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On Linear the Homeomorphism Between Function Spaces $C_p(X)$ and $C_{p,A}(X) \times C_p(A)$

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Abstract

In this paper, we investigate a linear homeomorphism between function spaces $C_p(X)$ and $C_{p,A}(X) \times C_p(A)$, where X is a normal space and A is a neighborhood retraction of X.

Key Words: Function Spaces; linear homeomorphism, pointwise topology.

1. Introduction

In [2] Jan Baars and J. D. Groot derived an isomorphical classification of the spaces $C_p(X)$, where X denotes any compact zero-dimensional space. In [6] J. Van Mill derived a isomorphical classification of the spaces $C_p(X)$, where X denotes any metrizable space. It has been proved in [6] that for metrizable spaces, there always exist an extender which is both linear and continuous.

First we fix some notation and give some definitions.

For a space X, we define $C(X)$ to be the set real-valued continuous functions on X, and $C(X)$ is vector space with the natural addition and scalar multiplication. For a covering $\mathcal{K}$ of X, we define a topology on $C(X)$ by taking the family of all sets

$$\langle f, \mathrm{K}, \delta \rangle = \{g \in C(X) : |f(x) - g(x)| < \delta, \text{ for every } x \in \mathrm{K}\},$$

where $f \in C(X)$, $\mathrm{K} \in \mathcal{K}$ and $\delta > 0$, as a subbase.

If $\mathcal{K}$ consists of all finite subsets of X, we denote $C(X)$ endowed with this topology by $C_p(X)$. The topology on $C_p(X)$ is called the pointwise convergence topology. It is well known or easy to prove that $C_p(X)$ is a topological vector space.

Let X be a space and $A \subset X$ closed. By $C_{p,A}(X)$, we denote the subspace of $C_p(X)$ of all functions vanishing on A. That is,

$$C_{p,A}(X) = \{f \in C_p(X) : f(A) = 0\}.$$

If A is singleton, say $\{a\}$,then we denote $C_{p,A}(X)$ simply by $C_{p,a}(X)$. Let X/A be the quotient space obtained from X by identifying A to a single point, say ∞. $C_{p,\infty}(X/A)$ is the space of $C_p(X/A)$ of all function vanishing at ∞. That is,

$$C_{p,\infty}(X/A) = \{f \in C_p(X/A) : f(\infty) = 0\}$$

Let the constant function with value 0 be denoted by $\underline{0}$.

Definition 1 *Let X be a space with subspace A. We say that A is a retract of X provided that there is a continuous function $r : X \to A$ such that r restricted to A is the identity on A. Such a function r is called a retraction.*

Lemma 1 [6] *Let X be a Hausdorff space with subspace A. If A is a retract of X then A is a closed subset of X.*

Proof. Let (X, τ) be a Hausdorff space and $r : X \to A$ be a retraction. We want to show that A is closed in X. Take any point x_0 in $X \backslash A$. Then $r(x_0) = a \in A$. Since r is a retraction it comes to be $x_0 \neq a$ and since X is a Hausdorff space, there are two open subsets $U \in \tau$ $(x_0 \in U)$ and $V \in \tau$ $(a \in V)$ such that $U \cap V = \varnothing$. That A is a subspace of X makes $A \cap V$ open in (A, τ_A), and so long as r is a continuous function, $r^{-1}(A \cap V)$ is open in (X, τ) and $x_0 \in r^{-1}(A \cap V)$. Let $W = U \cap r^{-1}(A \cap V)$. The set W is open in (X, τ), $x_0 \in W$ and $W \cap V = \varnothing$. Since r is a retraction, $r(W) \subset V$. So for every $x \in W$, we get $r(x) \neq x$ and thus we have $W \subset X \backslash A$. Thus $X \backslash A$ is open in (X, τ) and A is closed in X. □

Remark 1. The statement of Lemma 1 is not necessarily true if X is not Hausdorff. For instance, the subset Z_e of all even integers of the cofinite topology defined on Z is an example of a non-closed retract under the continuous function $f : Z \to Z_e$ where $f(2k-1) = 2k = f(2k)$ for each $k \in Z$. Notice that $f^{-1}(F)$ is finite whenever $F \subseteq Z_e$

is finite and thus f is continuous and furthermore cofinite topology determines a T_1 topological space on Z which is not T_2 (Hausdorff).

We say that A is a neighborhood retract of X provided that there exists a neighborhood U of A in X such that A is a retract of U.

Now we prove theorem 1 which will be used in the proof of the theorem 2.

Theorem 1 *Let X be a normal space and A be a neighborhood retract of X. Then there is a continuous linear and one to one function $\Phi : C_p(A) \to C_p(X)$ such that for each $f \in C(A)$, $\Phi(f)|_A = f$.*

Proof. Let U, including A, be an open subset of X and $r : U \to A$ be a retraction. Since X is a normal space, for an open subset W of X,

$$A \subseteq W \subseteq clW \subseteq U$$

A is closed in U because A is a retract of U. Then A is a closed subset of clW and also a closed subset of X. Hence A and $X\backslash W$ are two disjoint closed subset of X. Then for a continuous function

$$f_0 : X \to [0,1]$$

we get $f_0(A) = \{1\}$ and $f_0(X\backslash W) = \{0\}$. Define

$$\Phi(f)(x) = \left\{ \begin{array}{ccc} 0 & \text{if} & x \in X\backslash W \\ f_0(x) f(r(x)) & \text{if} & x \in W \end{array} \right\}$$

for $f \in C_p(A)$. We want to show that $\Phi(f) \in C_p(X)$. In other words,

$$\Phi(f) : X \to \mathbb{R}$$

is continuous. If $x = a \in A$ then

$$\Phi(f)(a) = f_0(a) f(r(a)) = 1f(a) = f(a).$$

From this, we get $\Phi(f)|_A = f$. $\Phi(f)$ is continuous on W as $\Phi(f)(x) = f_0(x) f(r(x))$ and W is open. Now take $x \in X\backslash W$. We claim that $\Phi(f)$ is continuous at $X\backslash W$. □

We prove this latter claim via the following two cases.

Case 1. Let $x \in clW \backslash W$ and let $(x_\mu)_{\mu\in\Gamma}$, which convergences to element, x be a net. We want to show that

$$(\Phi(f)(x_\mu))_{\mu\in\Gamma} \to \Phi(f)(x),$$

since x is an element of U and U is open; a tail of this net will be in U. For this reason, without lose of generality, we can assume that all the elements of this net are in U. As $x_\mu \to x$ and $r : U \to A$ are continuous,

$$(r(x_\mu))_{\mu\in\Gamma} \to r(x)$$

in A. Furthermore,

$$(f(r(x_\mu)))_{\mu\in\Gamma} \to f(r(x))$$

due to the continuity of $f : A \to \mathbb{R}$. Since $x \in X\backslash W$, $\Phi(f)(x) = 0$ and $f_0(x) = 0$. Then we get

$$(f_0(x_\mu))_{\mu\in\Gamma} \to 0.$$

On the other hand,

$$\Phi(f)(x_\mu) = \left\{ \begin{array}{ccc} 0 & \text{if} & x_\mu \in U\backslash W \\ f_0(x_\mu) f(r(x_\mu)) & \text{if} & x_\mu \in W \end{array} \right\}.$$

In every case, $\Phi(f)(x_\mu) = f_0(x_\mu) f(r(x_\mu))$. Then it is seen that

$$(\Phi(f)(x_\mu))_{\mu\in\Gamma} \to \Phi(f)(x) = 0.$$

Case 2. Let $x \in X\backslash clW$. Since $X\backslash clW$ is open, $\Phi(f) = \underline{0}$ is continuous on $X\backslash clW$.
We show that Φ is a linear. Let $f, g \in C_p(A)$, $\alpha, \beta \in \mathbb{R}$

$$\begin{aligned} \Phi(\alpha f + \beta g)(x) &= \left\{ \begin{array}{ccc} \alpha 0 + \beta 0 = 0 & \text{if} & x \in X\backslash W \\ f_0(x)(\alpha f + \beta g)(r(x)) & \text{if} & x \in W \end{array} \right\} \\ &= \left\{ \begin{array}{ccc} \alpha 0 & \text{if} & x \in X\backslash W \\ f_0(x)(\alpha f)(r(x)) & \text{if} & x \in W \end{array} \right\} \\ &\quad + \left\{ \begin{array}{ccc} \beta 0 & \text{if} & x \in X\backslash W \\ f_0(x)(\beta g)(r(x)) & \text{if} & x \in W \end{array} \right\} \\ &= \alpha \Phi(f)(x) + \beta \Phi(g)(x) \end{aligned}$$

Thus Φ is linear. Now let us show that $\Phi : C_p(A) \to C_p(X)$ is continuous. Since Φ is linear, $C_p(A)$ and $C_p(X)$ are topological vector spaces, it is sufficient to prove that Φ is continuous at $\underline{0}$.

$$\langle \underline{0}, \{x_0, x_1, ..., x_n\}, \varepsilon \rangle = \bigcap_{i=0}^{n} \langle \underline{0}, \{x_i\}, \varepsilon \rangle .$$

Let us choose $x_0 \in X$ and consider the open set

$$\langle \underline{0}, \{x_0\}, \varepsilon \rangle = \{ f \in C_p(X) : |f(x_0)| < \varepsilon \} = T$$

We want to show that

$$\Phi(\langle \underline{0}, \{a\}, \delta \rangle) = \Phi(\{ g \in C_p(A) : |g(a)| < \delta \}) \subseteq T$$

for $a \in A$ and $\delta > 0$. Let us assume that $a \in A$ and $g \in \langle \underline{0}, \{a\}, \delta \rangle$. Then

$$\Phi(g)(x_0) = \left\{ \begin{array}{ccc} 0 & \text{if} & x_0 \in X \backslash W \\ f_0(x_0)\, g(r(x_0)) & \text{if} & x_0 \in W \end{array} , \right\}$$

If $x_0 \in W$, then take $a = r(x_0)$ and $0 < \delta = \varepsilon / (f_0(x_0) + 1)$. Then we have a and $\delta > 0$. Because,

$$|\Phi(g)(x_0)| = |f_0(x_0)|\, |g(a)| < f(x_0) / (f(x_0) + 1) < 1.$$

Hence, $\Phi(g) \in T$. If $x_0 \in X \backslash W$ then $|\Phi(g)(x_0)| = 0 < 1$ for any a which is chosen from A. This implies $\Phi(g) \in T$. Therefore since $\Phi(g) \in T$ for $g \in \langle \underline{0}, \{a\}, \delta \rangle$, Φ is continuous on each two cases. Φ is one to one, it is seen by definition of Φ easily.

We now come to the following important theorem.

Theorem 2 *Let X be a normal space and let A be a neighborhood retract of X. Then*

$$C_p(X) \approx C_{p,A}(X) \times C_p(A) .$$

Proof. Define $G : C_p(X) \to C_p(A)$ by $G(f) = f|_A$. Notice that G is a continuous linear function. For $f \in C_p(X)$, $f \in C_{p,A}(X)$, if and only if $G(f) = \underline{0}$. By theorem 1, there is a continuous linear function $\Phi : C_p(A) \to C_p(X)$ such that for each $f \in C_p(A)$, $\Phi(f)|_A = f$. Notice that $G \circ \Phi = id_{C_p(A)}$.

Now define $\theta : C_p(X) \to C_{p,A}(X) \times C_p(A)$ by

$$\theta(f) = (f - (\Phi \circ G)(f), G(f)) .$$

We have to prove that θ is well-defined. Take an arbitrary $f \in C_p(X)$. It is obvious that $G(f) \in C_p(A)$ and that $f - (\Phi \circ G)(f) \in C_p(X)$. Furthermore,

$$G(f - (\Phi \circ G)(f)) = G(f) - (G \circ \Phi \circ G)(f) = G(f) - G(f) = \underline{0}$$

so $f - (\Phi \circ G)(f) \in C_{p,A}(X)$. That θ is continuous and linear is a triviality. We show that θ is a linear homeomorphism. For that, define

$$\Gamma : C_{p,A}(X) \times C_p(A) \to C_p(X)$$

By $\Gamma(f,h) = f + \Phi(h)$ it is trivial that Γ is well defined, continuous and linear. Furthermore, as is easily seen, $\Gamma \circ \theta = id_{C_p(X)}$ and we show that $\theta \circ \Gamma = id_{C_{p,A}(X) \times C_p(A)}$. Take $f \in C_{p,A}(X)$ and $h \in C_p(A)$. Notice that $G(f) = \underline{0}$ hence by linearity of Φ. $(\Phi \circ G)(f) = \Phi(\underline{0}) = \underline{0}$, so

$$\begin{aligned}(\theta \circ \Gamma)(f,h) &= \theta(f + \Phi(h)) = (f + \Phi(h) - (\Phi \circ G)(f + \Phi(h)), G(f + \Phi(h))) \\ &= (f + \Phi(h) - \underline{0} \; - \Phi(h), \underline{0} + h) \\ &= (f,h).\end{aligned}$$

Hence $\theta \circ \Gamma = id_{C_{p,A}(X) \times C_p(A)}$, i.e.,$\theta$ is a linear homeomorphism. □

Lemma 2 *Let X be a normal space and A be a neighborhood retract of X. Then*

$$C_{p,A}(X) \approx C_{p,\infty}(X/A).$$

Proof. Let $p : X \to X/A$ be the quotient map between X and X/A. For every function $f \in C_{p,A}(X)$ there is a unique function $g \in C_{p,\infty}(X/A)$ such that $g \circ p = f$. If we now define $\theta : C_{p,A}(X) \to C_{p,\infty}(X/A)$ by $\theta(f) = g$, then θ is a well-defined linear bijection. Since for $f \in C_{p,A}(X)$, $y_1, ..., y_n \in X/A$, $\delta > 0$ and $x_i \in p^{-1}(y_i)$, $(i \leq n)$ it is easily seen that

$$\theta(\langle f, \{x_1, ..., x_n\}, \delta\rangle) = \langle \theta(f), \{y_1, ..., y_n\}, \delta\rangle,$$

and it follows that θ is linear homeomorphism. □

From the last lemma and theorem 2, we have the useful following corollary.

Corollary 1 *Let X be a normal space and let A be a neighborhood retract of X. Then*

$$C_p(X) \approx C_{p,\infty}(X/A) \times C_{p,A}(X).$$

Proof. By lemma 2 and theorem 2

$$C_p(X) \approx C_{p,\infty}(X/A) \times C_{p,A}(X).$$

□

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