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On Generalization of The Quasi Homogeneous Riesz Potential

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Abstract

In this paper, a generalization of the quasi homogeneous Riesz Potential has been defined using non-isotropic quasi-distance and its L_p $(p \geq 1)$ continuity study.

Key words and phrases: Non-isotropic quasi-distance, Riesz Potential, Quasi-metric.

1. Introduction

Continuity properties of the classical Riesz Potential was studied in [1],[2],[3] and [4]. In this article we have defined a generalization of the quasi homogeneous Riesz Potential and we studied the L_p $(p \geq 1)-$ continuity of this potential.

2. Preliminaries

Firstly, we give some notations and definitions. We define a quasimetric (a non-isotropic quasi-distance) in $\mathbb{R}^n$ by

$$\|x-y\|_{\lambda,\gamma} = \left(|x_1-y_1|^{\frac{2\lambda_1}{\gamma}} + ... + |x_n-y_n|^{\frac{2\lambda_n}{\gamma}}\right)^{\frac{\gamma}{2n}\left|\frac{1}{\lambda}\right|}, \tag{1}$$

where $\lambda_1, ..., \lambda_n$ and γ are the positive real numbers and $\left|\frac{1}{\lambda}\right| = \frac{1}{\lambda_1} + ... + \frac{1}{\lambda_n}$. This quasimetric is named non-isotropic quasi-distance [5].

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For $\rho \in \mathbb{R}$ and $x \in \mathbb{R}^n$ define $\rho^{\frac{\gamma}{2\lambda}} x = (\rho^{\frac{\gamma}{2\lambda_1}} x_1, \rho^{\frac{\gamma}{2\lambda_2}} x_2, ..., \rho^{\frac{\gamma}{2\lambda_n}} x_n)$.

1. $\|x\|_{\lambda,\gamma} = 0 \Leftrightarrow x = \theta$
2. $\left\|\rho^{\frac{\gamma}{2\lambda}} x\right\|_{\lambda,\gamma} = |\rho|^{\frac{\gamma}{2n}\left|\frac{1}{\lambda}\right|} \|x\|_{\lambda,\gamma}$
3. $\|x+y\|_{\lambda,\gamma} \le C(\|x\|_{\lambda,\gamma} + \|y\|_{\lambda,\gamma})$,

where $\lambda_{\max} = \max\{\lambda_1, ..., \lambda_n\}$ and $C = 2^{(2\lambda_{\max}+\gamma)\frac{1}{2n}\left|\frac{1}{\lambda}\right|}$. It is clear that if $\lambda_1 = ... = \lambda_n = 2$ and $\gamma = 2$ then, quasimetric (a non-isotropic quasi-distance) is the Euclidean metric (Euclidean norm) on $\mathbb{R}^n$.

For a $f : \mathbb{R}^n \to \mathbb{R}$ and a $K : \mathbb{R}^n X \mathbb{R}^n \to \mathbb{R}^+$ we write

$$(I_\varphi^{\lambda,\gamma} f)(x) = \int\limits_{\mathbb{R}^n} f(y) K(x,y) dy, \tag{2}$$

where

$$K(x,y) = \frac{1}{\varphi(\|x-y\|_{\lambda,\gamma})}. \tag{3}$$

For $\mathbb{R}_0^+ = [0,\infty)$, function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ has the finite derivative and the following are valid:

$$\varphi(0) = 0 \tag{4}$$

$$\frac{\varphi'(t)}{\varphi(t)} \searrow on\ (0,\Delta)\ (\Delta > 0) \tag{5}$$

$$\frac{t\ \varphi'(t)}{\varphi(t)} \sim c > 0, t \to +0 \tag{6}$$

$$\int\limits_0^r \frac{t^{\gamma\left|\frac{1}{\lambda}\right|-1}}{\varphi^p(t)} dt = O\left(\frac{r^{\gamma\left|\frac{1}{\lambda}\right|}}{\varphi^p(r)}\right),\ r \to +0 \tag{7}$$

for any $\beta \in \mathbb{R}_0^+$,

$$\varphi(\beta t) = \beta^m \varphi(t), \quad m \in \mathbb{R}. \tag{8}$$

The operator $I_\varphi^{\lambda,\gamma} f$, when

$$\varphi(t) = t^{n-\alpha}, 0 < \alpha < n, t \in [0, +\infty) \tag{9}$$

is called the generalized Riesz potential with non-isotropic quasi-distance of function f. If φ satisfies the conditions (4), (5), (6), (7) and (8), then $I_\varphi^{\lambda,,\gamma} f$ is the generalization of the following quasi homogeneous Riesz potential

$$I^{\lambda,\gamma} f = \int_{\mathbb{R}^n} \|x - y\|_{\lambda,\gamma}^{\alpha-n} f(y)dy. \tag{10}$$

The L_p -continuity, or continuity in the L_p norm, of a real function g at point $x \in \mathbb{R}^n$ thus:

$$\left\{ \frac{1}{|B_r|} \int_{B_r} |g(x+s) - g(x)|^p \, ds \right\}^{\frac{1}{p}} \to 0, \ r \to +0, \tag{11}$$

where $|B_r|$ is the volume of a ball of radius r with center at zero [2].

Here we consider spherical coordinates by the following formulas:

$$y_1 = (\rho \cos\theta_1)^{\frac{\gamma}{\lambda_1}}, ..., y_n = (\rho \sin\theta_1, sin\theta_2, ..., \sin\theta_{n-1})^{\frac{\gamma}{\lambda_n}},$$

where we obtained that $\|x\|_\lambda = \rho^{\frac{\gamma}{n}\left|\frac{1}{\lambda}\right|}$. It can be seen that the Jacobian $J_\lambda(\rho, \theta)$ of this transformation is $J_\lambda(\rho, \theta) = \rho^{\gamma\left|\frac{1}{\lambda}\right|-1}\Omega_\lambda(\theta)$, where $\Omega_\lambda(\theta)$ is the bounded function and

$\Omega_\lambda(\theta) = \gamma^n.\frac{1}{\lambda_1}\frac{1}{\lambda_2}...\frac{1}{\lambda_n}(\cos\theta_1)^{\frac{\gamma}{\lambda_1}-1}.(\cos\theta_2)^{\frac{\gamma}{\lambda_2}-1}...(\cos\theta_{n-1})^{\frac{\gamma}{\lambda_n}-1}(\sin\theta_1)^{\frac{1}{\lambda_1}\frac{1}{\lambda_2}\cdots\frac{1}{\lambda_n}-1}\ldots$ $(\sin\theta_1)^{\frac{1}{\lambda_n}-1}$

Now we will prove that, depending on conditions imposed on f and φ, the operator $I_\varphi^{\lambda,\gamma} f$ is L_p continuous at every point at which it exists.

Theorem 2.1: If $p \geq 1$ and $I_\varphi^{\lambda,\gamma} f$ exists at point $x \in \mathbb{R}^n$, then $I_\varphi^{\lambda,\gamma} f$ is L_p continuous at x.

Proof: From (11), it is sufficient to consider the case when $x = 0$ and to prove that ([2] and [6])

$$\left\{ \int\limits_{\|s\|_{\lambda,\gamma}<r} \left| \left(I_{\varphi}^{\lambda,\gamma} f\right)(s) - \left(I_{\varphi}^{\lambda,\gamma} f\right)(0) \right| ds \right\}^{\frac{1}{p}} = o(r^{\frac{\gamma}{p}\left|\frac{1}{\lambda}\right|}) \tag{12}$$

holds when $r \to +0$. From (12), (2) and (3), we have following inequality:

$$\begin{aligned}
&\left\{ \int\limits_{\|s\|_{\lambda,\gamma}<r} \left| \left(I_{\varphi}^{\lambda,\gamma} f\right)(s) - \left(I_{\varphi}^{\lambda,\gamma} f\right)(0) \right|^p ds \right\}^{\frac{1}{p}} \\
&= \left\{ \int\limits_{\|s\|_{\lambda,\gamma}<r} \left| \int\limits_{\mathbb{R}^n} \left[\tfrac{1}{\varphi(\|s-y\|_{\lambda,\gamma})} - \tfrac{1}{\varphi(\|y\|_{\lambda,\gamma})} \right] f(y) dy \right|^p ds \right\}^{\frac{1}{p}} \\
&\le \left\{ \int\limits_{\|s\|_{\lambda,\gamma}<r} \left| \int\limits_{\|y\|_{\lambda,\gamma}>2r} \left[\tfrac{1}{\varphi(\|s-y\|_{\lambda,\gamma})} - \tfrac{1}{\varphi(\|y\|_{\lambda,\gamma})} \right] f(y) dy \right|^p ds \right\}^{\frac{1}{p}} \\
&+ \left\{ \int\limits_{\|s\|_{\lambda,\gamma}<r} \left| \int\limits_{\|y\|_{\lambda,\gamma}<2r} \left[\tfrac{1}{\varphi(\|s-y\|_{\lambda,\gamma})} - \tfrac{1}{\varphi(\|y\|_{\lambda,\gamma})} \right] f(y) dy \right|^p ds \right\}^{\frac{1}{p}} \\
&\le \left\{ \int\limits_{\|s\|_{\lambda,\gamma}<r} \left| \int\limits_{\|y\|_{\lambda,\gamma}<2r} \left[\tfrac{|f(y)|}{\varphi(\|s-y\|_{\lambda,\gamma})} \right] dy \right|^p ds \right\}^{\frac{1}{p}} \\
&+ \left\{ \int\limits_{\|s\|_{\lambda,\gamma}<r} \left| \int\limits_{\|y\|_{\lambda,\gamma}<2r} \left[\tfrac{|f(y)|}{\varphi(\|y\|_{\lambda,\gamma})} \right] dy \right|^p ds \right\}^{\frac{1}{p}} \\
&+ \left\{ \int\limits_{\|s\|_{\lambda,\gamma}<r} \left| \int\limits_{\|y\|_{\lambda,\gamma}>2r} \left[\tfrac{1}{\varphi(\|s-y\|_{\lambda,\gamma})} - \tfrac{1}{\varphi(\|y\|_{\lambda,\gamma})} \right] |f(y)| \, dy \right|^p ds \right\}^{\frac{1}{p}}. \\
&= \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3.
\end{aligned} \tag{13}$$

By Minkowsky inequality and (7) we have

$$\begin{aligned}\mathbf{I}_1 &\leq \int\limits_{\|y\|_{\lambda,\gamma}<2r} |f(y)| \left(\int\limits_{\|s\|_{\lambda,\gamma}<r} \frac{1}{\varphi^p(\|s-y\|_{\lambda,\gamma})} ds \right)^{\frac{1}{p}} dy \\ &\leq \int\limits_{\|y\|_{\lambda,\gamma\lambda}<2r} |f(y)| \left(\int\limits_{\|y-s\|_{\lambda}\leq 3Cr} \frac{ds}{\varphi^p(\|s-y\|_{\lambda,\gamma})} \right)^{\frac{1}{p}} dy\end{aligned}$$

$$\begin{aligned}&= \int\limits_{\|y\|_{\lambda,\gamma}<2r} |f(y)| \left(|B_{1,\lambda}| \int\limits_0^{3Cr} \frac{t^{\gamma\left|\frac{1}{\lambda}\right|-1}}{\varphi^p(t)} dt \right)^{\frac{1}{p}} dy \\ &= O\left(\frac{(3Cr)^{\frac{\gamma}{p}\left|\frac{1}{\lambda}\right|}}{\varphi^p(3Cr)} \varphi(2r) \int\limits_{\|y\|_{\lambda,\gamma}<2r} \frac{|f(y)|}{\varphi(\|y\|_{\lambda,\gamma})} dy \right) = o(r^{\frac{\gamma}{p}\left|\frac{1}{\lambda}\right|}), \quad (r\to 0),\end{aligned} \tag{14}$$

since the following is valid:

$$\int\limits_{\|x\|_{\lambda,\gamma}<r} h(\|x\|_{\lambda,\gamma}) dx = |B_{1,\lambda}| \int\limits_0^r t^{\gamma\left|\frac{1}{\lambda}\right|-1} h(t) dt, \tag{15}$$

if $t^{\gamma\left|\frac{1}{\lambda}\right|-1}h(t)$ is non-negative and measurable or integrable function at $(0,r)$, where, $|B_{1,\lambda}|$ denote the volume of the ball of $B_{1,\lambda}(x) = \left\{x \in \mathbb{R}^n : \|x\|_{\lambda,\gamma} < 1\right\}$.

For the $\mathbf{I}_2$

$$\mathbf{I}_2 \leq \int\limits_{\|y\|_{\lambda,\gamma}<2r} \frac{|f(y)|}{\varphi(\|y\|_{\lambda,\gamma})} \left(\int\limits_{\|s\|_{\lambda,\gamma}<r} ds \right)^{\frac{1}{p}} dy = o\left(\int\limits_{\|s\|_{\lambda,\gamma}<r} ds \right)^{\frac{1}{p}} = o\left(r^{\frac{\gamma}{p}\left|\frac{1}{\lambda}\right|} \right), \quad r\to +0. \tag{16}$$

If we use the Minkowski inequality at the $\mathbf{I}_3$, then we obtain,

$$
\begin{aligned}
\mathbf{I}_3 &= \left\{ \int\limits_{\|s\|_{\lambda,\gamma}<r} \left| \int\limits_{\|y\|_{\lambda,\gamma}>2r} \left[\tfrac{1}{\varphi(\|s-y\|_{\lambda,\gamma})} - \tfrac{1}{\varphi(\|y\|_{\lambda,\gamma})} \right] |f(y)|\, dy \right|^p ds \right\}^{\frac{1}{p}} \\
&= O\left\{ \int\limits_{\|s\|_{\lambda,\gamma}<r} \left| \int\limits_{\|y\|_{\lambda,\gamma}>2r} \frac{\varphi'(\|y\|_{\lambda,\gamma})}{\varphi^2(\|y\|_{\lambda,\gamma})} \|s\|_{\lambda,\gamma} |f(y)|\, dy \right|^p ds \right\}^{\frac{1}{p}} \\
&\leq O\left\{ \int\limits_{\|y\|_{\lambda,\gamma}>2r} \frac{\varphi'(\|y\|_{\lambda,\gamma})}{\varphi^2(\|y\|_{\lambda,\gamma})} |f(y)| \left[\int\limits_{\|s\|_{\lambda,\gamma}<r} \|s\|^p_{\lambda,\gamma}\, ds \right]^{\frac{1}{p}} dy \right\} \qquad (17) \\
&\leq O\left\{ \int\limits_{\|y\|_{\lambda,\gamma}>2r} \frac{\varphi'(\|y\|_{\lambda,\gamma})}{\varphi^2(\|y\|_{\lambda,\gamma})} |f(y)| .r. \left(\int\limits_0^r t^{\gamma\left|\frac{1}{\lambda}\right|-1} dt \right)^{\frac{1}{p}} dy \right\} \\
&= O\left\{ r^{\frac{\gamma}{p}\left|\frac{1}{\lambda}\right|+1} \int\limits_{\|y\|_{\lambda,\gamma}>2r} \frac{\varphi'(\|y\|_{\lambda,\gamma})}{\varphi^2(\|y\|_{\lambda,\gamma})} |f(y)|\, dy \right\}.
\end{aligned}
$$

for $0 < r < \min\{1, \Delta\}$, from (5) and (6) follows:

$$
\begin{aligned}
\int\limits_{\|y\|_{\lambda,\gamma}>2r} \frac{\varphi'(\|y\|_{\lambda,\gamma})|f(y)|}{\varphi(\|y\|_{\lambda,\gamma})^2} dy &= \left(\int\limits_{\|y\|_{\lambda,\gamma}>2\sqrt{r}} + \int\limits_{2\sqrt{r}\geq\|y\|_{\lambda,\gamma}>2r} \right) \frac{\varphi'(\|y\|_{\lambda,\gamma})|f(y)|}{\varphi(\|y\|_{\lambda,\gamma})^2} dy \\
&\leq \frac{\varphi'(2\sqrt{r})}{\varphi(2\sqrt{r})} \int\limits_{\|y\|_{\lambda,\gamma}>2\sqrt{r}} \frac{|f(y)|}{\varphi(\|y\|_{\lambda,\gamma})} dy \\
&\quad + \frac{\varphi'(2r)}{\varphi(2r)} \int\limits_{2\sqrt{r}\geq\|y\|_{\lambda,\gamma}>2r} \frac{|f(y)|}{\varphi(\|y\|_{\lambda,\gamma})} dy \qquad (18) \\
&\leq \frac{\varphi'(2\sqrt{r})}{\varphi(2\sqrt{r})} \int\limits_{\mathbb{R}^n} \frac{|f(y)|}{\varphi(\|y\|_{\lambda,\gamma})} dy \\
&\quad + \frac{\varphi'(2r)}{\varphi(2r)} \int\limits_{\|y\|_{\lambda,\gamma}<2\sqrt{r}} \frac{|f(y)|}{\varphi(\|y\|_{\lambda,\gamma})} dy \\
&= O\left(\frac{1}{2\sqrt{r}} \right) + o\left(\frac{1}{r} \right) = o\left(\frac{1}{r} \right).
\end{aligned}
$$

Then from (17) and (18) follows:

$$
\mathbf{I}_3 = o\left(r^{\frac{\gamma}{p}\left|\frac{1}{\lambda}\right|} \right), \quad (r \to +0) \qquad (19)
$$

Hence Theorem is obtained from (14), (16) and (19). □

Specially, if φ is defined by $\varphi = t^{n-\alpha}$, $\gamma = 2$ and $\lambda_1 = ... = \lambda_n = 2$, then $I_\varphi f$ is clasical Riesz potential of f. If φ is defined by $\varphi = t^{n-\alpha}$,i.e. if $I_\varphi^{\lambda,\gamma} f$ is a non-isotrop

Riesz potentials of f, then the conditions (4),(5),(6),(7) and (8) are valid if $1-\frac{1}{p} < \frac{\alpha}{p} < 1$, and if $I_{\varphi}^{\lambda,\gamma} f$ exists at point $x \in \mathbb{R}^n$, then the function $I_{\varphi}^{\lambda,\gamma} f$ is L_p continous at x.

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