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The Basis Number of the Semi-Composition Product of Some Graphs I

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Abstract

The basis number of a graph G is defined to be the least integer d such that there is a basis $\mathcal{B}$ of the cycle space of G such that each edge of G is contained in at most d members of $\mathcal{B}$. We investigate the basis number of the semi-composition product of two paths and a cycle with a path.

Key Words: Basis number; cycle space; fold; semi-composition product.

1. Introduction and Definitions

Graph products have been the impetus for several areas of research. The revival of interest seems to be mostly due to the algorithmic point of view and how particular graphical parameters interact with graph products. In recent years, there was a growing literature on the basis number of graphs. Even more recently, the most attention has been given to the basis number of graphs obtained from graph products. We refer the readers to the papers [1], [2], [3], [4], [7], [8], [9].

All graphs under consideration are undirected, finite and simple. Our terminology and notations will be standard except as indicated. For undefined terms, see [5]. We use the symbols $V(G)$ and $E(G)$, respectively, to denote the vertex set and edge set of G. Given a graph G, let $e_1, e_2, \ldots, e_{|E(G)|}$ be an ordering of its edges. Then a subset S of $E(G)$ corresponds to a $(0,1)$-vector $(b_1, b_2, \ldots, b_{|E(G)|})$ in the usual way with $b_i = 1$ if $e_i \in S$,

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and $b_i = 0$ if $e_i \notin S$. These vectors form an $|E(G)|$-dimensional vector space, denoted by $(Z_2)^{|E(G)|}$, over the field of integer numbers modulo 2. The vectors in $(Z_2)^{|E(G)|}$ which correspond to the cycles in G generate a subspace called the cycle space of G and denoted by $\mathcal{C}(G)$. We shall say that the cycles themselves, rather than the vectors corresponding to them, generate $\mathcal{C}(G)$. It is known that for a connected graph G

$$\dim \mathcal{C}(G) = |E(G)| - |V(G)| + 1. \tag{1}$$

Given any spanning tree T of G, every graph $T + e, e \notin T$, consists exactly one cycle C_e, and the collection of cycles $\{C_e : e \notin T\}$ forms a basis of $\mathcal{C}(G)$, called the fundamental basis corresponding to T. One can observe that each edge outside of T occurs in exactly one cycle of this basis, but each edge of T itself may occur in many cycles of the basis. This led Schmeichel [11]. to formally introduce the following definition:

Definition 1.1 *A basis $\mathcal{B}$ for $\mathcal{C}(G)$ is called a d-fold if each edge of G occurs in at most d of the cycles in the basis $\mathcal{B}$. The basis number $b(G)$ of G is the least non-negative integer d such that $\mathcal{C}(G)$ has a d-fold basis. The required basis of G is a basis $\mathcal{B}$ of $b(G)$-fold.*

We now give the definitions of the following four graph products.

Definition 1.2 *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. (1) The cartesian product $G^* = G \times H$ has the vertex set $V(G^*) = V(G) \times V(H)$ and the edge set $E(G^*) = \{(u_1, v_1)(u_2, v_2)|u_1u_2 \in E(G)$ and $v_1 = v_2$, or $u_1 = u_2$ and $v_1v_2 \in E(H)\}$. (2) The lexicographic (or composition) product $G^* = G[H]$ has the vertex set $V(G^*) = V(G) \times V(H)$ and edge set the $E(G^*) = \{(u_1, v_1)(u_2, v_2)|u_1u_2 \in E(G)$, or $u_1 = u_2$ and $v_1v_2 \in E(H)\}$. (3) The direct product $G^* = G \wedge H$ has the vertex set $V(G^*) = V(G) \times V(H)$ and the edge set $E(G^*) = \{(u_1, v_1)(u_2, v_2)|v_1v_2 \in E(H)$ and $u_1u_2 \in E(G)\}$. (4) The semi-composition product $G^* = G \odot H$ has the vertex set $V(G^*) = V(G) \times V(H)$ and the edge set $E(G^*) = \{(u_1, v_1)(u_2, v_2)|u_1 = u_2$ and $v_1v_2 \in E(H)$ or $u_1u_2 \in E(G)$ and $v_1v_2 \notin E(H)\}$* (see Figure 1). Note that $E(G \odot H) = E(G[H]) - E(G \wedge H)$.

One can notice that the cartesian and the direct products is commutative but the lexicographic and the semi-composition products are not commutative. Moreover,

$$\begin{aligned} d_{G\odot H}(u, v) &= |V(H)|d_G(u) + d_H(v) - d_G(u)d_H(v), \\ |E(G \odot H)| &= |E(G)||V(H)|^2 + |V(G)||E(H)| - 2|E(G)||E(H)|, \end{aligned}$$

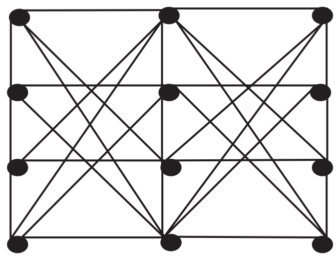

Figure 1. This is an example to explain the definition of the semi-composition product of two paths of order 3 and 4, $P_3 \odot P_4$.

where $d_G(x)$ is the degree of the vertex x in the graph G.

The first important result in the basis number was given by MacLane, in 1937 [10], who proved that $b(G) \leq 2$ if and only if G is non planar. In 1981 [11], Schmeichel proved that $b(P_2[N]) \leq 4$ where P_2 is a path of order two and N is a null graph.

The purpose of this paper is to find the basis number of the semi-composition product of two paths and a cycle with a path.

Throughout this work, for $B \subset \mathcal{C}(G)$, $f_B(e)$ stands for the number of cycles in B containing the edge e, $\mathcal{C}(B)$ stands for the subspace of $\mathcal{C}(G)$ generated by B and $E(B) = \cup_{c \in B} E(c)$.

2. Main Results

In this section, we investigate the basis number of the semi-composition product of two paths and a cycle with a path. In fact, we show, under some restrictions on their orders, the basis number is 4. Let $P_2 = ab$ be a path of order 2 and $U = \{u_1, u_2, ..., u_n\}$ be a set of vertices. Then the Schemichel basis, $\mathcal{B}$, (see [11], Theorem 2.4) of the cycle space $\mathcal{C}(P_2[N])$, where N is the null graph with vertex set U, is defined as follows:

$$\mathcal{B} = \{(a, u_j)(b, u_l)(a, u_{j+1})(b, u_{l+1})(a, u_j) : 1 \leq j, l \leq n-1\}.$$

Now, Let

$$\mathcal{A}_{ab}^{(1)} = \{(a, u_j)(b, u_l)(a, u_{j+1})(b, u_{l+1})(a, u_j) : 1 \leq j, l \leq n-1 \text{ and } |j-l| > 2\}.$$

Then the following assertions are easy to see (1) If $e = (a, u_j)(b, u_j)$, then $f_{\mathcal{A}_{ab}^{(1)}}(e) = 0$. (2) If $e = (a, u_1)(b, u_n)$ or $e = (a, u_n)(b, u_1)$ or $(a, u_j)(b, u_{j+2})$, or $(a, u_{j+2})(b, u_j)$, then $f_{\mathcal{A}_{ab}^{(1)}}(e) \leq 1$. (3) If $e = (a, u_1)(b, u_j)$ or $(a, u_j)(b, u_1)$ or $(a, u_j)(b, u_n)$ or$(a, u_j)(b, u_n)$, then $f_{\mathcal{A}_{ab}^{(1)}}(e) \leq 2$. (4) If $e = (a, u_j)(b, u_{j+3})$ or $(a, u_{j+3})(b, u_j)$, then $f_{\mathcal{A}_{ab}^{(1)}}(e) \leq 3$. (5) If $e \in E(\mathcal{A}_{ab}^{(1)})$ and is not any of the above forms, then $f_{\mathcal{A}_{ab}^{(1)}}(e) \leq 4$. Moreover, $|\mathcal{A}_{ab}^{(1)}| = |\mathcal{B}| - (5(n-5) + 14) = (n-3)(n-4)$.

The following lemma follows immediately from being that $\mathcal{A}_{ab}^{(1)} \subseteq \mathcal{B}$.

Lemma 2.1 *$\mathcal{A}_{ab}^{(1)}$ is a linearly independent set of cycles.*

We now define the following sets of 4-cycles:

$$\begin{aligned} \mathcal{A}_{ab}^{(2)} &= \{\mathcal{A}_2^{(j)} = (a, u_j)(b, u_j)(a, u_{j+2})(b, u_{j+2})(a, u_j) : 1 \leq j \leq n-2\}, \\ \mathcal{A}_{ab}^{(3)} &= \{\mathcal{A}_3^{(j)} = (b, u_n)(a, u_{n-j})(b, u_{n-j})(a, u_{n-j-2})(b, u_n) : 2 \leq j \leq n-3\}. \end{aligned}$$

Note that if $n < 5$, then $\mathcal{A}_{ab}^{(3)}$ contains no cycles.

Lemma 2.2 *$\mathcal{A}_{ab}^{(2)} \cup \mathcal{A}_{ab}^{(3)}$ is a linearly independent set of cycles.*

Proof. Since each cycle $\mathcal{A}_3^{(j)}$ contains $(a, u_{n-j})(b, u_{n-j})$, which is not in any other cycle of $\mathcal{A}_{ab}^{(3)}, \mathcal{A}_{ab}^{(3)}$ is linearly independent. Similarly, each cycle $\mathcal{A}_2^{(j)}$ contains $(b, u_j)(a, u_{j+2})$ which is not in any other cycle of $\mathcal{A}_{ab}^{(2)} \cup \mathcal{A}_{ab}^{(3)}$. Therefore, $\mathcal{A}_{ab}^{(2)} \cup \mathcal{A}_{ab}^{(3)}$ is a linearly independent set of cycles. The proof is complete. □

Lemma 2.3 *$\mathcal{C}(\mathcal{A}_{ab}^{(1)} \cup \mathcal{A}_{ab}^{(2)} \cup \mathcal{A}_{ab}^{(3)})$ is the direct sum of $\mathcal{C}(\mathcal{A}_{ab}^{(1)})$* and $\mathcal{C}(\mathcal{A}_{ab}^{(2)} \cup \mathcal{A}_{ab}^{(3)})$.

Proof. To prove the lemma, by Lemma 2.1 and Lemma 2.2, it suffices to show that any element of $\mathcal{C}(\mathcal{A}_{ab}^{(2)} \cup \mathcal{A}_{ab}^{(3)})$ can not be written as a linear combination of cycles from $\mathcal{A}_{ab}^{(1)}$. Let C be a nontrivial element of $\mathcal{C}(\mathcal{A}_{ab}^{(2)} \cup \mathcal{A}_{ab}^{(3)})$. Then $C = \sum_{j=1}^{k_1} \mathcal{A}_2^{(2_j)} + \sum_{j=1}^{k_2} \mathcal{A}_3^{(3_j)}$ (mod2) such that at least one of k_1 and k_2 is non zero. We consider two cases:

Case 1. C contains at least one edge of the form $(a, u_f)(b, u_f)$. Then C can not be written as a linear combination of cycles of $\mathcal{A}_{ab}^{(1)}$ because no cycle of $\mathcal{A}_{ab}^{(1)}$ contains such an edge.

Case 2. C contains no edges of the form $(a, u_f)(b, u_f)$. Since any linear combination of cycles of $\mathcal{A}_{ab}^{(3)}$ (or $\mathcal{A}_{ab}^{(2)}$) must contain at least one edge of the form $(a, u_f)(b, u_f)$ for some

f, as a result both of $k_1, k_2 \geq 1$. Now, $\mathcal{A}_{ab}^{(1)} \subseteq \mathcal{B}$, $\mathcal{C}(\mathcal{A}_{ab}^{(2)} \cup \mathcal{A}_{ab}^{(3)}) \subseteq \mathcal{C}(P_2[N])$ and $\mathcal{B}$ is a basis for $\mathcal{C}(P_2[N])$. Thus, to show that C can not be written as a linear combination of $\mathcal{A}_{ab}^{(1)}$, it suffices to show that the unique combination of C from cycles of $\mathcal{B}$ must contain at least one element of $\mathcal{B}-\mathcal{A}_{ab}^{(1)}$. Now each cycle of $\{\mathcal{A}_2^{(2_j)}\}_{j=1}^{k_1} \cup \{\mathcal{A}_3^{(3_j)}\}_{j=1}^{k_2}$ can be written uniquely as a linear combination of cycles of $\mathcal{B}$ as follows: For each $j = 1, 2, \ldots k_1$,

$$\mathcal{A}_2^{(2_j)} = S_1^{(2_j)} + S_2^{(2_j)} + S_3^{(2_j)} + S_4^{(2_j)} (\text{mod}2),$$

where

$$\begin{aligned}
S_1^{(2_j)} &= (a, u_{2_j})(b, u_{2_j})(a, u_{2_j+1})(b, u_{2_j+1})(a, u_{2_j}),\\
S_2^{(2_j)} &= (a, u_{2_j+1})(b, u_{2_j+1})(a, u_{2_j+2})(b, u_{2_j+2})(a, u_{2_j+1}),\\
S_3^{(2_j)} &= (a, u_{2_j})(b, u_{2_j+1})(a, u_{2_j+1})(b, u_{2_j+2})(a, u_{2_j}),\\
S_4^{(2_j)} &= (b, u_{2_j})(a, u_{2_j+1})(b, u_{2_j+1})(a, u_{2_j+2})(b, u_{2_j}).
\end{aligned}$$

And for each $j = 1, 2, \ldots, k_2$,

$$\mathcal{A}_3^{(3_j)} = \sum_{l=n-3_j}^{n-1} (S_{1_l}^{(3_j)} + S_{2_l}^{(3_j)})$$

where

$$\begin{aligned}
S_{1_l}^{(3_j)} &= (a, u_{n-3_j-2})(b, u_l)(a, u_{n-3_j-1})(b, u_{l+1})(a, u_{n-3_j-2}),\\
S_{2_l}^{(3_j)} &= (a, u_{n-3_j-1})(b, u_l)(a, u_{n-3_j})(b, u_{l+1})(a, u_{n-3_j-2}).
\end{aligned}$$

Note that $(a, u_{2_t+2})(b, u_{2_t}) \in E(S_4^{(2_t)})$ and $(a, u_{2_t+2})(b, u_{2_t}) \notin \cup_{j=1,\, j\neq t}^{k_1} \cup_{i=1}^{4} E(S_i^{(2_j)})$. Moreover, $(a, u_{2_t+2})(b, u_{2_t}) \notin \cup_{j=1}^{k_2} \cup_{l=n-3_j}^{n-1} (E(S_{1_l}^{(3_j)}) \cup E(S_{2_l}^{(3_j)}))$. Thus, $S_4^{(2_t)}$ appears only in the combination of $\mathcal{A}_2^{(2_t)}$. Therefore, $S_4^{(2_t)}$ appears in the combination of C and is an element of $\mathcal{B} - \mathcal{A}_{ab}^{(1)}$. Hence, C can not be written as a linear combination of cycles of $\mathcal{A}_{ab}^{(1)}$. The proof is complete. $\square$

We now define the following sets of cycles for $n \geq 5$:
$\mathcal{A}_{ab}^{(4)} = \{\mathcal{A}_4^{(j)} = (a,u_j)(a,u_{j+1})(b,u_{j+3})(a,u_j) : 1 \leq j \leq n-3\} \cup$
$\{\mathcal{A}_4^{(n-2)} = (a,u_{n-2})(a,u_{n-1})(b,u_{n-1})(b,u_n)(a,u_{n-2})\}$,
$\mathcal{A}_{ab}^{(5)} = \{\mathcal{A}_5^{(j)} = (a,u_{n-j})(a,u_{n-j-1})(b,u_{n-j-1})(b,u_{n-j-2})(a,u_{n-j}) : 0 \leq j \leq n-3\}$,
$\mathcal{A}_{ab}^{(6)} = \{(b,u_{n-1})(b,u_n)(a,u_2)(a,u_1)(b,u_{n-1})\}$,
$\mathcal{A}_{ab}^{(7)} = \{(b,u_{n-2})(b,u_{n-1})(a,u_1)(b,u_{n-2})\}$.

Lemma 2.4 $\cup_{k=1}^{7}\mathcal{A}_{ab}^{(k)}$ is a linearly independent set of cycles.

Proof. By Lemma 2.3, $\cup_{k=1}^{3}\mathcal{A}_{ab}^{(k)}$ is a linearly independent set of cycles. Since each cycle $\mathcal{A}_4^{(j)}$ contains $(a,u_j)(a,u_{j+1})$ which is not in any other cycle of $\cup_{k=1}^{4}\mathcal{A}_{ab}^{(k)}$, $(\cup_{k=1}^{4}\mathcal{A}_{ab}^{(k)}) - \{\mathcal{A}_4^{(n-2)}\}$ is linearly independent. Now, $\mathcal{A}_{ab}^{(6)}$ contains the edge $(b,u_{n-1})(b,u_n)$ which is not in any other cycle of $(\cup_{k=1}^{4}\mathcal{A}_{ab}^{(k)}) - \{\mathcal{A}_4^{(n-2)}\}$. Thus, $(\cup_{k=1}^{4}\mathcal{A}_{ab}^{(k)}) \cup \mathcal{A}_{ab}^{(6)} - \{\mathcal{A}_4^{(n-2)}\}$ is linearly independent. The cycle $\mathcal{A}_4^{(n-2)}$ contains the edge $(a,u_{n-2})(a,u_{n-1})$ which is not in any cycle of $(\cup_{k=1}^{4}\mathcal{A}_{ab}^{(k)}) \cup \mathcal{A}_{ab}^{(6)} - \{\mathcal{A}_4^{(n-2)}\}$. Hence, $(\cup_{k=1}^{4}\mathcal{A}_{ab}^{(k)}) \cup \mathcal{A}_{ab}^{(6)}$ is linearly independent. The cycle $\mathcal{A}_{ab}^{(7)}$ contains the edge $(b,u_{n-1})(b,u_{n-2})$ which is not in any cycle of $(\cup_{k=1}^{4}\mathcal{A}_{ab}^{(k)}) \cup \mathcal{A}_{ab}^{(6)}$. Hence, $(\cup_{k=1}^{4}\mathcal{A}_{ab}^{(k)}) \cup \mathcal{A}_{ab}^{(6)} \cup \mathcal{A}_{ab}^{(7)}$ is linearly independent. The cycle $\mathcal{A}_5^{(0)}$ contains the edge $(a,u_n)(a,u_{n-1})$ which is not in any cycle of $(\cup_{k=1}^{4}\mathcal{A}_{ab}^{(k)}) \cup \mathcal{A}_{ab}^{(6)} \cup \mathcal{A}_{ab}^{(7)}$. Thus, $(\cup_{k=1}^{4}\mathcal{A}_{ab}^{(k)}) \cup \mathcal{A}_{ab}^{(6)} \cup \mathcal{A}_{ab}^{(7)} \cup \{\mathcal{A}_5^{(0)}\}$ is linearly independent. Similarly, for each $j \geq 1$, the cycle $\mathcal{A}_5^{(j)}$ contains $(b,u_{n-j-1})(b,u_{n-j-2})$ which is not in any cycle of $(\cup_{k=1}^{4}\mathcal{A}_{ab}^{(k)}) \cup \mathcal{A}_{ab}^{(6)} \cup \mathcal{A}_{ab}^{(7)} \cup \{\mathcal{A}_5^{(0)}\}$. Therefore, $(\cup_{k=1}^{7}\mathcal{A}_{ab}^{(k)})$ is linearly independent set of cycles. The proof is complete. □

In the following work, for the simplicity, the edge $e = (a,u_j)(b,u_i) \in P_2[N]$ is said to be of length $l(e) = |i-j|$ for any i, j. Set

$$\mathcal{A}_{ab}^{(8)} = \{\mathcal{A}_8^{(j)} = (b,u_j)(b,u_{j+1})(a,u_{j+3})(b,u_j) : 1 \leq j \leq n-3\}.$$

Lemma 2.5 $\mathcal{C}(\cup_{k=1}^{8}\mathcal{A}_{ab}^{(k)} - \{\mathcal{A}_2^{(n-2)}, \mathcal{A}_5^{(0)}, \mathcal{A}_8^{(n-3)}, \mathcal{A}_8^{(1)}\})$ is the direct sum of $\mathcal{C}(\cup_{k=1}^{7}\mathcal{A}_{ab}^{(k)} - \{\mathcal{A}_2^{(n-2)}, \mathcal{A}_5^{(0)}\})$ and $\mathcal{C}(\mathcal{A}_{ab}^{(8)} - \{\mathcal{A}_8^{(n-3)}, \mathcal{A}_8^{(1)}\})$.

Proof. Since $E(\mathcal{A}_8^{(i)}) \cap E(\mathcal{A}_8^{(k)}) = \phi$ for each $i \neq k$, $\mathcal{A}_{ab}^{(8)}$ is linearly independent. Therefore, it suffices to show that any linear combination C of $\mathcal{A}_{ab}^{(8)} - \{\mathcal{A}_8^{(n-3)}, \mathcal{A}_8^{(1)}\}$ can not be written as a linear combination of cycles of $(\cup_{k=1}^{7}\mathcal{A}_{ab}^{(k)}) - \{\mathcal{A}_2^{(n-2)}, \mathcal{A}_5^{(0)}\}$. Let C be a linear combination of cycles of $\mathcal{A}_{ab}^{(8)} - \{\mathcal{A}_8^{(n-3)}, \mathcal{A}_8^{(1)}\}$. Then C is either a cycle or an edge disjoint union of cycles each of which is a cycle of $\mathcal{A}_{ab}^{(8)} - \{\mathcal{A}_8^{(n-3)}, \mathcal{A}_8^{(1)}\}$. Therefore, C contains an edge of the form $e_0 = (b, u_j)(a, u_{j+3})$, which is of a longest length in C, for some $2 \leq j \leq n-4$. Thus, if C is a sum modulo 2 of some cycles of $(\cup_{k=1}^{7}\mathcal{A}_{ab}^{(k)}) - \{\mathcal{A}_2^{(n-2)}, \mathcal{A}_5^{(0)}\}$, say of $R = \{R_1, R_2, \ldots, R_s\}$, then there must be at least one cycle of R, say R_1, contains this edge. Note that there are at most three cycles of $(\cup_{k=1}^{7}\mathcal{A}_{ab}^{(k)}) - \{\mathcal{A}_2^{(n-2)}, \mathcal{A}_5^{(0)}\}$ contain this edge, we list them as follows:

$R_1^{(1)} = (b, u_j)(a, u_{j+3})(b, u_{j+1})(a, u_{j+4})(b, u_j)$,

$R_1^{(2)} = (b, u_{j-1})(a, u_{j+2})(b, u_j)(a, u_{j+3})(b, u_{j-1})$,

$R_1^{(3)} = (b, u_{j-1})(a, u_{j+3})(b, u_j)(a, u_{j+4})(b, u_{j-1})$.

Therefore, R_1 must be one of $R_1^{(1)}, R_1^{(2)}$ and $R_1^{(3)}$. To this end, we choose R_1 to be the cycle of $R_1^{(1)}, R_1^{(2)}$ and $R_1^{(3)}$ which belongs to R and has an edge of a longest length among the edges of $R_1^{(1)} \cup R_1^{(2)} \cup R_1^{(3)}$. Thus, only one of the following holds:

(i) $R_1^{(3)} \in R$. Then we choose $R_1 = R_1^{(3)}$. Note that $(a, u_{j+4})(b, u_{j-1}) \in E(R_1)$ which is the longest edge of R_1. Moreover, $(a, u_{j+4})(b, u_{j-1}) \notin E(C)$.

(ii) $R_1^{(3)} \notin R$ and $R_1^{(2)} \in R$. Then we choose $R_1 = R_1^{(2)}$. Note that $(a, u_{j+3})(b, u_{j-1}) \in E(R_1)$ which is the longest edge of R_1. Moreover, $(a, u_{j+3})(b, u_{j-1}) \notin E(C)$.

(iii) $R_1^{(3)}, R_1^{(2)} \notin R$ and $R_1^{(1)} \in R$. Then we choose $R_1 = R_1^{(1)}$. Note that $(a, u_{j+4})(b, u_j) \in E(R_1)$ which is the longest edge of R_1. Moreover, $(a, u_{j+4})(b, u_j) \notin E(C)$.

We notice that in either of the above (i), (ii) and (iii) holds, we get the following: R_1 contains an edge of the longest length of the form $e_1 = (a, u_{k_1})(b, u_{j_1})$ where $k_1 \geq j+3$, $j \geq j_1$, $l(e_1) \geq 4$ and $l(e_1) > l(e_0)$. Moreover, $e_1 \notin E(C)$. Thus, without loss of generality, we may assume that any of (i), (ii) and (iii) holds, say (i).

Since $e_1 = (a, u_{j+4})(b, u_{j-1}) \in E(R_1)$ and $e_1 \notin E(C)$, as a result there must be a cycle of R, say R_2, which contains e_1 as an edge. Now, e_1 belongs to at most four cycles of $(\cup_{k=1}^{7}\mathcal{A}_{ab}^{(k)}) - \{\mathcal{A}_2^{(n-2)}, \mathcal{A}_5^{(0)}\}$, we list them as follows:

$R_2^{(1)} = (b, u_{j-1})(a, u_{j+3})(b, u_j)(a, u_{j+4})(b, u_{j-1})$,

$R_2^{(2)} = (b, u_{j-1})(a, u_{j+4})(b, u_j)(a, u_{j+5})(b, u_{j-1})$,

$R_2^{(3)} = (b, u_{j-2})(a, u_{j+3})(b, u_{j-1})(a, u_{j+4})(b, u_{j-2})$,

$R_2^{(4)} = (b, u_{j-2})(a, u_{j+4})(b, u_{j-1})(a, u_{j+5})(b, u_{j-2})$.

Since $R_2^{(1)} = R_1$, R_2 must be one of $R_2^{(2)}, R_2^{(3)}$ and $R_2^{(4)}$. As in the above, we choose the cycle which belongs to R and has a longest edge. Thus, only one of the following holds:

(I) $R_2^{(4)} \in R$. Then we choose $R_2 = R_2^{(4)}$. Note that $(a, u_{j+5})(b, u_{j-2})$ is the longest edge in $R_1 + R_2$ (mod2) which is longer than the longest edge in R_1 by 2. Moreover, $(a, u_{j+5})(b, u_{j-2}) \notin E(C)$.

(II) $R_2^{(4)} \notin R$ and $R_2^{(3)} \in R$. Then we choose $R_2 = R_2^{(3)}$. Note that $(a, u_{j+4})(b, u_{j-2})$ is the longest edge in $R_1 + R_2$ (mod2) which is longer than the longest edge in R_1 by 1. Moreover, $(a, u_{j+4})(b, u_{j-2}) \notin E(C)$

(III) $R_2^{(4)}, R_2^{(3)} \notin R$ and $R_2^{(2)} \in R$. Then we choose $R_2 = R_2^{(2)}$. Note that $(a, u_{j+5})(b, u_{j-1})$ is the longest edge in $R_1 + R_2$ (mod2) which is longer than the longest edge in R_1 by 1. Moreover, $(a, u_{j+5})(b, u_{j-1}) \notin E(C)$.

We notice that in either of the above (I), (II) and (III) holds, we get the following. $R_1 + R_2$ (mod2) contains an edge of the longest length of the form $e_2 = (a, u_{k_2})(b, u_{j_2})$ where $k_2 \geq k_1$, $j_1 \geq j_2$ and $l(e_2) > l(e_1) > l(e_0)$. Moreover, $e_2 \notin E(C)$. Thus, without loss of generality, we may assume that any of (I), (II) and (III) holds, say (I).

Since $e_2 = (a, u_{j+5})(b, u_{j-2})$ is an edge of $R_1 + R_2$ (mod2) and $e_2 \notin E(C)$, as a result there must be a cycle of R, say R_3, which contains e_2 as an edge. Now, e_2 belongs to at most four cycles of $(\cup_{k=1}^{7} \mathcal{A}_{ab}^{(k)}) - \{\mathcal{A}_2^{(n-2)}, \mathcal{A}_5^{(0)}\}$. By using the same arguments as in the above cases, we get that the longest edge of $R_1 + R_2 + R_3$ (mod2) has the form $e_3 = (a, u_{k_2})(b, u_{j_2})$ where $k_3 \geq k_2$, $j_2 \geq j_3$ and $l(e_3) > l(e_2) > l(e_1) > l(e_0)$. Moreover, $e_3 \notin E(C)$.

By continuing in this process, there must be the least integer $r \leq s$ such that $R_r \in R$ and the longest edge of $R_r + R_{r-1} + \cdots + R_1$ (mod2), which is of R_r, has the form $e_r = (a, u_h)(b, u_1)$ or $(a, u_n)(b, u_g)$ where $h \geq 5$ and $g \leq n-4$. Moreover, $l(e_r) > l(e_{r-1}) > \ldots > l(e_0)$. Now we consider two cases:

Case a. $(a, u_h)(b, u_1)$ is the longest edge of $R_r + R_{r-1} + \cdots + R_1$ (mod2). Note that, $(a, u_h)(b, u_1)$ belongs to exactly two cycles of $(\cup_{k=1}^{7} \mathcal{A}_k) - \{\mathcal{A}_2^{(n-2)}, \mathcal{A}_5^{(0)}\}$ we list them as follows:

$R_{r+1}^{(1)} = (b, u_1)(a, u_h)(b, u_2)(a, u_{h+1})(b, u_1)$,

$R_{r+1}^{(2)} = (b, u_1)(a, u_{h-1})(b, u_2)(a, u_h)(b, u_1)$.

Since $(a, u_h)(b, u_1)$ is an edge of $R_r + R_{r-1} + \cdots + R_1$ (mod2), R_r must be one of $R_{r+1}^{(1)}$ and $R_{r+1}^{(2)}$. If $R_{r+1}^{(1)} = R_r$, then $(a, u_{h+1})(b, u_1)$ is an edge of $R_r + R_{r-1} + \cdots + R_1$ (mod2) which is longer than $(a, u_h)(b, u_1)$, a contradiction. Therefore, $R_r = R_{r+1}^{(2)}$. Since $(a, u_h)(b, u_1)$ is an edge of $R_r + R_{r-1} + \cdots + R_1$ (mod2) and $(a, u_h)(b, u_1) \notin E(C)$, $R_{r+1}^{(1)} \in R$, say $R_{r+1} = R_{r+1}^{(1)}$. Thus, $(a, u_{h+1})(b, u_1)$ is an edge of $R_{r+1} + R_r + R_{r-1} + \cdots + R_1$ (mod2) and not in $E(C)$. Since $(a, u_{h+1})(b, u_1)$ belongs to exactly two cycles of $(\cup_{k=1}^{7} \mathcal{A}_{ab}^{(k)}) - \{\mathcal{A}_2^{(n-2)}, \mathcal{A}_5^{(0)}\}$, in fact, in R_{r+1} and in $R_{r+2}^{(1)} = (b, u_1)(a, u_{h+1})(b, u_2)(a, u_{h+2})(b, u_1)$. Thus, $R_{r+2}^{(1)}$ must be in R. say $R_{r+2} = R_{r+2}^{(1)}$. So, $(a, u_{h+2})(b, u_1)$ is an edge of $R_{r+2} + R_{r+1} + \cdots + R_1$ (mod2), but $(a, u_{h+2})(b, u_1) \notin E(C)$. By continuing in this way , there must be an integer z such that $R_z = (b, u_1)(a, u_{n-1})(b, u_2)(a, u_n)(b, u_1) \in R$. Thus, $(a, u_n)(b, u_1) \in R_z + R_{z-1} + \cdots + R_1$ (mod2). Since R_z is the only cycle of $(\cup_{k=1}^{7} \mathcal{A}_{ab}^{(k)}) - \{\mathcal{A}_2^{(n-2)}, \mathcal{A}_5^{(0)}\}$ which contains the edge $(a, u_n)(b, u_1)$, $(a, u_n)(b, u_1) \in R_1 + R_2 + \cdots + R_s$ (mod2). This contradicts the fact that $(a, u_n)(b, u_1) \notin E(C)$.

Case b. $(a, u_n)(b, u_g)$ is the longest edge of $R_r + R_{r-1} + \cdots + R_1$ (mod2). Then we use the same arguments as in Case a, to get the same contradiction. The proof is complete. □

Lemma 2.6 $\cup_{k=1}^{8} \mathcal{A}_{ab}^{(k)}$ is a linearly independent set of cycles.

Proof. By the proof of Lemma 2.5, $(\cup_{k=1}^{8} \mathcal{A}_{ab}^{(k)}) - \{\mathcal{A}_2^{(n-2)}, \mathcal{A}_5^{(0)}, \mathcal{A}_8^{(1)}, \mathcal{A}_8^{(n-3)}\}$ is linearly independent. $\mathcal{A}_8^{(n-3)}$ contains the edge $(b, u_{n-2})(a, u_n)$ which is not in any cycle of $(\cup_{k=1}^{8} \mathcal{A}_{ab}^{(k)}) - \{\mathcal{A}_2^{(n-2)}, \mathcal{A}_5^{(0)}, \mathcal{A}_8^{(1)}, \mathcal{A}_8^{(n-3)}\}$. Thus $(\cup_{k=1}^{8} \mathcal{A}_{ab}^{(k)}) - \{\mathcal{A}_2^{(n-2)}, \mathcal{A}_5^{(0)}, \mathcal{A}_8^{(1)}\}$ is linearly independent. $\mathcal{A}_5^{(0)}$ contains the edge $(a, u_{n-1})(a, u_n)$ which is not in any cycle of $(\cup_{k=1}^{8} \mathcal{A}_{ab}^{(k)}) - \{\mathcal{A}_2^{(n-2)}, \mathcal{A}_5^{(0)}, \mathcal{A}_8^{(1)}\}$. Thus, $(\cup_{k=1}^{8} \mathcal{A}_{ab}^{(k)}) - \{\mathcal{A}_2^{(n-2)}, \mathcal{A}_8^{(1)}\}$ is linearly independent. $\mathcal{A}_2^{(n-2)}$ contains the edge $(a, u_n)(b, u_n)$ which is not in any cycle of $(\cup_{k=1}^{8} \mathcal{A}_{ab}^{(k)}) - \{\mathcal{A}_2^{(n-2)}, \mathcal{A}_8^{(1)}\}$. Thus, $(\cup_{k=1}^{8} \mathcal{A}_{ab}^{(k)}) - \{\mathcal{A}_8^{(1)}\}$ is linearly independent. Finally, assume that $\mathcal{A}_8^{(1)}$ is a linear combination of cycles from $(\cup_{k=1}^{8} \mathcal{A}_{ab}^{(k)}) - \{\mathcal{A}_8^{(1)}\}$, say $T = \{T_1, T_2, \ldots, T_y\}$. Since $\mathcal{A}_8^{(1)}$contains the edge $(b, u_1)(b, u_2)$ and the only cycle of $(\cup_{k=1}^{8} \mathcal{A}_{ab}^{(k)}) - \{\mathcal{A}_8^{(1)}\}$ contains such an edge is $\mathcal{A}_5^{(n-3)}$, as a result $\mathcal{A}_5^{(n-3)}$ must belong to

T , say $T_1 = \mathcal{A}_5^{(n-3)}$. Since $(a,u_3)(b,u_1) \in E(\mathcal{A}_5^{(n-3)})$ and $(a,u_3)(b,u_1) \notin E(\mathcal{A}_8^{(1)})$ and since the only cycle of $(\cup_{k=1}^{8} \mathcal{A}_{ab}^{(k)}) - \{\mathcal{A}_5^{(n-3)}\}$ contains such an edge is $\mathcal{A}_2^{(1)}$, it implies that $\mathcal{A}_2^{(1)} \in T$, say $\mathcal{A}_2^{(1)} = T_2$. Since $(a,u_1)(b,u_1) \in \mathcal{A}_2^{(1)}$ and no other cycles of $\cup_{k=1}^{8} \mathcal{A}_{ab}^{(k)}$ contain such an edge, we have that $(a,u_1)(b,u_1)$ is an edge of $T_1 + T_2 + \ldots + T_y$ (mod2) but $(a,u_1)(b,u_1) \notin E(\mathcal{A}_8^{(1)})$. That is a contradiction. The proof is complete. □

Now, let $P_m = a_1a_2\ldots a_m$ and $P_n = u_1u_2\ldots u_n$ be two paths with m,n vertices. Then, the following lemma is a straightforward from equation (1.1) and noting that $|E(P_m \odot P_n)| = (m-1)n^2 - (n-1)(m-2)$ and $|V(P_m \odot P_n)| = mn$.

Lemma 2.7 $\dim C(P_m \odot P_n) = (n-1)^2(m-1)$.

Lemma 2.8 For any integer $n \geq 2$, $b(P_2 \odot P_n) \leq 4$.

Proof. To prove the lemma, it suffices to exhibit a 4-fold basis. The lemma is clear for $n = 2, 3$, and 4. For $n \geq 5$, define $\mathcal{B}(P_2 \odot P_n) = \cup_{k=1}^{8} \mathcal{A}_{ab}^{(k)}$. By Lemma 2.6, $\mathcal{B}(P_2 \odot P_n)$ is linearly independent. Since

$$|\mathcal{A}_{ab}^{(1)}| = (n-3)(n-4),$$

$$|\mathcal{A}_{ab}^{(2)}| = |\mathcal{A}_{ab}^{(4)}| = |\mathcal{A}_{ab}^{(5)}| = (n-2),$$

$$|\mathcal{A}_{ab}^{(3)}| = (n-4),$$

$$|\mathcal{A}_{ab}^{(6)}| = |\mathcal{A}_{ab}^{(7)}| = 1,$$

and

$$|\mathcal{A}_{ab}^{(8)}| = (n-3),$$

we have,

$$\begin{aligned}|\mathcal{B}(P_2 \odot P_n)| &= |\cup_{k=1}^{8} \mathcal{A}_k| \\ &= \sum_{i=1}^{8} |\mathcal{A}_k| \\ &= (n-3)(n-4) + 3(n-2) + (n-4) + 1 + 1 + (n-3) \\ &= (n-1)^2 \\ &= \dim\ \mathcal{C}(P_2 \odot P_n).\end{aligned}$$

Therefore, $\mathcal{B}(P_2 \odot P_n)$ is a basis of $\mathcal{C}(P_2 \odot P_n)$. We now show that $\mathcal{B}(P_2 \odot P_n)$ is of fold 4. Let $e \in E(P_2 \odot P_n)$. Then (1) If $e = (a, u_j)(b, u_j)$ where $j \neq n-1$, then $f_{\mathcal{A}_{ab}^{(1)}}(e) = f_{\mathcal{A}_{ab}^{(4)}}(e) = f_{\mathcal{A}_{ab}^{(6)}}(e) = f_{\mathcal{A}_{ab}^{(7)}}(e) = f_{\mathcal{A}_{ab}^{(8)}}(e) = 0, f_{\mathcal{A}_{ab}^{(2)}}(e) \leq 2, f_{\mathcal{A}_{ab}^{(3)}}(e) \leq 1, f_{\mathcal{A}_{ab}^{(5)}}(e) \leq 1$. (2) If $e = (a, u_{n-1})(b, u_{n-1})$, then $f_{\mathcal{A}_{ab}^{(1)}}(e) = f_{\mathcal{A}_{ab}^{(3)}}(e) = f_{\mathcal{A}_{ab}^{(6)}}(e) = f_{\mathcal{A}_{ab}^{(7)}}(e) = f_{\mathcal{A}_{ab}^{(8)}}(e) = 0, f_{\mathcal{A}_{ab}^{(2)}}(e) \leq 1, f_{\mathcal{A}_{ab}^{(4)}}(e) = 1, f_{\mathcal{A}_{ab}^{(5)}}(e) \leq 1$. (3) If $e = (a, u_n)(b, u_j)$ or $(b, u_1)(a, u_i)$ or$(a, u_1)(b, u_l)$ or$(b, u_n)(a, u_f)$ for $j < n-3, i > 3, 3 < l < n-1, 2 < j < n-3$, then $f_{\mathcal{A}_{ab}^{(2)}}(e) = f_{\mathcal{A}_{ab}^{(3)}}(e) = f_{\mathcal{A}_{ab}^{(3)}}(e) = f_{\mathcal{A}_{ab}^{(4)}}(e) = f_{\mathcal{A}_{ab}^{(5)}}(e) = f_{\mathcal{A}_{ab}^{(6)}}(e) = f_{\mathcal{A}_{ab}^{(7)}}(e) = f_{\mathcal{A}_{ab}^{(8)}}(e) = 0, f_{\mathcal{A}_{ab}^{(1)}}(e) \leq 4$. (4) If $e = (a, u_1)(b, u_n)$ or $(a, u_2)(b, u_n)$ or$(a, u_1)(b, u_{n-1})$, then $f_{\mathcal{A}_{ab}^{(2)}}(e) = f_{\mathcal{A}_{ab}^{(3)}}(e) = f_{\mathcal{A}_{ab}^{(3)}}(e) = f_{\mathcal{A}_{ab}^{(4)}}(e) = f_{\mathcal{A}_{ab}^{(5)}}(e) = f_{\mathcal{A}_{ab}^{(8)}}(e) = 0, f_{\mathcal{A}_{ab}^{(6)}}(e) = f_{\mathcal{A}_{ab}^{(7)}}(e) = 1, f_{\mathcal{A}_{ab}^{(1)}}(e) \leq 2$. (5) If $e = (a, u_j)(b, u_{j+2})$ for $j < n-2$, then $f_{\mathcal{A}_{ab}^{(5)}}(e) = f_{\mathcal{A}_{ab}^{(6)}}(e) = f_{\mathcal{A}_{ab}^{(7)}}(e) = f_{\mathcal{A}_{ab}^{(8)}}(e) = 0, f_{\mathcal{A}_{ab}^{(1)}}(e) \leq 1, f_{\mathcal{A}_{ab}^{(2)}}(e) = 1, f_{\mathcal{A}_{ab}^{(3)}}(e) \leq 1, f_{\mathcal{A}_{ab}^{(4)}}(e) \leq 1$. (6) If $e = (a, u_{n-2})(b, u_n)$, then $f_{\mathcal{A}_{ab}^{(1)}}(e) = f_{\mathcal{A}_{ab}^{(5)}}(e) = f_{\mathcal{A}_{ab}^{(6)}}(e) = f_{\mathcal{A}_{ab}^{(7)}}(e) = f_{\mathcal{A}_{ab}^{(8)}}(e) = 0, f_{\mathcal{A}_{ab}^{(2)}}(e) \leq 1, f_{\mathcal{A}_{ab}^{(3)}}(e) \leq 1, f_{\mathcal{A}_{ab}^{(4)}}(e) \leq 2$.(7) If $e = (a, u_{j+2})(b, u_j)$, then $f_{\mathcal{A}_{ab}^{(3)}}(e) = f_{\mathcal{A}_{ab}^{(4)}}(e) = f_{\mathcal{A}_{ab}^{(6)}}(e) = f_{\mathcal{A}_{ab}^{(7)}}(e) = 0, f_{\mathcal{A}_{ab}^{(1)}}(e) \leq 1, f_{\mathcal{A}_{ab}^{(2)}}(e) = 1, f_{\mathcal{A}_{ab}^{(5)}}(e) \leq 1,\ f_{\mathcal{A}_{ab}^{(8)}}(e) \leq 1$. (8) If $e = (a, u_j)(b, u_{j+3})$ where $j < n-3$, then $f_{\mathcal{A}_{ab}^{(2)}}(e) = f_{\mathcal{A}_{ab}^{(3)}}(e) = f_{\mathcal{A}_{ab}^{(5)}}(e) = f_{\mathcal{A}_{ab}^{(6)}}(e) = f_{\mathcal{A}_{ab}^{(7)}}(e) = f_{\mathcal{A}_{ab}^{(8)}}(e) = 0, f_{\mathcal{A}_{ab}^{(1)}}(e) \leq 3, f_{\mathcal{A}_{ab}^{(4)}}(e) \leq 1$. (9) If $e = (a, u_{n-3})(b, u_n)$, then $f_{\mathcal{A}_{ab}^{(2)}}(e) = f_{\mathcal{A}_{ab}^{(5)}}(e) = f_{\mathcal{A}_{ab}^{(6)}}(e) = f_{\mathcal{A}_{ab}^{(7)}}(e) = f_{\mathcal{A}_{ab}^{(8)}}(e) = 0, f_{\mathcal{A}_{ab}^{(1)}}(e) \leq 2, f_{\mathcal{A}_{ab}^{(3)}}(e) \leq 1, f_{\mathcal{A}_{ab}^{(4)}}(e) \leq 1$.(10) If $e = (a, u_{j+3})(b, u_j)$ then $f_{\mathcal{A}_{ab}^{(2)}}(e) = f_{\mathcal{A}_{ab}^{(3)}}(e) = f_{\mathcal{A}_{ab}^{(4)}}(e) = f_{\mathcal{A}_{ab}^{(5)}}(e) = f_{\mathcal{A}_{ab}^{(6)}}(e) = f_{\mathcal{A}_{ab}^{(7)}}(e) = 0, f_{\mathcal{A}_{ab}^{(1)}}(e) \leq 3, f_{\mathcal{A}_{ab}^{(8)}}(e) \leq 1$. (11) If $e = (a, u_j)(a, u_{j+1})$ where $j \neq 1$ then $f_{\mathcal{A}_{ab}^{(1)}}(e) = f_{\mathcal{A}_{ab}^{(2)}}(e) = f_{\mathcal{A}_{ab}^{(3)}}(e) = f_{\mathcal{A}_{ab}^{(6)}}(e) = f_{\mathcal{A}_{ab}^{(7)}}(e) = f_{\mathcal{A}_{ab}^{(8)}}(e) = 0, f_{\mathcal{A}_{ab}^{(4)}}(e) =$

$f_{\mathcal{A}_{ab}^{(5)}}(e) = 1$. (12) If $e = (a, u_1)(a, u_2)$, then $f_{\mathcal{A}_{ab}^{(1)}}(e) = f_{\mathcal{A}_{ab}^{(2)}}(e) = f_{\mathcal{A}_{ab}^{(3)}}(e) = f_{\mathcal{A}_{ab}^{(5)}}(e) = f_{\mathcal{A}_{ab}^{(7)}}(e) = f_{\mathcal{A}_{ab}^{(8)}}(e) = 0, f_{\mathcal{A}_{ab}^{(4)}}(e) = f_{\mathcal{A}_{ab}^{(6)}}(e) = 1$.(13) If $e = (b, u_j)(b, u_{j+1})$ where $j \neq n-1, n-2$ then $f_{\mathcal{A}_{ab}^{(1)}}(e) = f_{\mathcal{A}_{ab}^{(2)}}(e) = f_{\mathcal{A}_{ab}^{(3)}}(e) = f_{\mathcal{A}_{ab}^{(4)}}(e) = f_{\mathcal{A}_{ab}^{(6)}}(e) = f_{\mathcal{A}_{ab}^{(7)}}(e) = 0, f_{\mathcal{A}_{ab}^{(5)}}(e) = f_{\mathcal{A}_{ab}^{(8)}}(e) = 1$. (14) If $e = (b, u_{n-2})(b, u_{n-1})$, then $f_{\mathcal{A}_{ab}^{(1)}}(e) = f_{\mathcal{A}_{ab}^{(2)}}(e) = f_{\mathcal{A}_{ab}^{(3)}}(e) = f_{\mathcal{A}_{ab}^{(4)}}(e) = f_{\mathcal{A}_{ab}^{(6)}}(e) = f_{\mathcal{A}_{ab}^{(8)}}(e) = 0, f_{\mathcal{A}_{ab}^{(5)}}(e) = f_{\mathcal{A}_{ab}^{(7)}}(e) = 1$. (15) If $e = (b, u_{n-1})(b, u_n)$, then $f_{\mathcal{A}_{ab}^{(1)}}(e) = f_{\mathcal{A}_{ab}^{(2)}}(e) = f_{\mathcal{A}_{ab}^{(3)}}(e) = f_{\mathcal{A}_{ab}^{(5)}}(e) = f_{\mathcal{A}_{ab}^{(7)}}(e) = f_{\mathcal{A}_{ab}^{(8)}}(e) = 0, f_{\mathcal{A}_{ab}^{(4)}}(e) = f_{\mathcal{A}_{ab}^{(6)}}(e) = 1$. (16) If $e \in E(P_2 \odot P_n)$ and not of any form of the above, then $f_{\mathcal{A}_{ab}^{(2)}}(e) = f_{\mathcal{A}_{ab}^{(3)}}(e) = f_{\mathcal{A}_{ab}^{(4)}}(e) = f_{\mathcal{A}_{ab}^{(5)}}(e) = f_{\mathcal{A}_{ab}^{(6)}}(e) = f_{\mathcal{A}_{ab}^{(7)}}(e) = f_{\mathcal{A}_{ab}^{(8)}}(e) = 0, f_{\mathcal{A}_{ab}^{(1)}}(e) \leq 4$. The proof is complete. □

Theorem 2.1 *For any $n, m \geq 2$, we have $b(P_m \odot P_n) \leq 4$. Moreover, the equality holds if $n \geq 14$ and $m \geq 2$.*

Proof. To prove that $b(P_m \odot P_n) \leq 4$, it suffices to exhibit a 4-fold basis. Let $P_2^{(i)} = a_i a_{i+1}$ for each $i \leq m-1$. Define $\mathcal{B}(P_m \odot P_n) = \cup_{i=1}^{m-1} \mathcal{B}(P_2^{(i)} \odot P_n)$ where $\mathcal{B}(P_2^{(i)} \odot P_n)$ is the basis of $\mathcal{C}(P_2^{(i)} \odot P_n)$ as in Lemma 2.8. We proceed using induction on m to show that $\mathcal{B}(P_m \odot P_n)$ is linearly independent. If $m = 2$, then $\mathcal{B}(P_m \odot P_n) = \mathcal{B}(P_2^{(1)} \odot P_n)$ which is linearly independent, by Lemma 2.8. Now, $\mathcal{B}(P_m \odot P_n) = (\cup_{i=1}^{m-2} \mathcal{B}(P_2^{(i)} \odot P_n)) \cup \mathcal{B}(P_2^{(m-1)} \odot P_n)$. By induction step and Lemma 2.8, each of $\cup_{i=1}^{m-2} \mathcal{B}(P_2^{(i)} \odot P_n)$ and $\mathcal{B}(P_2^{(m-1)} \odot P_n)$ is linearly independent. Since $E(\cup_{i=1}^{m-2}(P_2^{(i)} \odot P_n)) \cap E(P_2^{(m-1)} \odot P_n) = E(a_{m-1} \times P_n)$ which is an edge set of a path, as a result any non trivial linear combination of cycles of $\mathcal{B}(P_2^{(m-1)} \odot P_n)$ can not be written as a linear combination of $\cup_{i=1}^{m-2} \mathcal{B}(P_2^{(i)} \odot P_n)$. Thus, $\mathcal{B}(P_m \odot P_n)$ is a linearly independent set. Since,

$$\begin{aligned} |\mathcal{B}(P_m \odot P_n)| &= \sum_{i=1}^{m-1} |\mathcal{B}(P_2^{(i)} \odot P_n)| \\ &= \sum_{i=1}^{m-1} (n-1)^2 \\ &= (n-1)^2(m-1) \\ &= \dim \mathcal{C}(P_m \odot P_n), \end{aligned}$$

$\mathcal{B}(P_m \odot P_n)$ is a basis of $\mathcal{C}(P_m \odot P_n)$. Let $e \in E(P_m \odot P_n)$. (1) If $e \in E(a_i \times P_n)$ for some $2 \leq i \leq m-1$, then $f_{\mathcal{B}(P_m \odot P_n)}(e) = \sum_{k=1}^{m-1} f_{\mathcal{B}(P_2^{(k)} \odot P_n)}(e) = f_{\mathcal{B}(P_2^{(i-1)} \odot P_n)}(e) + f_{\mathcal{B}(P_2^{(i)} \odot P_n)}(e) \leq 2+2=4$. (2) If $e \in E(a_1 \times P_n)$, then $f_{\mathcal{B}(P_m \odot P_n)}(e) = f_{\mathcal{B}(P_2^{(1)} \odot P_n)}(e) \leq 2$. (3) If $e \in E(a_m \times P_n)$, then $f_{\mathcal{B}(P_m \odot P_n)}(e) = f_{\mathcal{B}(P_2^{(m-1)} \odot P_n)}(e) \leq 2$. (4) If $e \notin E(\cup_{i=1}^{m} a_i \times P_n)$, then e belongs only to $P_2^{(i_0)} \odot P_n$ for some i_0 and so , by Lemma 2.8, $f_{\mathcal{B}(P_m \odot P_n)}(e) = f_{\mathcal{B}(P_2^{(i_0)} \odot P_n)}(e) \leq 4$.

On the other hand, to show that $b(P_m \odot P_n) \geq 4$ for any $n \geq 14$ and $m \geq 2$, we have to exclude any possibility for the cycle space $\mathcal{C}(P_m \odot P_n)$ to have a 3-fold basis for any $n \geq 14$ and $m \geq 2$. Suppose that $\mathcal{B}$ is a 3-fold basis of the cycle space, then we have the following three cases:

Case 1. Suppose that $\mathcal{B}$ consists only of 3-cycles. Then $|\mathcal{B}| \leq 3m(n-1)$ because any 3-cycle must contain an edge of the form $(a_i, u_j)(a_i, u_{j+1})$ for some $1 \leq i \leq m, 1 \leq j \leq n-1$ and each edge is of fold at most 3. That is equivalent to the inequality $(m-1)(n-1)^2 \leq 3m(n-1)$ which implies that $m(n-4) \leq n-1$. But $m \geq 2$, so $n \leq 7$. This is a contradiction.

Case 2. Suppose that $\mathcal{B}$ consists only of cycles of length greater than or equal to 4. Then $4|\mathcal{B}| \leq 3|E(P_m \odot P_n)|$ because the length of each cycle of $\mathcal{B}$ greater than or equal to 4 and each edge is of fold at most 3. Thus, $4(m-1)(n-1)^2 \leq 3(m-1)n^2 - (n-1)(m-2)$ which is equivalent to $m(n^2 - 7n + 3) \leq n^2 - 6n + 2$. But $m \geq 2$, so $n \leq 7$. This is a contradiction.

Case 3. Suppose that $\mathcal{B}$ consists of s 3-cycles and t cycles of length greater than or equal to 4. Then as in Case 1 $s \leq 3m(n-1)$. Since the length of each cycle of s is 3 and each cycle of t is at least 4 and the fold of each edge is at most 3, we have that $4t + 3s \leq 3|E(P_m \odot P_n)|$. But $t = |\mathcal{B}| - s = |E(P_m \odot P_n)| - |V(P_m \odot P_n)| + 1 - s$, so $4t + 3s = 4|E(P_m \odot P_n)| - 4|V(P_m \odot P_n)| + 4 - s \leq 3|E(P_m \odot P_n)|$ which implies that $|E(P_m \odot P_n)| - 4|V(P_m \odot P_n)| + 4 = (m-1)(n-1)^2 - 3nm + 3 \leq s \leq 3m(n-1)$. Thus, $(m-1)(n-1)^2 - 3nm + 3 \leq 3m(n-1)$ which is equivalent to $m(n^2 - 8n + 4) \leq n^2 - 2n - 2$. But $m \geq 2$, so $n \leq 13$. This is a contradiction. The proof is complete. □

Let $C_m = a_1 a_2 \ldots a_m a_1$ be a cycle of order m. Then, the following lemma is a straightforward from equation (1.1) and noting that $|E(C_m \odot P_n)| = mn^2 - m(n-1)$ and $|V(C_m \odot P_n)| = mn$.

Lemma 2.9 $\dim C(C_m \odot P_n) = m(n-1)^2 + 1$.

Theorem 2.2 *For any $n \geq 2, m \geq 3$, we have $b(C_m \odot P_n) \leq 4$. Moreover, the equality holds if $n \geq 8$ and $m \geq 4$.*

Proof. To prove that $b(C_m \odot P_n) \leq 4$, it suffices to exhibit a 4-fold basis. Let $P_2^{(i)} = a_i a_{i+1}$ for each $i \leq m-1$ and $P_2^{(m)} = a_m a_1$. Define $\mathcal{B} = \cup_{i=1}^{m} \mathcal{B}(P_2^{(i)} \odot P_n)$ where $\mathcal{B}(P_2^{(i)} \odot P_n)$ is the basis of $\mathcal{C}(P_2^{(i)} \odot P_n)$ as in Lemma 2.8. By Theorem 2.1, $\cup_{i=1}^{m-1} \mathcal{B}(P_2^{(i)} \odot P_n)$ is a linearly independent set. Since $E(a_1 \times P_m) \cup E(a_m \times P_m)$ is an edge set of a forest, any linear combination of cycles of $\mathcal{B}(P_2^{(m)} \odot P_n)$ must contain at least one edge of the form $(a_1, u_j)(a_m, u_l)$ for some j, l which is not in any cycle of $\cup_{i=1}^{m-1} \mathcal{B}(P_2^{(i)} \odot P_n)$. Thus, $\mathcal{B}$ is linearly independent. Now, consider the following cycle:

$$C = (a_1, u_1)(a_2, u_1) \ldots (a_{m-1}, u_1)(a_m, u_1)(a_1, u_1).$$

We now show that C is independent of cycles of $\cup_{i=1}^{m} \mathcal{B}(P_2^{(i)} \odot P_n)$. Suppose that C is a sum modulo 2 of cycles of $\cup_{i=1}^{m} \mathcal{B}(P_2^{(i)} \odot P_n)$. Then

$$C = \sum_{j=1}^{m} b_j \pmod 2$$

where b_j is a linear combination of cycles of $\mathcal{B}(P_2^{(j)} \odot P_n)$. Thus,

$$b_1 = C + \sum_{j=2}^{m} b_j \pmod 2.$$

Therefore,

$$E(b_1) = E(C \oplus b_2 \oplus \cdots \oplus b_m) \subseteq E(\mathcal{B}(P_2^{(1)} \odot P_n)) \cap (E(\cup_{i=2}^{m} \mathcal{B}(P_2^{(i)} \odot P_n)) \cup E(C))$$

where $\oplus$ is the ring sum. But,

$$\begin{aligned} E(\mathcal{B}(P_2^{(1)} \odot P_n)) \cap (E(\cup_{i=2}^{m} \mathcal{B}(P_2^{(i)} \odot P_n)) \cup E(C)) &= E(a_1 \times P_n) \cup E(a_2 \times P_n) \\ &\cup \{(a_1, u_1)(a_2, u_1)\}, \end{aligned}$$

which is an edge set of a tree. This contradicts the fact that b_1 is a cycle or an edge disjoint union of cycles. Therefore, $\mathcal{B}(C_m \odot P_n) = \mathcal{B} \cup \{C\}$ is linearly independent. Since,

$$\begin{aligned} |\mathcal{B}(C_m \odot P_n)| &= \sum_{i=1}^{m} |\mathcal{B}(P_2^{(i)} \odot P_n)| + |C| \\ &= \sum_{i=1}^{m} (n-1)^2 + 1 \\ &= (n-1)^2 m + 1 \\ &= \dim \mathcal{C}(C_n \odot P_m), \end{aligned}$$

$\mathcal{B}(C_m \odot P_n)$ is a basis of $\mathcal{C}(C_m \odot P_n)$. It is easy to show that $\mathcal{B}(C_m \odot P_n)$ is a 4-fold basis.

On the other hand, to show that $b(C_m \odot P_n) \geq 4$, we follow, more or less, the same arguments as in the three cases of Theorem 2.1. The proof is complete. $\square$

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