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On Space of Parabolic Potentials Associated with the Singular Heat Operator

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Abstract

Anisotropic spaces $L^{\alpha}_{p,\gamma}$ of parabolic Bessel potentials, associated with the singular heat operator $I-\Delta_\gamma+\frac{\partial}{\partial t}$, where $\Delta_\gamma=\sum\limits_{k=1}^{n}\frac{\partial^2}{\partial x_k^2}+\frac{2\gamma}{x_n}\cdot\frac{\partial}{\partial x_n}$, are introduced, and making use of special wavelet-type transform, a characterization of these spaces is obtained.

Key Words: Generalized translation, Fourier-Bessel transform, parabolic potential, wavelet transform.

1. Introduction

The classical Jones-Sampson parabolic Bessel potentials $\mathcal{H}^{\alpha} f$, $(\alpha>0)$ are defined in the Fourier terms by

$$F\left[\mathcal{H}^{\alpha} f\right](x,t)=(1+|x|^2+it)^{-\frac{\alpha}{2}}F\left[f\right](x,t)\ , \tag{1.1}$$

where $x\in\mathbb{R}^n$, $t\in\mathbb{R}^1$; F is the Fourier transform. These potentials are interpreted as negative (fractional) powers of the heat operator $I+\Delta+\frac{\partial}{\partial t}$. Here, $\Delta=\sum\limits_{k=1}^{n}\frac{\partial^2}{\partial x_k^2}$ is the Laplacean and I is an identity operator. Parabolic potentials were introduced by B. F. Jones [8] and C. H. Sampson [13] and studied in [5, 6, 7, 10]. The space of parabolic

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Bessel potentials

$$L_p^\alpha = \{f: \ f = \mathcal{H}^\alpha\varphi \ , \varphi \in L_p(\mathbb{R}^{n+1})\} \ \ , \ \ 1 < p < \infty \tag{1.2}$$

were introduced by C. H. Sampson [13], studied by R. Bagby [5], V. R. Gopala Rao [7], S. Chanillo [6] and generalized by Nogin and Rubin [10].

Singular parabolic equations were studied by many authors (see, e.g. [4] and references therein). The relevant singular parabolic potentials, associated with the singular heat operator, $I - \Delta_\gamma + \frac{\partial}{\partial t}$, where $\Delta_\gamma = \sum\limits_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} + \frac{2\gamma}{x_n} \cdot \frac{\partial}{\partial x_n}$, $(\gamma > 0)$ were introduced and studied by I. A. Aliev [3]. These potentials are defined in terms of the Fourier-Bessel transform F_γ by

$$F_\gamma \left[\mathcal{H}_\gamma^\alpha f\right](x,t) = (1 + |x|^2 + it)^{-\frac{\alpha}{2}} F_\gamma\,[f]\,(x,t) \ , \ (x \in \mathbb{R}_+^n \ , \ t \in \mathbb{R}^1 \ , \ \alpha > 0). \tag{1.3}$$

The wavelet approach to these potentials was studied by I. A. Aliev and B. Rubin [1, 2]. In this paper we introduce the spaces of singular parabolic potentials

$$L_{p,\gamma}^\alpha = \{f: \ f = \mathcal{H}_\gamma^\alpha\varphi \ , \varphi \in L_p(\ \mathbb{R}_+^n \times \mathbb{R}^1; \ x_n^{2\gamma} dxdt \ )\} \tag{1.4}$$

and give the "wavelet-type" characterization of these spaces. In subsequent publications we plan to give some applications of our results to singular heat equations.

2. Preliminaries

Let $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x = (x_1, x_2, \ldots, x_{n-1}, x_n), \ \ x_n > 0\}$;
$\mathbb{R}_+^n \times \mathbb{R}^1 = \left\{(x,t) : x \in \mathbb{R}_+^n \ , \ t \in \mathbb{R}^1\right\}$; and let $S^+ = S(\mathbb{R}_+^n \times \mathbb{R}^1)$ be the class of Schwartz test functions on $\mathbb{R}_+^n \times \mathbb{R}^1$, which are even with respect to x_n. The Fourier-Bessel transform of $f(x,t)$ and its inverse are defined by

$$(F_\gamma f)(y,\tau) = \int\limits_{\mathbb{R}_+^n \times \mathbb{R}^1} f(x,t) e^{-i(x'\cdot y' + t\tau)} j_{\gamma-\frac{1}{2}}(x_n y_n) d\nu(x) dt, \tag{2.1}$$

$$(F_\gamma^{-1} f)(y,\tau) = c(n,\gamma)(F_\gamma f)(-y_1, \ldots, -y_{n-1}, y_n, -\tau), \tag{2.2}$$

where $x' \cdot y' = x_1 y_1 + \cdots + x_{n-1} y_{n-1}$; $d\nu(x) = x_n^{2\gamma} dx = x_n^{2\gamma} dx_1 \ldots dx_n, \gamma > 0$; $j_\lambda(z) = 2^\lambda \Gamma(\lambda+1) z^{-\lambda} J_\lambda(z)$ is the normalized Bessel function such that $j_\lambda(0) = 1$ (see [9, 1, 3]); and $c(n,\gamma) = \left[(2\pi)^n 2^{2\gamma-1} \Gamma^2(\gamma + \frac{1}{2})\right]^{-1}$.

We need the following weighted L_p-spaces:

$$L_{p,\gamma} \equiv L_p(\mathbb{R}^n_+ \times \mathbb{R}^1, d\nu(x)dt) = \Big\{ f : \|f\|_{p,\gamma} = \Big(\int\limits_{\mathbb{R}^n_+ \times \mathbb{R}^1} |f(x,t)|^p d\nu(x)dt \Big)^{\frac{1}{p}} < \infty \Big\}$$

$1 \leq p < \infty$. (In the case $p = \infty$, we identify $L_{p,\gamma}$ with C^0-the corresponding space of continuous functions vanishing at infinity).

For $x \in \mathbb{R}^n_+$, $y \in \mathbb{R}^n_+$ and $t, \tau \in \mathbb{R}^1$, the *generalized translation* of $f : \mathbb{R}^n_+ \times \mathbb{R}^1 \to \mathbb{C}$ is defined by

$$T^{y,\tau} f(x,t) = \frac{\Gamma(\gamma + \frac{1}{2})}{\Gamma(\gamma)\Gamma(\frac{1}{2})} \int\limits_0^\pi f(x' - y'; \sqrt{x_n^2 - 2x_n y_n \cos\beta + y_n^2}; t - \tau) \sin^{2\gamma - 1} \beta d\beta \quad (2.3)$$

(cf. [9, 1, 3]). Here we actually deal with the ordinary translation in x' and t, and with the generalized translation in x_n. It is known that for $1 \leq p < \infty$,

$$\|T^{y,\tau} f\|_{p,\gamma} \leq \|f\|_{p,\gamma} \ , \ \ (\forall (y,\tau) \in \mathbb{R}^n_+ \times \mathbb{R}^1); \quad (2.4)$$

$$\|T^{y,\tau} f - f\|_{p,\gamma} \to 0 \ \text{ as } \ |y| + |\tau| \to 0. \quad (2.5)$$

The generalized convolution associated with the generalized translation (2.3) is defined as

$$(f \circledast g)(x,t) = \int\limits_{\mathbb{R}^n_+ \times \mathbb{R}^1} g(y,\tau) \left(T^{y,\tau} f(x,t) \right) d\nu(y) d\tau. \quad (2.6)$$

It is known that (see, e.g. [9, 1]) $F_\gamma(f \circledast g) = F_\gamma(f) F_\gamma(g)$, $(f, g \in L_{1,\gamma})$, and

$$\|f \circledast g\|_{r,\gamma} \leq \|f\|_{p,\gamma} \cdot \|g\|_{q,\gamma} \ , \ \ 1 \leq p,q,r \leq \infty \ , \ \ \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1. \quad (2.7)$$

We need below the generalized Gauss-Weierstrass kernel:

$$W_\gamma(y,s) = c(n,\gamma)(2s)^{-\frac{(n+2\gamma)}{2}} \exp(-|y|^2/4s) \ , \ \ y \in \mathbb{R}^n_+ \ , \ \ s > 0; \quad (2.8)$$

$c(n,\gamma)$ being defined as in (2.2) (see [14] for $n = 1$ and [1, 3] for any $n \geq 1$).

Lemma 2.1 (see [1])*:*

$$\textbf{1)}\ F_{\gamma,y\to x}\left(W_\gamma(y,s)\right)(x)=\exp(-s|x|^2),\ (\forall s>0); \tag{2.9}$$

$F_{\gamma,y\to x}$ being the Fourier-Bessel transform in $y\in\mathbb{R}^n_+$.

$$\textbf{2)}\ W_\gamma(\lambda^{\frac{1}{2}}y,\lambda s)=\lambda^{-\gamma-\frac{n}{2}}W_\gamma(y,s),\ (\forall y\in\mathbb{R}^n_+,\ s>0,\ \lambda>0); \tag{2.10}$$

in particular, $W_\gamma(\lambda^{\frac{1}{2}}y,\lambda)=\lambda^{-\gamma-\frac{n}{2}}W_\gamma(y,1)$.

$$\textbf{3)}\ \int\limits_{\mathbb{R}^n_+}W_\gamma(y,s)d\nu(y)=1,\ (\forall s>0). \tag{2.11}$$

The generalized parabolic potentials $\mathcal{H}^\alpha_\gamma f$, initially defined by (1.3), can be represented as an integral operator [1, 3]

$$(\mathcal{H}^\alpha_\gamma f)(x,t)=\frac{1}{\Gamma(\alpha/2)}\int\limits_{\mathbb{R}^n_+\times\mathbb{R}^1}\tau^{\frac{\alpha}{2}-1}e^{-\tau}W_\gamma(y,\tau)\left(T^{y,\tau}f(x,t)\right)d\nu(y)d\tau, \tag{2.12}$$

which is clear in terms of Fourier-Bessel transform. Here and on, we suppose that $W_\gamma(y,\tau)$ is extended by zero to $\tau\le 0$.

By setting $h_\alpha(x,t)=\frac{1}{\Gamma(\alpha/2)}t_+^{\frac{\alpha}{2}-1}e^{-t}W_\gamma(x,t)$ with $t_+^{\frac{\alpha}{2}-1}=t^{\frac{\alpha}{2}-1}$ if $t>0$ and $t_+^{\frac{\alpha}{2}-1}=0$ if $t\le 0$, we have $(\mathcal{H}^\alpha_\gamma f)(x,t)=(h_\alpha\circledast f)(x,t)$.

From Young's inequality (2.7), and the fact that $\|h_\alpha\|_{1,\gamma}=1$, it follows that

$$\|\mathcal{H}^\alpha_\gamma f\|_{p,\gamma}\le\|f\|_{p,\gamma}\ ,\ \ 1\le p\le\infty. \tag{2.13}$$

Definition 2.2 *The spaces of singular parabolic potentials is defined by*

$$L^\alpha_{p,\gamma}=\left\{f:\mathbb{R}^n_+\times\mathbb{R}^1\to\mathbb{C}\ \middle|\ f=\mathcal{H}^\alpha_\gamma\varphi,\ \varphi\in L_{p,\gamma}\right\},\ 1\le p<\infty$$

with the norm $\|f\|_{L^\alpha_{p,\gamma}}=\|\varphi\|_{p,\gamma}$.

Now, as in [1, p. 6], we define a special wavelet-type transform needed in Section 3.

Definition 2.3 *Let μ be a finite (signed) Borel measure on $\mathbb{R}^1$ such that supp $\mu\subset[0,\infty)$ and $\mu\left(\mathbb{R}^1\right)=0$. Let the generalized Gauss-Weierstrass kernel $W_\gamma(y,\tau)$ be extended by zero* *to* $\tau \le 0$. *The generalized anisotropic and weighted wavelet transform of* $f : \mathbb{R}^n_+ \times \mathbb{R}^1 \to \mathbb{C}$ *is defined by*

$$(V_\mu f)(x,t;\eta) = \int\limits_{\mathbb{R}^n_+ \times \mathbb{R}^1} \left(T^{\sqrt{\eta}y,\eta\tau} f(x,t)\right) W_\gamma(y,\tau) e^{-\eta\tau} d\nu(y) d\mu(\tau)$$

$$= \int\limits_{\mathbb{R}^n_+ \times [0,\infty)} \left(T^{\sqrt{\eta}y,\eta\tau} f(x,t)\right) W_\gamma(y,\tau) e^{-\eta\tau} d\nu(y) d\mu(\tau), \ (\eta > 0). \tag{2.14}$$

Remark 2.4 Using (2.10) and changing variables, we have

$$(V_\mu f)(x,t;\eta) = \int\limits_{\mathbb{R}^n_+ \times [0,\infty)} \left(T^{\sqrt{\eta\tau}y,\eta\tau} f(x,t)\right) W_\gamma(y,1) e^{-\eta\tau} d\nu(y) d\mu(\tau). \tag{2.15}$$

Remark 2.5 The Minkowski inequality with (2.4) and (2.11) yields that for any fixed $\eta > 0$

$$\| (V_\mu f)(\cdot,\cdot;\eta) \|_{p,\gamma} \le \|\mu\| . \|f\|_{p,\gamma} \ \text{ with } \ \|\mu\| \equiv |\mu|(\mathbb{R}^1) < \infty.$$

The next lemma shows that the potentials $\mathcal{H}^\alpha_\gamma f$ can be represented via the wavelet-type transform (2.14). From now on, the notation $\int\limits_a^b g(t)d\mu(t)$ designates $\int\limits_{[a,b)} g(t)d\mu(t)$. If $\lim\limits_{t\to a^+} g(t) = \infty$, then it is assumed that $\mu(\{0\}) = 0$ and therefore $\int\limits_a^b g(t)d\mu(t) = \int\limits_{(a,b)} g(t)d\mu(t)$.

Lemma 2.6 *Let* $f \in L_{p,\gamma}, \ \ 1 \le p \le \infty$ *(where* $L_{\infty,\gamma} = C^0$*-the class of continuous functions vanishing at infinity). Further let* μ *be a (signed) Borel measure supported by* $[0,\infty)$, *such that*

$$\int\limits_0^\infty \tau^{-\frac{\alpha}{2}} d|\mu|(\tau) < \infty \quad \text{and} \quad c(\alpha,\mu) \overset{def}{=} \int\limits_0^\infty \tau^{-\frac{\alpha}{2}} d\mu(\tau) \ne 0, \ \ (\alpha > 0) . \tag{2.16}$$

Then

$$(\mathcal{H}^\alpha_\gamma f)(x,t) = \frac{1}{\Gamma(\alpha/2) c(\alpha,\mu)} \int\limits_0^\infty \eta^{\frac{\alpha}{2}-1} (V_\mu f)(x,t;\eta) d\eta . \tag{2.17}$$

Proof. From (2.16) it follows that $\mu(\{0\}) = 0$. By making use (2.15) and Fubini's theorem, we have

$$\int_0^\infty \eta^{\frac{\alpha}{2}-1} (V_\mu f)(x,t;\eta) d\eta$$

$$= \int_{\mathbb{R}^n_+ \times (0,\infty)} W_\gamma(y,1) \left(\int_0^\infty T^{\sqrt{\eta\tau}y,\eta\tau} f(x,t) e^{-\eta\tau} \eta^{\frac{\alpha}{2}-1} d\eta \right) d\nu(y) d\mu(\tau)$$

$$\left(\text{we put } \eta = \frac{s}{\tau},\ d\eta = \frac{ds}{\tau};\ y = \frac{1}{\sqrt{s}}u,\ d\nu(y) = \left(\frac{1}{\sqrt{s}}\right)^{n+2\gamma} d\nu(u) \right)$$

$$= \int_{\mathbb{R}^n_+ \times (0,\infty)} s^{-\frac{n}{2}-\gamma} W_\gamma(\frac{1}{\sqrt{s}}u,1) (T^{u,s} f(x,t)) s^{\frac{\alpha}{2}-1} e^{-s} d\nu(u) ds \int_0^\infty \tau^{-\frac{\alpha}{2}} d\mu(\tau)$$

$$= c(\alpha,\mu) \int_{\mathbb{R}^n_+ \times (0,\infty)} s^{-\frac{n}{2}-\gamma} W_\gamma(\frac{1}{\sqrt{s}}u,1) (T^{u,s} f(x,t)) s^{\frac{\alpha}{2}-1} e^{-s} d\nu(u) ds$$

$$\overset{(2.10)}{=} c(\alpha,\mu) \int_{\mathbb{R}^n_+ \times \mathbb{R}^1} W_\gamma(u,s) (T^{u,s} f(x,t)) s^{\frac{\alpha}{2}-1} e^{-s} d\nu(u) ds$$

$$\overset{(2.12)}{=} \Gamma(\frac{\alpha}{2}) c(\alpha,\mu) (\mathcal{H}^\alpha_\gamma f)(x,t).$$

□

We need in Section 3 the following lemmas.

Lemma 2.7 ([11], p. 8) *Let $\lambda > 0$ and μ be a finite Borel measure on $\mathbb{R}^1$ such that supp $\mu \subset [0,\infty)$, and*

a) $\int_0^\infty s^j d\mu(s) = 0,\ \ j = 0,1,\ldots,[\lambda]$ *($[\lambda]$ is the integer part of λ) ,*

b) $\int_0^\infty s^\beta d|\mu|(s) < \infty$ *for some $\beta > \lambda$.*

Denote by

$$\left(I^{\lambda+1}\mu\right)(s)=\frac{1}{\Gamma(\lambda+1)}\int_0^s (s-t)^{\lambda}d\mu(t) \qquad (2.18)$$

the Riemann-Liouville fractional integral of the measure μ. *Then*

$$\left(I^{\lambda+1}\mu\right)(s)=\left\{\begin{array}{ccc} O(s^{\lambda}) & , & s\to 0 \\ O(s^{-\delta}) & , & s\to\infty \end{array}\right\}, \qquad (2.19)$$

where $\delta=\min\{\beta-\lambda,1+[\lambda]-\lambda\}$, $(\delta\in(0,1])$. *Moreover,*

$$d(\lambda,\mu)\overset{def}{\equiv}\int_0^\infty \left(I^{\lambda+1}\mu\right)(s)\frac{ds}{s}=\left\{\begin{array}{ccc} \Gamma(-\lambda)\int\limits_0^\infty s^{\lambda}d\mu(s) & , & if\ \ \lambda\notin\mathbb{N} \\ \frac{(-1)^{\lambda+1}}{\lambda!}\int\limits_0^\infty s^{\lambda}\log s\ d\mu(s) & , & if\ \ \lambda\in\mathbb{N} \end{array}\right\}. \qquad (2.20)$$

Lemma 2.8 ([1], p. 13) *Let the wavelet-type transform* V_μ *and generalized parabolic potential operators* $\mathcal{H}_\gamma^\alpha$ *be defined as (2.14) and (2.12), respectively. Then for any* $g\in L_{p,\gamma},\ \ 1<p<\infty$,

$$V_\mu\left(\mathcal{H}_\gamma^\alpha g\right)(x,t;\eta)=(g\circledast h_\eta^{\frac{\alpha}{2}})(x,t), \qquad (2.21)$$

where

$$h_\eta^{\frac{\alpha}{2}}(x,t)=e^{-t}W_\gamma(x,t)\eta^{\frac{\alpha}{2}-1}\left(I^{\frac{\alpha}{2}}\mu\right)(t/\eta), \qquad (2.22)$$

and

$$(I^{\frac{\alpha}{2}}\mu)(t)=\frac{1}{\Gamma(\frac{\alpha}{2})}\int_0^t (t-\tau)^{\frac{\alpha}{2}-1}d\mu(\tau)$$

is the Riemann-Liouville fractional integral of order $\frac{\alpha}{2}$ *and of measure* μ.

Lemma 2.9 *Let* $\lambda_\alpha(t)=\frac{1}{t}(I^{\frac{\alpha}{2}+1}\mu)(t)$ *and* $I^{\frac{\alpha}{2}+1}\mu$ *be the Riemann-Liouville fractional integral of order* $\frac{\alpha}{2}+1$ *of measure* μ. *Let further* $d(\frac{\alpha}{2},\mu)$ *be defined as in (2.20). Denote*

$$\psi_\varepsilon(x,t)=\frac{1}{d(\frac{\alpha}{2},\mu)}W_\gamma(x,t)\ \frac{1}{\varepsilon}\ \lambda_\alpha(\frac{t}{\varepsilon}), \quad (\varepsilon>0;\ t>0,\ x\in\mathbb{R}_+^n).$$

Then

$$\int\limits_{\mathbb{R}^n_+\times(0,\infty)} \psi_\varepsilon(x,t)d\nu(x)dt = 1,\ \forall\varepsilon > 0. \tag{2.23}$$

Proof. Owing to (2.11) and (2.20), by Fubini's theorem it follows that

$$\begin{aligned}\int\limits_{\mathbb{R}^n_+\times(0,\infty)} \psi_\varepsilon(x,t)d\nu(x)dt &= \frac{1}{d(\frac{\alpha}{2},\mu)}\int\limits_0^\infty \lambda_\alpha(t)\Big(\int\limits_{\mathbb{R}^n_+} W_\gamma(x,t\varepsilon)d\nu(x)\Big)dt \\ &= \frac{1}{d(\frac{\alpha}{2},\mu)}\int\limits_0^\infty \lambda_\alpha(t)dt = 1.\end{aligned}$$

□

3. "Wavelet-type" characterization of the spaces $\mathbf{L}^\alpha_{p,\gamma}$

The main result of the paper is the following.

Theorem 3.1 *Let $\alpha > 0,\ \gamma > 0,\ 1 < p < \infty$ and μ be a finite (signed) Borel measure on $\mathbb{R}^1$ such that supp $\mu \in [0,\infty)$ and*

$$\int\limits_0^\infty t^j d\mu(t) = 0,\quad j = 0,1,\dots,[\frac{\alpha}{2}],\ ([\frac{\alpha}{2}] \text{ is the integer part of } \frac{\alpha}{2}); \tag{3.1}$$

$$\int\limits_0^\infty t^\beta d|\mu|(t) < \infty \quad \text{for some} \quad \beta > \alpha/2. \tag{3.2}$$

Then

$$L^\alpha_{p,\gamma} = \left\{ f \in L_{p,\gamma}\ :\ \sup_{\varepsilon>0}\left\| \int\limits_\varepsilon^\infty \frac{(V_\mu f)(x,t;\eta)}{\eta^{1+\frac{\alpha}{2}}}d\eta \right\|_{p,\gamma} < \infty \right\}.$$

Proof. Here and on, the abbreviation $\langle f\,,\,w\rangle$ will denote the value of distribution f at a test function $w\in S^{+}$. If f is a regular distribution (e.g. $f\in L_{p,\gamma}$), then

$$\langle f\,,\,w\rangle=\int\limits_{\mathbb{R}^n_+\times\mathbb{R}^1} f(x,t)\overline{w(x,t)}d\nu(x)dt.$$

The parabolic potentials $\mathcal{H}^{\alpha}_{\gamma}f$, $(\alpha>0)$ of distribution f are interpreted as a distribution defined by duality: $\langle\mathcal{H}^{\alpha}_{\gamma}f\,,\,w\rangle=\langle f\,,\,\tilde{\mathcal{H}}^{\alpha}_{\gamma}w\rangle$, where $\tilde{\mathcal{H}}^{\alpha}_{\gamma}w=U\mathcal{H}^{\alpha}_{\gamma}Uw$, $(Uw)(x,t)=w(-x,-t)$; (w is even with respect to x_n).

For good f the above equality is the consequence of the identity

$$\langle u\circledast\varphi\,,\,w\rangle=\langle u\,,\,\varphi_{\text{-}}\circledast w\rangle,\ \varphi,w\in S^{+}, \tag{3.3}$$

where $\varphi_{\text{-}}(x,t)\equiv(U\varphi)(x,t)=\varphi(-x,-t)$.

For arbitrary $f\in L_{p,\gamma},(1<p<\infty)$ the result follows by density.

To prove the theorem it suffices to show the equivalence

$$f=\mathcal{H}^{\alpha}_{\gamma}g\Longleftrightarrow\sup_{\varepsilon>0}\left\|\int\limits_{\varepsilon}^{\infty}(V_{\mu}f)(x,t;\eta)\frac{d\eta}{\eta^{1+\frac{\alpha}{2}}}\right\|_{p,\gamma}<\infty, \tag{3.4}$$

for some $g\in L_{p,\gamma}$.

Let $f=\mathcal{H}^{\alpha}_{\gamma}g$, $g\in L_{p,\gamma}$. It follows from (2.13) that $f\in L_{p,\gamma}$, and therefore the wavelet-type transform $V_{\mu}f$ is well defined (see Remark 2.5). Denote

$$(D^{\alpha}_{\varepsilon}f)(x,t)=\frac{1}{d(\frac{\alpha}{2},\mu)}\int\limits_{\varepsilon}^{\infty}(V_{\mu}f)(x,t;\eta)\frac{d\eta}{\eta^{1+\frac{\alpha}{2}}},\quad(\varepsilon>0).$$

Assuming $f=\mathcal{H}^{\alpha}_{\gamma}g$, $g\in L_{p,\gamma}$, we first show that

$$(D^{\alpha}_{\varepsilon}f)(x,t)=e^{-t}\ \psi_{\varepsilon}(x,t)\circledast g, \tag{3.5}$$

where

$$\psi_{\varepsilon}(x,t)=\frac{1}{d(\frac{\alpha}{2},\mu)}W_{\gamma}(x,t)\ \frac{1}{\varepsilon}\ \lambda_{\alpha}(\frac{t}{\varepsilon}), \tag{3.6}$$

$\lambda_{\alpha}(t)=\frac{1}{t}(I^{\frac{\alpha}{2}+1}\mu)(t)$, $I^{\frac{\alpha}{2}+1}\mu$ is the Riemann-Liouville fractional integral of μ (see (2.18)), and $W_{\gamma}(x,t)$ is extended by zero to $t\le 0$.

Using Lemma 2.8 we have

$$d(\frac{\alpha}{2},\mu)(D_{\varepsilon}^{\alpha}f)(x,t)=\int_{\varepsilon}^{\infty}(V_{\mu}f)(x,t;\eta)\frac{d\eta}{\eta^{1+\frac{\alpha}{2}}}\overset{(2.21)}{=}\int_{\varepsilon}^{\infty}(g\circledast h_{\eta}^{\frac{\alpha}{2}})(x,t)\frac{d\eta}{\eta^{1+\frac{\alpha}{2}}}$$

$$=\int_{\varepsilon}^{\infty}\frac{d\eta}{\eta^{1+\frac{\alpha}{2}}}\int_{\mathbb{R}_{+}^{n}\times\mathbb{R}^{1}}e^{-\tau}\ W_{\gamma}(y,\tau)\eta^{\frac{\alpha}{2}-1}\left(I^{\frac{\alpha}{2}}\mu\right)(\frac{\tau}{\eta})\left(T^{x,t}g(y,\tau)\right)d\nu(y)d\tau$$

(we use Fubini's theorem and the convention $W_{\gamma}(y,\tau)=0$ for $\tau\leq 0$)

$$=\int_{\mathbb{R}_{+}^{n}\times(0,\infty)}\left(T^{x,t}g(y,\tau)\right)\phi_{\varepsilon}(y,\tau)d\nu(y)d\tau.$$

Here,

$$\begin{aligned}\phi_{\varepsilon}(y,\tau)&=\int_{\varepsilon}^{\infty}\frac{1}{\eta^{1+\frac{\alpha}{2}}}\ e^{-\tau}\ W_{\gamma}(y,\tau)\ \eta^{\frac{\alpha}{2}-1}\left(I^{\frac{\alpha}{2}}\mu\right)(\frac{\tau}{\eta})d\eta\\&=\frac{1}{\Gamma(\frac{\alpha}{2})}\ e^{-\tau}\ W_{\gamma}(y,\tau)\int_{\varepsilon}^{\infty}\frac{d\eta}{\eta^{1+\frac{\alpha}{2}}}\eta^{\frac{\alpha}{2}-1}\int_{0}^{\frac{\tau}{\eta}}\left(\frac{\tau}{\eta}-\rho\right)^{\frac{\alpha}{2}-1}d\mu(\rho)\\&=\frac{1}{\Gamma(\frac{\alpha}{2})}\ e^{-\tau}\ W_{\gamma}(y,\tau)\int_{\varepsilon}^{\infty}\frac{d\eta}{\eta^{1+\frac{\alpha}{2}}}\int_{0}^{\infty}(\tau-\eta\rho)_{+}^{\frac{\alpha}{2}-1}d\mu(\rho)\\&=\frac{1}{\Gamma(\frac{\alpha}{2})}\ e^{-\tau}\ W_{\gamma}(y,\tau)\int_{0}^{\infty}\left(\int_{\varepsilon}^{\infty}\frac{(\tau-\eta\rho)_{+}^{\frac{\alpha}{2}-1}}{\eta^{1+\frac{\alpha}{2}}}d\eta\right)d\mu(\rho).\end{aligned}$$

Setting $\eta=\frac{\tau}{\rho}\,\frac{1}{\xi+1}$, after simple calculations we have

$$\int_{\varepsilon}^{\infty}\frac{(\tau-\eta\rho)_{+}^{\frac{\alpha}{2}-1}}{\eta^{1+\frac{\alpha}{2}}}d\eta\equiv\int_{0}^{\frac{\tau}{\rho}}\frac{(\tau-\eta\rho)^{\frac{\alpha}{2}-1}}{\eta^{1+\frac{\alpha}{2}}}d\eta=\frac{2}{\alpha\tau}\left(\frac{\tau}{\varepsilon}-\rho\right)_{+}^{\frac{\alpha}{2}}.$$

Further,

$$\frac{1}{\Gamma(\frac{\alpha}{2})}\int\limits_0^\infty\left(\int\limits_\varepsilon^\infty\frac{(\tau-\eta\rho)_+^{\frac{\alpha}{2}-1}}{\eta^{1+\frac{\alpha}{2}}}d\eta\right)d\mu(\rho)=\frac{1}{\Gamma(\frac{\alpha}{2})}\int\limits_0^\infty\frac{2}{\alpha\,\tau}\left(\frac{\tau}{\varepsilon}-\rho\right)_+^{\frac{\alpha}{2}}d\mu(\rho)$$

$$=\frac{1}{\frac{\alpha}{2}\Gamma(\frac{\alpha}{2})}\cdot\frac{1}{\tau}\int\limits_0^{\frac{\tau}{\varepsilon}}\left(\frac{\tau}{\varepsilon}-\rho\right)^{\frac{\alpha}{2}}d\mu(\rho)=\frac{1}{\varepsilon}\;\lambda_\alpha(\frac{t}{\varepsilon}),$$

where $\lambda_\alpha(t)=\frac{1}{t}(I^{\frac{\alpha}{2}+1}\mu)(t),\ I^{\frac{\alpha}{2}+1}\mu$ is defined as in (2.18).

Hence, $(D_\varepsilon^\alpha f)(x,t)=e^{-t}\psi_\varepsilon(x,t)\circledast g$, and ψ_ε is defined by (3.6). Now, using Young's inequality (2.7) we have

$$\|D_\varepsilon^\alpha f\|_{p,\gamma}\le\|\psi_\varepsilon\|_{1,\gamma}\cdot\|g\|_{p,\gamma}\,;$$

$$\|\psi_\varepsilon\|_{1,\gamma}=c\int\limits_{\mathbb{R}_+^n\times(0,\infty)}e^{-t}\;W_\gamma(x,t)\frac{1}{\varepsilon}\left|\lambda_\alpha(\frac{t}{\varepsilon})\right|d\nu(x)dt$$

$$=c\int\limits_0^\infty e^{-t}\;\frac{1}{\varepsilon}\left|\lambda_\alpha(\frac{t}{\varepsilon})\right|dt\int\limits_{\mathbb{R}_+^n}W_\gamma(x,t)d\nu(x)$$

$$\overset{(2.11)}{=}c\int\limits_0^\infty e^{-t}\;\frac{1}{\varepsilon}\left|\lambda_\alpha(\frac{t}{\varepsilon})\right|dt=c\int\limits_0^\infty e^{-t\varepsilon}\left|\lambda_\alpha(t)\right|dt$$

$$\le c\int\limits_0^\infty\left|\lambda_\alpha(t)\right|dt=c\int\limits_0^\infty\frac{1}{t}\left|(I^{\frac{\alpha}{2}+1}\mu)(t)\right|dt\overset{(2.19)}{<}\infty.$$

Hence, $\|D_\varepsilon^\alpha f\|_{p,\gamma}\le c.\,\|g\|_{p,\gamma}\implies\sup\limits_{\varepsilon>0}\|D_\varepsilon^\alpha f\|_{p,\gamma}<\infty.$

Let now $f\in L_{p,\gamma}$, $1<p<\infty$ and $\sup\limits_{\varepsilon>0}\|D_\varepsilon^\alpha f\|_{p,\gamma}<\infty$. We want to show that $f=\mathcal{H}_\gamma^\alpha g$, for some $g\in L_{p,\gamma}$. Since the Schwartz space S^+ is dense in $L_{p,\gamma}$, it sufficies to show that

$$\langle f\,,\,w\rangle=\langle\mathcal{H}_\gamma^\alpha g\,,\,w\rangle,\quad\forall w\in S^+\tag{3.7}$$

for some $g \in L_{p,\gamma}$. Since $\sup\limits_{\varepsilon>0} \|D_\varepsilon^\alpha f\|_{p,\gamma} < \infty$, a function $g \in L_{p,\gamma}$ and a sequence $\varepsilon_k \to 0$, $(k \to \infty)$ exist by Banach-Alaoglu theorem, such that $\langle D_{\varepsilon_k}^\alpha f\,,\, w\rangle \to \langle g\,,\, w\rangle$ as $k \to \infty$ for any $w \in L_{p',\gamma}$, $\frac{1}{p'} + \frac{1}{p} = 1$ (in particular, for all $w \in S^+$).

We want to prove that the function $g \in L_{p,\gamma}$ satisfies the equality (3.7). For this g and any Schwartz function $w \in S^+$ we have

$$\langle \mathcal{H}_\gamma^\alpha g\,,\, w\rangle = \langle g\,,\, \tilde{\mathcal{H}}_\gamma^\alpha w\rangle = \lim_{k\to\infty} \langle D_{\varepsilon_k}^\alpha f\,,\, \tilde{\mathcal{H}}_\gamma^\alpha w\rangle = \lim_{k\to\infty} \langle f\,,\, \tilde{D}_{\varepsilon_k}^\alpha \tilde{\mathcal{H}}_\gamma^\alpha w\rangle, \tag{3.8}$$

where $\tilde{D}_{\varepsilon_k}^\alpha \varphi = U D_{\varepsilon_k}^\alpha U\varphi$ and $\tilde{\mathcal{H}}_\gamma^\alpha w = U\mathcal{H}_\gamma^\alpha U w$.

Since $(Uw)(x,t) = w(-x,-t)$, then $U^2 = E$ (identity operator) and therefore (3.8) yields that

$$\langle \mathcal{H}_\gamma^\alpha g\,,\, w\rangle = \lim_{k\to\infty} \langle f\,,\, U D_{\varepsilon_k}^\alpha \mathcal{H}_\gamma^\alpha U w\rangle\,. \tag{3.9}$$

Set $Uw = v$. It is clear that $Uw \in S^+$ if $w \in S^+$. We first show that

$$\lim_{k\to\infty} \left\| D_{\varepsilon_k}^\alpha \mathcal{H}_\gamma^\alpha v - v \right\|_{q,\gamma} = 0, \quad \forall v \in S^+, \forall q \in (1,\infty).$$

By (3.5), $D_{\varepsilon_k}^\alpha \mathcal{H}_\gamma^\alpha v = e^{-t}\ \psi_{\varepsilon_k}(x,t) \circledast v$, where ψ_{ε_k} is defined as in (3.6). Hence

$$D_{\varepsilon_k}^\alpha \mathcal{H}_\gamma^\alpha v(x,t) = e^{-t}\ \psi_{\varepsilon_k}(x,t) \circledast v \ = \int\limits_{\mathbb{R}_+^n\times(0,\infty)} e^{-t}\psi_{\varepsilon_k}(y,\tau)\left(T^{y,\tau} v(x,t)\right) d\nu(y)d\tau$$

$$\Big(\text{we set } \tau = \rho\varepsilon_k\,,\ \ d\tau = \varepsilon_k d\rho\,;\ \ y = \sqrt{\varepsilon_k}z\,,\ \ d\nu(y) = \varepsilon_k^{\gamma+\frac{n}{2}} d\nu(z) \text{ and use (2.10)}\Big)$$

$$= \int\limits_{\mathbb{R}_+^n\times(0,\infty)} e^{-\varepsilon_k\rho}\psi_1(z,\rho)\left(T^{\sqrt{\varepsilon_k}z,\varepsilon_k\rho} v(x,t)\right) d\nu(z)d\rho,$$

where $\psi_1(z,\rho)=\frac{1}{d(\frac{\alpha}{2},\mu)}.W_\gamma(z,\rho)\lambda_\alpha(\rho)$. Further, owing to (2.23) we have

$$D^\alpha_{\varepsilon_k}\mathcal{H}^\alpha_\gamma v(x,t)-v(x,t)=\int\limits_{\mathbb{R}^n_+\times(0,\infty)} e^{-\varepsilon_k\rho}\psi_1(z,\rho)\left(T^{\sqrt{\varepsilon_k}z,\varepsilon_k\rho}v(x,t)\right)d\nu(z)d\rho$$

$$-\int\limits_{\mathbb{R}^n_+\times(0,\infty)}\psi_1(z,\rho)v(x,t)d\nu(z)d\rho=\int\limits_{\mathbb{R}^n_+\times(0,\infty)}\left(e^{-\varepsilon_k\rho}-1\right)\psi_1(z,\rho)\left(T^{\sqrt{\varepsilon_k}z,\varepsilon_k\rho}v(x,t)\right)$$

$$\times d\nu(z)d\rho\ +\int\limits_{\mathbb{R}^n_+\times(0,\infty)}\psi_1(z,\rho)\left(T^{\sqrt{\varepsilon_k}z,\varepsilon_k\rho}v(x,t)-v(x,t)\right)d\nu(z)d\rho.$$

By making use the Minkowski inequality we have

$$\left\|D^\alpha_{\varepsilon_k}\mathcal{H}^\alpha_\gamma v(x,t)-v(x,t)\right\|_{q,\gamma}\le\int\limits_{\mathbb{R}^n_+\times(0,\infty)}(1-e^{-\varepsilon_k\rho})|\psi_1(z,\rho)|\left\|T^{\sqrt{\varepsilon_k}z,\varepsilon_k\rho}v(x,t)\right\|_{q,\gamma}$$

$$\times d\nu(z)d\rho\ +\int\limits_{\mathbb{R}^n_+\times(0,\infty)}|\psi_1(z,\rho)|\left\|T^{\sqrt{\varepsilon_k}z,\varepsilon_k\rho}v(x,t)-v(x,t)\right\|_{q,\gamma}d\nu(z)d\rho.$$

Owing to (2.4), (2.5) and the Lebesque dominated convergence theorem, it follows that the right-hand side tends to zero as $\varepsilon_k\to0$. Thus

$$\lim_{\varepsilon_k\to0}\left\|UD^\alpha_{\varepsilon_k}\mathcal{H}^\alpha_\gamma Uw-w\right\|_{q,\gamma}=0,\quad\forall w\in S^+\,. \tag{3.10}$$

Now let us show that for $f\in L_{p,\gamma}$ and any $w\in S^+$ the equality

$$\lim_{\varepsilon_k\to0}\langle f\,,\,UD^\alpha_{\varepsilon_k}\mathcal{H}^\alpha_\gamma Uw\rangle=\langle f\,,\,w\rangle \tag{3.11}$$

holds. The Hölder inequality yields

$$\left|\langle f\,,\,UD^\alpha_{\varepsilon_k}\mathcal{H}^\alpha_\gamma Uw\rangle-\langle f\,,\,w\rangle\right|\ \le\ \|f\|_{p,\gamma}\left\|UD^\alpha_{\varepsilon_k}\mathcal{H}^\alpha_\gamma Uw-w\right\|_{q,\gamma},\ \ \frac{1}{p}+\frac{1}{q}=1.$$

From (3.10) it follows that the right-hand side of last expression tends to zero as $\varepsilon_k\to0$. Thus (3.11) holds.

Now the equalities (3.9) and (3.11) show that for any $w\in S^+$

$$\langle\mathcal{H}^\alpha_\gamma g\,,\,w\rangle=\langle f\,,\,w\rangle,$$

and as a result, $f(x,t) = \mathcal{H}_\gamma^\alpha g(x,t)$ for almost all $(x,t) \in \mathbb{R}_+^n \times \mathbb{R}^1$.

The proof of the Theorem is completed. □

The following theorem contains a description of the space $L_{p,\gamma}^\alpha$ in terms of convergence in the $L_{p,\gamma}$-norm of the "truncated" integrals $D_\varepsilon^\alpha f$ as $\varepsilon \to 0$.

Theorem 3.2 *Let a measure μ satisfies the conditions (3.1) and (3.2) of Theorem 3.1. Then $f \in L_{p,\gamma}^\alpha$, $(1 < p < \infty)$ if and only if $f \in L_{p,\gamma}$ and the family of truncated integrals*

$$(D_\varepsilon^\alpha f)(x,t) = \frac{1}{d(\frac{\alpha}{2},\mu)} \int_\varepsilon^\infty (V_\mu f)(x,t;\eta) \frac{d\eta}{\eta^{1+\alpha/2}}$$

converges in $L_{p,\gamma}$-norm as $\varepsilon \to 0$.

Proof. Let $f \in L_{p,\gamma}$ and the family $D_\varepsilon^\alpha f$ converges in $f \in L_{p,\gamma}$-norm as $\varepsilon \to 0$. Then there exist a constant $c > 0$ such that $\sup\limits_{\varepsilon>0} \|D_\varepsilon^\alpha f\|_{p,\gamma} \le c$ and therefore, by Theorem 3.1, f belongs to $L_{p,\gamma}^\alpha$. Conversely, let $f \in L_{p,\gamma}^\alpha$. Then there exist $g \in L_{p,\gamma}$ such that $f = \mathcal{H}_\gamma^\alpha g$. Using this representation of f we have by (3.5) that

$$(D_\varepsilon^\alpha f)(x,t) = e^{-t}\ \psi_\varepsilon(x,t) \circledast g = \int\limits_{\mathbb{R}_+^n \times (0,\infty)} e^{-\tau} \psi_\varepsilon(y,\tau)\,(T^{y,\tau} g(x,t))\, d\nu(y) d\tau,$$

where the function ψ_ε is defined by (3.6).

By setting $y = \sqrt{\varepsilon} z$, $\tau = \varepsilon\rho$, $d\nu(y)d\tau = \varepsilon^{\nu+\frac{n}{2}+1} d\nu(z) d\rho$, and using (2.10) we have

$$(D_\varepsilon^\alpha f)\,(x,t) = \int\limits_{\mathbb{R}_+^n \times (0,\infty)} e^{-\varepsilon\rho} \psi_1(z,\rho) \left(T^{\sqrt{\varepsilon} z, \varepsilon\rho} g(x,t)\right) d\nu(z) d\rho, \tag{3.12}$$

where the function $\psi_1 = \psi_\varepsilon|_{\varepsilon=1}$. Further, by (2.23) it follows that

$$g(x,t) = \int\limits_{\mathbb{R}_+^n \times (0,\infty)} \psi_1(z,\rho) g(x,t) d\nu(z) d\rho. \tag{3.13}$$

Using (3.12) , (3.13) and Minkowski inequality we get

$$\|D_\varepsilon^\alpha f - g\|_{p,\gamma} \le \int\limits_{\mathbb{R}_+^n \times (0,\infty)} |e^{-\varepsilon\rho} - 1|\, |\psi_1(z,\rho)|\, \|T^{\sqrt{\varepsilon}z,\varepsilon\rho} g(x,t)\|_{p,\gamma}\, d\nu(z) d\rho$$

$$+ \int\limits_{\mathbb{R}_+^n \times (0,\infty)} |\psi_1(z,\rho)|\, \|T^{\sqrt{\varepsilon}z,\varepsilon\rho} g(x,t) - g(x,t)\|_{p,\gamma}\, d\nu(z) d\rho.$$

Now by virtue of (2.4), (2.5) and the Lebesgue theorem on dominated convergence, it follows that $\lim\limits_{\varepsilon\to 0} \|D_\varepsilon^\alpha f - g\|_{p,\gamma} = 0$. The proof is completed. □

Remark 3.3 Take a measure $\mu = \sum\limits_{k=0}^{l} (-1)^k \binom{l}{k} \delta_k$, where $l > \frac{\alpha}{2}$ and $\delta_k = \delta_k(t)$ is the unit mass at $t = k$, $(k = 0, 1, \dots, l)$, that is $\langle \delta_k, w\rangle = w(k)$, $(k = 0, 1, \dots, l)$. It is well known that (see, e.g. [12], p.116-117)

$$\int\limits_0^\infty t^m d\mu(t) \equiv \sum_{k=0}^{l} (-1)^k \binom{l}{k} k^m = 0, \quad \forall m = 0, 1, \dots, l-1.$$

It is also clear that $|\mu|(\mathbb{R}^1) < \infty$ and supp $\mu \subset [0,\infty)$. Thus the measure μ satisfies all the conditions of Theorem 3.1.

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