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Self-Adjoint Boundary Value Problems on Time Scales and Symmetric Green's Functions

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Abstract

In this note, higher order self-adjoint differential expressions on time scales, and associated with them self-adjoint boundary conditions, are discussed. The symmetry peoperty of the corresponding Green's functions is emphasized.

Key Words: Time scales, self-adjoint differential expressions, self-adjoint boundary conditions.

1. Introduction

In [3] self-adjoint boundary value problems (BVPs) for second order differential equations on time scales were introduced and examined by making use of both delta and nabla derivatives. Next some BVPs for higher order equations on time scales involving delta and nabla derivatives at the same time were investigated in [1, 2] where, however, the considered BVPs turned out, in general, to be nonselfadjoint because their Green's functions were found nonsymmetric. Therefore it remained unclear as to how to place the successive delta and nabla derivatives for higher order to get self-adjoint differential expressions that can yield symmetric Green's functions. In this paper we offer a solution to this problem indicating two classes of higher order differential equations on time scales. These classes of equations can be formulated as follows.

Let $\mathbf{T}$ be a time scale, $p_0, p_1, ..., p_n$ are real-valued right-dense continuous functions

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defined on $\mathbf{T}$ with $p_0(t) \neq 0$ for all $t \in \mathbf{T}$, and $a \in \mathbf{T}^{\kappa^n}$, $b \in \mathbf{T}_{\kappa^n}$, with $a < b$. For notation's sake, by $f^{\Delta^{-1}\nabla}$ and $f^{\nabla^{-1}\Delta}$ we mean the function f.

Then any $2n$th order differential expression

$$\begin{aligned} Ly(t) &= \sum_{i=0}^{n} (-1)^{n-i} \left[p_i(t) y^{\Delta^{n-i-1}\nabla}(t)\right]^{\nabla^{n-i-1}\Delta} \\ &= (-1)^n \left[p_0(t) y^{\Delta^{n-1}\nabla}(t)\right]^{\nabla^{n-1}\Delta} + \cdots - \left[p_{n-3}(t) y^{\Delta^2\nabla}(t)\right]^{\nabla^2\Delta} \\ &\quad + \left[p_{n-2}(t) y^{\Delta\nabla}(t)\right]^{\nabla\Delta} - \left[p_{n-1}(t) y^{\nabla}(t)\right]^{\Delta} + p_n(t) y(t) \end{aligned} \tag{1}$$

is formally self-adjoint with respect to the inner product

$$\langle y, z \rangle = \int_a^b y(t) z(t) \Delta t; \tag{2}$$

that is, the identity

$$\langle Ly, z \rangle = \langle y, Lz \rangle$$

holds provided that y and z satisfy some appropriate self-adjoint boundary conditions at a and b.

Similarly, the differential expression

$$\begin{aligned} My(t) &= \sum_{i=0}^{n} (-1)^{n-i} \left[p_i(t) y^{\nabla^{n-i-1}\Delta}(t)\right]^{\Delta^{n-i-1}\nabla} \\ &= (-1)^n \left[p_0(t) y^{\nabla^{n-1}\Delta}(t)\right]^{\Delta^{n-1}\nabla} + \cdots - \left[p_{n-3}(t) y^{\nabla^2\Delta}(t)\right]^{\Delta^2\nabla} \\ &\quad + \left[p_{n-2}(t) y^{\nabla\Delta}(t)\right]^{\Delta\nabla} - \left[p_{n-1}(t) y^{\Delta}(t)\right]^{\nabla} + p_n(t) y(t) \end{aligned} \tag{3}$$

is formally self-adjoint with respect to the inner product

$$\langle y, z \rangle = \int_a^b y(t) z(t) \nabla t.$$

In the present paper, we give a detailed presentation for differential expression (1). Differential expression (3) can be examined similarly.

The paper is organized as follows. In Section 2, some time scale essentials are included for the convenience of the reader. Next in Section 3, we consider the differential expression (1) and prove its formally self-adjointness with respect to the inner product (2). Finally, in Section 4 a definition of boundary conditions which are self-adjoint with respect to the differential expression (1) is given and the symmetry property of the corresponding Green's functions is emphasized.

2. Preliminaries on Time Scale Calculus

A *time scale* ($\mathbf{T}$) is an arbitrary nonempty closed subset of the real numbers. The calculus of time sacles was initiated by Aulbach and Hilger [4, 8] in order to create a theory that can unify and extend discrete and continuous analysis. The real numbers ($\mathbf{R}$), the integers ($\mathbf{Z}$), the natural numbers ($\mathbf{N}$), the nonnegative integers ($\mathbf{N}_0$), the $h-$numbers ($h\mathbf{Z} = \{hk : k \in \mathbf{Z}\}$, where $h > 0$ is a fixed real number), and the $q-$numbers ($\mathbf{K}_q = q^{\mathbf{Z}} \cup \{0\} = \{q^k : k \in \mathbf{Z}\} \cup \{0\}$, where $q > 1$ is a fixed real number) are examples of time scales, as are

$$[0, 1] \cup [2, 3],\ [0, 1] \cup \mathbf{N}, \text{ and the Cantor set,}$$

where $[0, 1]$ and $[2, 3]$ are real number intervals. In [4, 8] Aulbach and Hilger introduced also dynamic equations (Δ-differential equations) on time scales in order to unify and extend the theory of ordinary differential equations, difference (h-difference) equations, and $q-$difference equations. For a general introduction to the calculus of time scales we refer the reader to the textbooks by Bohner and Peterson [5, 6]. Here we give only those notions and facts connected to time scales, which we need for our purpose in this paper.

Any time scale $\mathbf{T}$ is a complete metric space with the metric (distance) $d(t, s) =\mid t-s \mid$ for $t, s \in \mathbf{T}$. Consequently, according to the well-known theory of general metric spaces, we have for $\mathbf{T}$ the fundamental concepts such as open balls (intervals), neighborhoods of points, open sets, closed sets, compact sets, and so on. In particular, for a given number $\delta > 0$, the $\delta-$*neighborhood* $U_\delta(t)$ of a given point $t \in \mathbf{T}$ is the set of all points $s \in \mathbf{T}$ such that $d(t, s) < \delta$. By a *neighborhood* of a point $t \in \mathbf{T}$ is meant an arbitrary set in $\mathbf{T}$ containing a δ-neighborhood of the point t. Also we have for functions $f : \mathbf{T} \to \mathbf{R}$ the concepts of limit, continuity, and the properties of continuous functions on general complete metric spaces (note that, in particular, any function $f : \mathbf{Z} \to \mathbf{R}$ is continuous at each point of $\mathbf{Z}$). The main task is to introduce and investigate the concept of derivative for functions $f : \mathbf{T} \to \mathbf{R}$. This proves to be possible due to the special structure of the metric space $\mathbf{T}$. In definition of the derivative an important role play the so-called forward and backward jump operators.

Definition 1 *For $t \in \mathbf{T}$ we define the forward jump operator $\sigma : \mathbf{T} \to \mathbf{T}$ by*

$$\sigma(t) = \inf \ \{s \in \mathbf{T} : s > t\},$$

while the backward jump operator $\rho : \mathbf{T} \to \mathbf{T}$ is defined by

$$\rho(t) = \sup \ \{s \in \mathbf{T} : s < t\}.$$

In this definition we put in addition $\sigma(\max \mathbf{T}) = \max \mathbf{T}$ if there exists a finite $\max \mathbf{T}$, and $\rho(\min \mathbf{T}) = \min \mathbf{T}$ if there exists a finite $\min \mathbf{T}$. Obviously both $\sigma(t)$ and $\rho(t)$ are in $\mathbf{T}$ when $t \in \mathbf{T}$. This is because of our assumption that $\mathbf{T}$ is a closed subset of $\mathbf{R}$.

Let $t \in \mathbf{T}$. If $\sigma(t) > t$, we say that t is *right-scattered*, while if $\rho(t) < t$ we say that t is *left-scattered*. Also, if $t < \max \mathbf{T}$ and $\sigma(t) = t$, then t is called *right-dense*, and if $t > \min \mathbf{T}$ and $\rho(t) = t$, then t is called *left-dense*. Points that are right-scattered and left-scattered at the same time are called *isolated*. Finally, the *graininess functions* $\mu, \nu : \mathbf{T} \to [0, \infty)$ are defined by

$$\mu(t) = \sigma(t) - t \text{ and } \nu(t) = t - \rho(t) \text{ for all } t \in \mathbf{T}.$$

Example 2 *If $\mathbf{T} = \mathbf{R}$, then $\sigma(t) = \rho(t) = t$ and $\mu(t) = \nu(t) = 0$. If $\mathbf{T} = h\mathbf{Z}$, then $\sigma(t) = t + h$, $\rho(t) = t - h$, and $\mu(t) = \nu(t) = h$. On the other hand, if $\mathbf{T} = \mathbf{K}_q$ then we have $\sigma(t) = qt$, $\rho(t) = q^{-1}t$, $\mu(t) = (q-1)t$, and $\nu(t) = (1 - q^{-1})t$.*

Let $\mathbf{T}^\kappa$ denote Hilger's truncated above (kappen = lop off) set consisting of $\mathbf{T}$ except for a possible left-scattered maximal point. Similarly, $\mathbf{T}_\kappa$ is the truncated below set obtained from $\mathbf{T}$ by deleting a possible right-scattered minimal point. In addition we use the notation $\mathbf{T}^{\kappa^2} = (\mathbf{T}^\kappa)^\kappa$, etc.

Definition 3 *(Delta Derivative). Let $f : \mathbf{T} \to \mathbf{R}$ be a function and $t \in \mathbf{T}^\kappa$. Then the delta derivative (or Δ−derivative) of f at the point t is defined to be the number $f^\Delta(t)$*

(provided it exists) with the property that for each $\varepsilon > 0$ *there is a neghborhood* U *of* t *in* $\mathbf{T}$ *such that*

$$| f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s] | \leq \varepsilon | \sigma(t) - s | \text{ for all } s \in U. \tag{4}$$

Remark 1 *If* $t \in \mathbf{T} \setminus \mathbf{T}^{\kappa}$, *then* $f^{\Delta}(t)$ *is not uniquely defined, since for such a point* t *small neighborhoods* U *of* t *consist only of* t, *and besides, we have* $\sigma(t) = t$. *Therefore (4) holds for an arbitrary number* $f^{\Delta}(t)$. *This is a reason why we omit a maximal left-scattered point.*

Definition 4 *(Nabla Derivative). If* $t \in \mathbf{T}_{\kappa}$, *then we define the nabla derivative (or* ∇*-derivative) of* $f : \mathbf{T} \to \mathbf{R}$ *at* t *to be the number* $f^{\nabla}(t)$ *(provided it exists) with the property that for each* $\varepsilon > 0$ *there is a neighborhood* U *of* t *in* $\mathbf{T}$ *such that*

$$| f(\rho(t)) - f(s) - f^{\nabla}(t)[\rho(t) - s] | \leq \varepsilon | \rho(t) - s | \text{ for all } s \in U.$$

Example 5 *If* $\mathbf{T} = \mathbf{R}$, *then* $f^{\Delta}(t) = f^{\nabla}(t) = f^{'}(t)$, *the ordinary derivative of* f *at* t. *If* $\mathbf{T} = h\mathbf{Z}$,*then*

$$f^{\Delta}(t) = \frac{f(t+h) - f(t)}{h} \text{ and } f^{\nabla}(t) = \frac{f(t) - f(t-h)}{h}.$$

If $\mathbf{T} = \mathbf{K}_q$, *then*

$$f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q-1)t} \text{ and } f^{\nabla}(t) = \frac{f(t) - f(q^{-1}t)}{(1-q^{-1})t}$$

for all $t \neq 0$, *and*

$$f^{\Delta}(0) = f^{\nabla}(0) = \lim_{s \to 0} \frac{f(s) - f(0)}{s}$$

provided that this limit exists.

Among the important properties of the delta and nabla differentiations on $\mathbf{T}$ we have the product rule: If $f, g : \mathbf{T} \to \mathbf{R}$ are Δ-differentiable functions at $t \in \mathbf{T}^{\kappa}$,then so is their product fg and we have

$$\begin{aligned} (fg)^{\Delta}(t) &= f^{\Delta}(t)g(t) + f(\sigma(t))g(t) \qquad (5) \\ &= f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)). \end{aligned}$$

Also, if $f, g : \mathbf{T} \to \mathbf{R}$ are ∇-differentiable at $t \in \mathbf{T}_\kappa$, then so is their product fg and we have

$$\begin{aligned}(fg)^\nabla(t) &= f^\nabla(t)g(t) + f(\rho(t))g^\nabla(t) \\ &= f(t)g^\nabla(t) + f^\nabla(t)g(\rho(t)).\end{aligned} \tag{6}$$

In the next theorem we give a relationship between the delta and nabla derivatives (see [3]).

Theorem 6 *(i) If $f : \mathbf{T} \to \mathbf{R}$ is Δ-differentiable on $\mathbf{T}^\kappa$ and if f^Δ is continuous on $\mathbf{T}^\kappa$, then f is ∇-differentiable on $\mathbf{T}_\kappa$ and*

$$f^\nabla(t) = f^\Delta(\rho(t)) \text{ for all } t \in \mathbf{T}^\kappa.$$

(ii) If $f : \mathbf{T} \to \mathbf{R}$ is ∇-differentiable on $\mathbf{T}_\kappa$ and if f^∇ is continuous on $\mathbf{T}_\kappa$, then f is Δ-differentiable on $\mathbf{T}^\kappa$ and

$$f^\Delta(t) = f^\nabla(\sigma(t)) \text{ for all } t \in \mathbf{T}^\kappa.$$

Now we introduce the concept of integral for functions $f : \mathbf{T} \to \mathbf{R}$.

If $a, b \in \mathbf{T}$ with $a \le b$ we define the closed interval $[a, b]$ in $\mathbf{T}$ by

$$[a, b] = \{t \in \mathbf{T} : a \le t \le b\}.$$

Open and half-open intervals etc. are defined accordingly. Below all our intervals will be time scale intervals.

Definition 7 *(Delta Integral). Let $[a, b]$ be a closed bounded interval in $\mathbf{T}$. A function $F : [a, b] \to \mathbf{R}$ is called a delta antiderivative of a function $f : [a, b) \to \mathbf{R}$ provided F is continuous on $[a, b]$ and delta differentiable on $[a, b)$, and $F^\Delta(t) = f(t)$ for all $t \in [a, b)$. Then we define the $\Delta-$integral from a to b of f by*

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

Definition 8 *(Nabla Integral). Let $[a, b]$ be a closed bounded interval in $\mathbf{T}$. A function $\Phi : [a, b] \to \mathbf{R}$ is called a nabla antiderivative of a function $f : (a, b] \to \mathbf{R}$ provided Φ is* *continuous on* $[a,b]$ *and nabla differentiable on* $(a,b]$, *and* $\Phi^{\nabla}(t) = f(t)$ *for all* $t \in (a,b]$. *Then we define the* $\nabla-$*integral from* a *to* b *of* f *by*

$$\int_a^b f(t)\nabla t = \Phi(b) - \Phi(a).$$

Example 9 *Let* $a, b \in \mathbf{T}$ *with* $a < b$. *Then we have the following:*

(i) If $\mathbf{T} = \mathbf{R}$, *then*

$$\int_a^b f(t)\Delta t = \int_a^b f(t)\nabla t = \int_a^b f(t)dt,$$

where the integral on the right is the ordinary integral.

(ii) If $[a,b]$ *consists of only isolated points, then*

$$\int_a^b f(t)\Delta t = \sum_{t\in[a,b)} \mu(t)f(t) \ \text{and} \ \int_a^b f(t)\nabla t = \sum_{t\in(a,b]} \nu(t)f(t),$$

where $\mu(t) = \sigma(t) - t$ *and* $\nu(t) = t - \rho(t)$. *In particular, if* $\mathbf{T} = \mathbf{Z}$, *then*

$$\int_a^b f(t)\Delta t = \sum_{k=a}^{b-1} f(k) \ \text{and} \ \int_a^b f(t)\nabla t = \sum_{k=a+1}^{b} f(k).$$

If $\mathbf{T} = h\mathbf{Z}$, *then*

$$\int_a^b f(t)\Delta t = h\sum_{t\in[a,b)} f(t) \ \text{and} \ \int_a^b f(t)\nabla t = h\sum_{t\in(a,b]} f(t).$$

If $\mathbf{T} = \mathbf{K}_q$, *then*

$$\int_a^b f(t)\Delta t = (q-1)\sum_{t\in[a,b)} tf(t) \ \text{and} \ \int_a^b f(t)\nabla t = (1-q^{-1})\sum_{t\in(a,b]} tf(t).$$

Definition 10 *A function* $f : \mathbf{T} \to \mathbf{R}$ *is right-dense continuous (or rd-continuous) provided it is continuous at all right-dense points of* $\mathbf{T}$ *and its left-sided limits exist (finite) at left-dense points of* $\mathbf{T}$. *The set of all right-dense continuous functions on* $\mathbf{T}$ *is denoted by* $C_{rd}(\mathbf{T})$. *Similarly, a function* $f : \mathbf{T} \to \mathbf{R}$ *is left-dense continuous provided it is continuous at all left-dense points of* $\mathbf{T}$ *and its right-sided limits exist (finite) at right-dense points of* $\mathbf{T}$. *The set of all left-dense continuous functions on* $\mathbf{T}$ *is denoted by* $C_{ld}(\mathbf{T})$.

All rd-continuous bounded functions on $[a, b)$ are delta integrable from a to b, and all ld-continuous bounded functions on $(a, b]$ are nabla integrable from a to b. For a more general treatment of the delta and nabla integrals on time scales (Riemann and Lebesgue integration on time scales), see [7] and [6, Chapter 5].

The following relationship between the delta and nabla integrals follows from Definition 7 and Definiton 8 by using Theorem 6.

Theorem 11 *If the function $f : \mathbf{T} \to \mathbf{R}$ is continuous, then for all $a, b \in \mathbf{T}$ with $a < b$ we have*

$$\int_a^b f(t)\Delta t = \int_a^b f(\rho(t))\nabla t \text{ and } \int_a^b f(t)\nabla t = \int_a^b f(\sigma(t))\Delta t. \tag{7}$$

Indeed, if $F : \mathbf{T} \to \mathbf{R}$ is a Δ-antiderivative for f, then $F^{\Delta}(t) = f(t)$ for all $t \in \mathbf{T}^{\kappa}$, and by Theorem 6(i) we have

$$F^{\nabla}(t) = F^{\Delta}(\rho(t)) = f(\rho(t)) \text{ for all } t \in \mathbf{T}_{\kappa},$$

so that F is a ∇-antiderivative for $f(\rho(t))$. Therefore

$$\int_a^b f(\rho(t))\nabla t = F(b) - F(a) = \int_a^b f(t)\Delta t.$$

The second formula of (7) can be proved in a similar manner by using Theorem 6(ii).

From (5), (6), and (7) we have the following integration by parts formulas: If the functions $f, g : \mathbf{T} \to \mathbf{R}$ are delta and nabla differentiable with continuous derivatives, then

$$\int_a^b f^{\Delta}(t)g(t)\Delta t = f(t)g(t) \mid_a^b - \int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t, \tag{8}$$

$$\int_a^b f^{\nabla}(t)g(t)\nabla t = f(t)g(t) \mid_a^b - \int_a^b f(\rho(t))g^{\nabla}(t)\nabla t, \tag{9}$$

$$\int_a^b f^{\Delta}(t)g(t)\Delta t = f(t)g(t) \mid_a^b - \int_a^b f(t)g^{\nabla}(t)\nabla t, \tag{10}$$

$$\int_a^b f^{\nabla}(t)g(t)\nabla t = f(t)g(t) \mid_a^b - \int_a^b f(t)g^{\Delta}(t)\Delta t. \tag{11}$$

3. Self-Adjoint Differential Expressions

Consider $2n$th order differential expression (1), in which the coefficients $p_i : \mathbf{T} \to \mathbf{R}$ are right-dense continuous for $0 \le i \le n$ and $p_0(t) \neq 0$ for all $t \in \mathbf{T}$. Set $\mathbf{T}^* = \mathbf{T}^{\kappa^n}_{\kappa^n} = \mathbf{T}^{\kappa^n} \cap \mathbf{T}_{\kappa^n}$.

Definition 12 *Let Ω be the linear set of all functions $y : \mathbf{T} \to \mathbf{R}$ such that the function*

$$\left(p_i y^{\Delta^{n-i-1}\nabla}\right)^{\nabla^{n-i-1}\Delta}$$

is defined on $\mathbf{T}^{\kappa^{n-i}}_{\kappa^{n-i}}$ and is right-dense continuous for $0 \le i \le n$.

For each $y \in \Omega$ the expression Ly is defined and presents a right-dense continuous function on $\mathbf{T}^*$.

The differential expression (1) takes the form

$$Ly(t) = -\left[p_0(t)y^{\nabla}(t)\right]^{\Delta} + p_1(t)y(t) \tag{12}$$

if $n = 1$, and the form

$$Ly(t) = \left[p_0(t)y^{\Delta\nabla}(t)\right]^{\nabla\Delta} - \left[p_1(t)y^{\nabla}\right]^{\Delta} + p_2(t)y(t) \tag{13}$$

if $n = 2$.

Theorem 13 *Let $a, b \in \mathbf{T}$ be such that $a \in \mathbf{T}^{\kappa^n}, b \in \mathbf{T}_{\kappa^n}$, and $a < b$. Then for all functions $y, z \in \Omega$ we have the Lagrange identity, also called the Green formula,*

$$\int_a^b (Ly)z\Delta t = [y,z]_a^b + \int_a^b y(Lz)\Delta t, \tag{14}$$

where $[y,z]_a^b = [y,z](b) - [y,z](a)$ and

$$\begin{aligned}[y,z](t) \quad = \quad & \sum_{k=0}^{n-1}(-1)^k \left\{ y^{\Delta^{n-k-1}}(t)\sum_{i=0}^{k}(-1)^i\left[p_i(t)z^{\Delta^{n-i-1}\nabla}(t)\right]^{\nabla^{k-i}} \right. \\ & \left. -z^{\Delta^{n-k-1}}(t)\sum_{i=0}^{k}(-1)^i\left[p_i(t)y^{\Delta^{n-i-1}\nabla}(t)\right]^{\nabla^{k-i}} \right\}. \end{aligned} \tag{15}$$

In particular, if $n = 1$ then

$$[y, z](t) = p_0(t)[y(t)z^{\nabla}(t) - y^{\nabla}(t)z(t)],$$

and if $n = 2$ then

$$\begin{aligned} [y, z](t) &= \left[p_0(t)y^{\Delta\nabla}(t)\right]^{\nabla} z(t) - y(t)\left[p_0(t)z^{\Delta\nabla}(t)\right]^{\nabla} \\ &\quad - p_0(t)[y^{\Delta\nabla}(t)z^{\Delta}(t) - y^{\Delta}(t)z^{\Delta\nabla}(t)] - p_1(t)[y^{\nabla}(t)z(t) - y(t)z^{\nabla}(t)]. \end{aligned}$$

Proof. Consider only the cases $n = 1$ and $n = 2$. The case of arbitrary n can be considered in a similar manner.

If $n = 1$, then $Ly(t)$ is defined as in (12) and we have

$$\int_a^b (Ly)z\Delta t = -\int_a^b \left[p_0(t)y^{\nabla}\right]^{\Delta} z(t)\Delta t + \int_a^b p_1(t)y(t)z(t)\Delta t. \tag{16}$$

Next, using first the integration by parts formula (10) and then (11), we obtain

$$\begin{aligned} \int_a^b \left[p_0(t)y^{\nabla}\right]^{\Delta} z(t)\Delta t &= p_0(t)y^{\nabla}(t)z(t) \mid_a^b - \int_a^b p_0(t)y^{\nabla}(t)z^{\nabla}(t)\nabla t \\ &= p_0(t)y^{\nabla}(t)z(t) \mid_a^b - p_0(t)y(t)z^{\nabla}(t) \mid_a^b + \int_a^b y(t)\left[p_0(t)z^{\nabla}\right]^{\Delta} \Delta t. \end{aligned}$$

Substituting this in the right-hand side of (16) we complete the proof for the case $n = 1$.

If $n = 2$, then $Ly(t)$ is defined as in (13). For the term

$$L_1 y(t) = \left[p_0(t)y^{\Delta\nabla}(t)\right]^{\nabla\Delta},$$

applying the integration by parts formula (8) and using Theorem 6(ii), we have

$$\begin{aligned} \int_a^b (L_1 y)z\Delta t &= \int_a^b \left[p_0(t)y^{\Delta\nabla}(t)\right]^{\nabla\Delta} z(t)\Delta t \\ &= \left[p_0(t)y^{\Delta\nabla}(t)\right]^{\nabla} z(t) \mid_a^b - \int_a^b \left[p_0 y^{\Delta\nabla}\right]^{\nabla} (\sigma(t))z^{\Delta}(t)\Delta t \\ &= \left[p_0(t)y^{\Delta\nabla}(t)\right]^{\nabla} z(t) \mid_a^b - \int_a^b \left[p_0 y^{\Delta\nabla}\right]^{\Delta} (t)z^{\Delta}(t)\Delta t. \end{aligned}$$

Next, using the integration by parts formula (10) we get

$$\int_a^b \left[p_0 y^{\Delta\nabla}\right]^{\Delta}(t) z^{\Delta}(t)\Delta t = p_0(t) y^{\Delta\nabla}(t) z^{\Delta}(t)\mid_a^b - \int_a^b p_0(t) y^{\Delta\nabla}(t) z^{\Delta\nabla}(t)\nabla t.$$

Now using the integration by parts formula (9) and Theorem 6(i) we find

$$\begin{aligned}\int_a^b p_0(t) y^{\Delta\nabla}(t) z^{\Delta\nabla}(t)\nabla t &= p_0(t) y^{\Delta}(t) z^{\Delta\nabla}(t)\mid_a^b - \int_a^b \left[y^{\Delta}\right](\rho(t))\left[p_0 z^{\Delta\nabla}\right]^{\nabla}(t)\nabla t \\ &= p_0(t) y^{\Delta}(t) z^{\Delta\nabla}(t)\mid_a^b - \int_a^b y^{\nabla}(t)\left[p_0 z^{\Delta\nabla}\right]^{\nabla}(t)\nabla t.\end{aligned}$$

Finally, applying the integration by parts formula (11) we get

$$\int_a^b y^{\nabla}(t)\left[p_0 z^{\Delta\nabla}\right]^{\nabla}(t)\nabla t = y(t)\left[p_0(t) z^{\Delta\nabla}(t)\right]^{\nabla}\mid_a^b - \int_a^b y(t)\left[p_0(t) z^{\Delta\nabla}(t)\right]^{\nabla\Delta}\Delta t.$$

Gathering these formulas all together we obtain

$$\begin{aligned}\int_a^b (L_1 y) z\Delta t &= \left[p_0(t) y^{\Delta\nabla}(t)\right]^{\nabla} z(t)\mid_a^b - p_0(t) y^{\Delta\nabla}(t) z^{\Delta}(t)\mid_a^b \\ + p_0(t) y^{\Delta}(t) z^{\Delta\nabla}(t) &\mid_a^b - y(t)\left[p_0(t) z^{\Delta\nabla}(t)\right]^{\nabla}\mid_a^b + \int_a^b y(L_1 z)\Delta t.\end{aligned}$$

Consequently, taking into account also the calculations in the case $n=1$ done above, we complete the proof for the case $n=2$. $\square$

Remark 2 *The identity (14) shows that the differential expression Ly is formally self-adjoint with respect to the inner product (2).*

For each function $y \in \Omega$, at $t \in \mathbf{T}^*$ set

$$\begin{aligned} y^{[k]} &= y^{\Delta^k},\ 0 \le k \le n-1,\ y^{[0]} = y^{\Delta^0} = y, \\ y^{[n]} &= p_0 y^{\Delta^{n-1}\nabla}, \\ y^{[n+k]} &= p_k y^{\Delta^{n-k-1}\nabla} - \left(y^{[n+k-1]}\right)^{\nabla},\ 1 \le k \le n-1, \\ y^{[2n]} &= p_n y - \left(y^{[2n-1]}\right)^{\Delta}. \end{aligned}$$

The functions $y^{[i]}, 0 \le i \le 2n$, we call the *quasi-derivatives* of y related to the expression Ly.

It follows that

$$\begin{aligned} y^{[n+j]} &= \sum_{i=0}^{j} (-1)^{j-i} \left[p_i(t) y^{\Delta^{n-i-1}\nabla}(t) \right]^{\nabla^{j-i}}, \ 0 \le j \le n-1, \\ y^{[2n]} &= \sum_{i=0}^{n} (-1)^{n-i} \left[p_i(t) y^{\Delta^{n-i-1}\nabla}(t) \right]^{\nabla^{n-i-1}\Delta} = Ly(t). \end{aligned}$$

Therefore the function $[y,z](t)$ defined by (15) can be represented in the form

$$[y,z](t) = \sum_{k=1}^{n} \left\{ y^{[k-1]}(t) z^{[2n-k]}(t) - y^{[2n-k]}(t) z^{[k-1]}(t) \right\}. \tag{17}$$

4. Self-Adjoint Boundary Conditions and Green's Function

Let $a, b \in \mathbf{T}$ be such that $a \in \mathbf{T}^{\kappa^n}$, $b \in \mathbf{T}_{\kappa^n}$, and $a < b$. If y and z are real valued right-dense continuous functions and bounded on $[a,b)$, define their inner product to be

$$\langle y, z \rangle = \int_a^b y(t) z(t) \Delta t.$$

Suppose that $p_n : [a,b) \to \mathbf{R}$ is a right-dense continuous and bounded function, and for $0 \le i \le n-1, p_i : [\rho^{n-i-1}(a), b] \to \mathbf{R}$ is right-dense continuous with $p_0(t) \ne 0$ on $[\rho^{n-1}(a), b]$.

Definition 14 *Denote by $\Omega[a,b)$ the linear set of all right-dense continuous functions $y : [\rho^n(a), \sigma^{n-1}(b)] \to \mathbf{R}$ such that*

(i) for $0 \le i \le n-1$ the function $\left[p_i(t) y^{\Delta^{n-i-1}\nabla}(t) \right]^{\nabla^{n-i-1}}$ is defined for $t \in [a,b]$,

(ii) for $0 \le i \le n-1$ the function $\left[p_i(t) y^{\Delta^{n-i-1}\nabla}(t) \right]^{\nabla^{n-i-1}\Delta}$ is defined for $t \in [a,b)$ and is right-dense continuous and bounded on $[a,b)$.

For $y \in \Omega[a, b)$ let

$$Ly(t) = \sum_{i=0}^{n} (-1)^{n-i} \left[p_i(t) y^{\Delta^{n-i-1}\nabla}(t) \right]^{\nabla^{n-i-1}\Delta},\ t \in [a, b). \tag{18}$$

Then Ly is right-dense continuous and bounded on $[a, b)$. Together with the differential expression (18) define the boundary conditions

$$U_j(y) : \ = \sum_{k=1}^{2n} \alpha_{jk} y^{[k-1]}(a) + \sum_{k=1}^{2n} \beta_{jk} y^{[k-1]}(b) = 0,\ 1 \le j \le 2n, \tag{19}$$

where α_{jk} , $\beta_{jk}, 1 \le j, k \le 2n$ are given real numbers.

Definition 15 *The boundary conditions (19) are self-adjoint with respect to the differential expression (18) if and only if*

$$\langle Ly, z \rangle = \langle y, Lz \rangle \tag{20}$$

for all functions $y, z \in \Omega[a, b)$ satisfying the boundary conditions (19).

By the Lagrange identity (14) we have

$$\langle Ly, z \rangle - \langle y, Lz \rangle = [y, z]_a^b,$$

where $[y, z]$ is as defined in (15) or (17). Therefore boundary conditions (19) are self-adjoint if and only if

$$[y, z]_a^b = 0$$

for all functions $y, z \in \Omega[a, b)$ satisfying (19). For example, the boundary conditions

$$y^{[k]}(a) = y^{[k]}(b) = 0,\ 0 \le k \le n-1,$$

and also the boundary conditions

$$y^{[k]}(a) = y^{[k]}(b),\ 0 \le k \le 2n-1,$$

are self-adjoint.

The boundary value problem

$$Ly(t) = 0,\ U_j(y) = 0,\ 1 \le j \le 2n \tag{21}$$

has a Green's function $G(t,s)$ if for any right-dense continuous and bounded functon $g:[a,b)\to \mathbf{R}$ the nonhomogeneous boundary value problem

$$Ly(t)=g(t),\ U_j(y)=0,\ 1\le j\le 2n,$$

has a unique solution $y:[\rho^n(a),\sigma^{n-1}(b)]\to \mathbf{R}$ which is given by

$$y(t)=\int_a^b G(t,s)g(s)\Delta s.$$

Let Λ be a differential operator generated by the differential expression Ly and the boundary conditions $U_j(y)=0, 1\le j\le 2n$. Then the domain of definition $D(\Lambda)$ of the operator Λ consists of all functions $y\in\Omega[a,b)$ satisfying the boundary conditions (19), and $\Lambda y=Ly$ for all $y\in D(\Lambda)$. Existence of the Green's function $G(t,s)$ for (21) means that the corresponding operator Λ has an inverse Λ^{-1} given by

$$(\Lambda^{-1}g)(t)=\int_a^b G(t,s)g(s)\Delta s,\ t\in[\rho^n(a),\sigma^{n-1}(b)], \tag{22}$$

for all bounded right-dense continuous functions $g:[a,b)\to\mathbf{R}$.

Suppose that the boundary conditions (19) are self-adjoint with respect to Ly. Then (20) implies that the operator Λ is self-adjoint (symmetric):

$$\langle \Lambda y,z\rangle=\langle y,\Lambda z\rangle \text{ for all } y,z\in D(\Lambda).$$

It easily follows that the operator Λ^{-1} (provided it exists) is also symmetric:

$$\left\langle \Lambda^{-1}f,g\right\rangle=\left\langle f,\Lambda^{-1}g\right\rangle \text{ for all } f,g\in C_{rd}[a,b). \tag{23}$$

Now (22) and (23) yield that the Green's function $G(t,s)$, provided it exists, of the self-adjoint bundary value problem (21) must be symmetric, i.e.

$$G(t,s)=G(s,t) \text{ for } t,s\in[a,b).$$

Example 16 *Consider the second order self-adjoint differential expression $Ly(t)$ as given in (12). It is easy to see that the boundary conditions*

$$\alpha y(a)-\beta y^{[1]}(a)=0,\ \gamma y(b)+\delta y^{[1]}(b)=0, \tag{24}$$

where $\alpha, \beta, \gamma, \delta$ are real numbers such that $|\alpha| + |\beta| \neq 0$ and $|\gamma| + |\delta| \neq 0$, are self-adjoint with respect to the differential expression $Ly(t)$. Denote by φ and ψ the solutions of the equation

$$-\left[p_0(t)y^{\nabla}(t)\right]^{\Delta} + p_1(t)y(t) = 0, \ t \in [a, b),$$

under the initial conditions

$$\varphi(a) = \beta, \ \varphi^{[1]}(a) = \alpha;$$

$$\psi(b) = \delta, \ \psi^{[1]}(b) = -\gamma,$$

and set

$$\omega = W_t(\varphi, \psi) = \varphi(t)\psi^{[1]}(t) - \varphi^{[1]}(t)\psi(t),$$

the Wronskian of solutions φ and ψ, which is independent of t. If $\omega \neq 0$, then the Green's function of the BVP (12), (24) exists and is given by (see [3])

$$G(t, s) = -\frac{1}{\omega}\begin{cases} \varphi(t)\psi(s), & t \leq s, \\ \varphi(s)\psi(t), & s \leq t. \end{cases}$$

Obviously, $G(t, s)$ is symmetric: $G(t, s) = G(s, t)$.

Remark 3 *In [2] it is shown (Example 18) that in the case $\mathbf{T} = \mathbf{Z}$ the Green's function of*

$$Ly(t) = y^{\Delta^2\nabla^2}(t) \tag{25}$$

with the boundary conditions

$$y(a) = y^{\Delta}(a) = y^{\Delta^2}(b) = y^{\Delta^2\nabla}(b) = 0 \tag{26}$$

is not symmetric. Notice that the expression (25) is in the form (13) with $p_0(t) \equiv 1$ and $p_1(t) = p_2(t) \equiv 0$, since in the case $\mathbf{T} = \mathbf{Z}$ the operators Δ and ∇ commute. However, the boundary conditions (26) are not self-adjoint. This is why the Green's function turned out to be nonsymmetric. The boundary conditions

$$y(a) = y^{\Delta}(a) = y^{\Delta\nabla}(b) = y^{\Delta\nabla^2}(b) = 0,$$

in contrast to the boundary conditions (26), are self-adjoint. Note also that if we replace in the self-adjoint boundary conditions for $\mathbf{T} = \mathbf{R}$ the real derivative with the delta or nabla derivative, the resulting boundary conditions need not be self-adjoint.

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