}$, $l = 1, 2$.

Now it is easy to see that under the conditions (3.2)–(3.3), the estimates (1.6)–(1.7) hold trivially. In fact, (1.7) follows by the first inequality in (3.2). On the other hand, (1.6) follows by (3.2)–(3.3) and an integration by parts. For the details see pages 525-526 in ([2]). Hence the result follows by Theorem 1.1.

By arguing inductively using Theorem 1.1, we can also obtain the following:

Corollary 3.4. *If* ϕ_1 and ϕ_2 are real valued polynomials, then the corresponding operators T_{Ω,ϕ_1,ϕ_2} *and* $(T_{\Omega,\phi_1,\phi_2})^*$ *are bounded on* $L^p(\mathbf{R}^n \times \mathbf{R}^m)$ *for* $p \in (1,\infty)$ provided that $\Omega \in L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$.

We should mention here that the result of Corollary 3.4 still holds if $\phi_1$and ϕ_2 are allowed to be generalized polynomials (For more details see[4]).

Final Remark. By minor modifications of the argument in this paper, one can easily notice that the operators discussed here can be allowed to be rough in the radial direction. More specifically, we are able to consider operators of the form

$$(T_{\Omega,\phi_1,\phi_2,h}f)(x,y) = \text{p.v.}\int_{\mathbf{R}^n\times\mathbf{R}^m} f(x-\phi_1(|u|)u', y-\phi_2(|v|)v')\,|u|^{-n}\,|v|^{-m}\,h(|u|,|v|)\Omega\,(u,v)\,dudv,$$

where $h : \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}$ is a measurable function that satisfies

$$\sup_{R_1,R_2>0}\left[(R_1R_2)^{-1}\int_0^{R_1}\int_0^{R_2} |h\,(t,s)|^{\gamma}\,dtds\right]^{\frac{1}{\gamma}} < \infty \tag{3.4}$$

for some $\gamma > 1$. The only difference here between the results for $T_{\Omega,\phi_1,\phi_2,h}$ and those for T_{Ω,ϕ_1,ϕ_2} is that the interval of p for which the later is bounded on L^p is $\left|\frac{1}{p}-\frac{1}{2}\right| < \min\left\{\frac{1}{2}, \frac{1}{\gamma'}\right\}$. For more information on operators of this form, we refer the reader to consult [1], [2], [6], [9], [10].

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