

Submanifolds of Riemannian Product Manifolds

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Abstract

In this paper, we study the geometry of the semi-invariant submanifolds of a Riemannian product manifold. Fundamental properties of these type submanifolds such as the integrability of the distributions D , D^\perp and mixed-geodesic property are studied. Finally, necessary and sufficient conditions are given on a semi-invariant submanifold of Riemannian product manifold to be D -geodesic and D^\perp -geodesic.

Key Words: Riemannian Product Manifold, Real Space Form, Locally Riemannian Product and Curvature-Invariant Submanifold.

1. Introduction

The geometry of a submanifold (M, g) of a Riemannian product manifold $(\bar{M}_1 \times \bar{M}_2, \bar{g}_1 \times \bar{g}_2)$ has been studied by many geometers. In particular, Matsumoto ([4]) proved that (M, g) is a locally Riemannian product manifold of Riemannian manifolds (M_1, g_1) and (M_2, g_2) , if (M, g) is an invariant submanifold of a Riemannian product manifold $(\bar{M}_1 \times \bar{M}_2, \bar{g}_1 \times \bar{g}_2)$. Then, Senlin Xu and Yilong Ni ([5]) updated of the Matsumoto's Theorem and proved that (M_1, g_1) and (M_2, g_2) are pseudo-umbilical submanifolds of (\bar{M}_1, \bar{g}_1) and (\bar{M}_2, \bar{g}_2) , respectively, if (M, g) is an invariant pseudo-umbilical submanifold of $(\bar{M}_1 \times \bar{M}_2, \bar{g}_1 \times \bar{g}_2)$. They also demonstrated that M is isometric to the production of its two totally-geodesic submanifolds (M_1, g_1) and (M_2, g_2) which are submanifolds of (\bar{M}_1, \bar{g}_1) and (\bar{M}_2, \bar{g}_2) , respectively.

In [1], we have shown that (M, g) is a pseudo-umbilical submanifold of $(\bar{M}_1 \times \bar{M}_2, \bar{g}_1 \times \bar{g}_2)$, if (M_1, g_1) and (M_2, g_2) are pseudo-umbilical submanifolds of (\bar{M}_1, \bar{g}_1)

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and (\bar{M}_2, \bar{g}_2) , respectively. Moreover, necessary and sufficient conditions are given on an invariant submanifold of a Riemannian product manifold to be curvature-invariant submanifold and totally geodesic submanifold.

As was done in [6], we generalize the geometry of invariant submanifolds of a Riemannian product manifold to the geometry of semi-invariant submanifolds of a Riemannian product manifold. We show that a semi-invariant submanifold (M, g) of a Riemannian product manifold $(\bar{M}_1 \times \bar{M}_2, \bar{g}_1 \times \bar{g}_2)$ is a locally Riemannian product manifold if and only if $A_{FD^\perp} D = 0$ which is equivalent to $\nabla f = 0$, or $Bh(TM, D) = 0$. Furthermore, necessary and sufficient conditions are given on distributions D and D^\perp of a semi-invariant submanifold M to be integrable. Finally, we studied totally-umbilical semi-invariant submanifolds in any positively or negatively curved Riemannian product manifold $(\bar{M}_1 \times \bar{M}_2, \bar{g}_1 \times \bar{g}_2)$.

In this paper, we further our work with a study of the integrability conditions of distributions D and D^\perp from a different point of view. Necessary and sufficient conditions are given of semi-invariant submanifold to be D -geodesic(D^\perp -geodesic) and mixed-geodesic submanifold. Moreover, we have studied semi-invariant submanifolds which are curvature-invariant in any positively or negatively curved Riemannian product manifold $(\bar{M}_1 \times \bar{M}_2, \bar{g}_1 \times \bar{g}_2)$. Moreover, we have constructed an example for semi-invariant submanifold of Riemannian product manifold to illustrate our results.

2. Preliminaries

In this section, we give the definitions and terminology used throughout this paper. We recall some necessary facts and formulas from the theory of submanifolds in any Riemannian manifold. For an arbitrary submanifold M of any Riemannian manifold \bar{M} , The Gauss and Weingarten formulas are respectively given by formulas

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{1}$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla_X^\perp V \tag{2}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(TM^\perp)$, where $\bar{\nabla}, \nabla$ denote the Levi-Civita connections on \bar{M} and M , respectively. Moreover, $h : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM^\perp)$ is the second fundamental form of M in \bar{M} , ∇^\perp is the normal connection on the normal bundle $\Gamma(TM^\perp)$ and A_V is the shape operator of M with respect to V . Furthermore, A_V and h are related

by formula

$$g(A_V X, Y) = g(h(X, Y), V) \quad (3)$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(TM^\perp)$, where g denotes the Riemannian metric on M as well as \bar{M} .

Now, we denote the Riemannian curvature tensors of the connections $\bar{\nabla}$ and ∇ by \bar{R} and R , respectively. Then the equations of Gauss, Codazzi and Ricci are, respectively, given by formulas

$$\begin{aligned} g(\bar{R}(X, Y)Z, W) &= g(R(X, Y)Z, W) + g(h(X, W), h(Y, Z)) \\ &- g(h(X, Z), h(Y, W)), \end{aligned} \quad (4)$$

$$g(\bar{R}(X, Y)\xi, \eta) = g(\bar{R}(X, Y)^\perp \xi, \eta) - g([A_\xi, A_\eta]X, Y) \quad (5)$$

and

$$\{\bar{R}(X, Y)Z\}^\perp = (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z) \quad (6)$$

for any $X, Y, Z, W \in \Gamma(TM)$ and $\xi, \eta \in \Gamma(TM^\perp)$, where $\{\bar{R}(X, Y)Z\}^\perp$ denotes the normal component of $\bar{R}(X, Y)Z$ and the covariant derivative $\bar{\nabla}h$ is defined by

$$(\bar{\nabla}_X h)(Y, Z) = \nabla_X^\perp h(Y, Z) - h(\nabla_X Y, Z) - h(\nabla_X Z, Y)$$

for any $X, Y, Z \in \Gamma(TM)$. We recall that M is said to be curvature-invariant submanifold if $\bar{R}(X, Y)Z \in \Gamma(TM)$, i.e., we have $\{\bar{R}(X, Y)Z\}^\perp = 0$ for any $X, Y, Z \in \Gamma(TM)$ [8].

Definition 2.1 *Let M be an n -dimensional submanifold of any Riemannian manifold \bar{M} . The mean-curvature vector field H of M is defined by formula*

$$H = \frac{1}{n} \sum_{j=1}^n h(e_j, e_j),$$

where, $\{e_j\}$, $1 \leq j \leq n$, is a locally orthonormal basis of $\Gamma(TM)$. If a submanifold M has one of the conditions

$$h = 0, \quad H = 0, \quad h(X, Y) = g(X, Y)H, \quad g(h(X, Y), H) = \lambda g(X, Y), \quad \lambda \in C^\infty(M, \mathbb{R}),$$

then it is said to be totally geodesic, minimal, totally-umbilical and pseudo-umbilical submanifold, respectively[2].

3. The Riemannian Product of the Riemannian Manifolds

Let (\bar{M}_1, \bar{g}_1) and (\bar{M}_2, \bar{g}_2) be the Riemannian manifolds with dimensions n_1, n_2 , respectively and $\bar{M}_1 \times \bar{M}_2$ be the Riemannian product manifold of Riemannian manifolds \bar{M}_1 and \bar{M}_2 . We denote the projections mappings of $\Gamma(T(\bar{M}_1 \times \bar{M}_2))$ onto $\Gamma(T\bar{M}_1)$ and $\Gamma(T\bar{M}_2)$ by π_* and σ_* , respectively. Then we have

$$\pi_* + \sigma_* = I, \quad \pi_*^2 = \pi_*, \quad \sigma_*^2 = \sigma_*, \quad \text{and} \quad \pi_* \times \sigma_* = \sigma_* \times \pi_* = 0.$$

The Riemannian metric tensor of the Riemannian product manifold $\bar{M} = \bar{M}_1 \times \bar{M}_2$ is given by

$$g(X, Y) = \bar{g}_1(\pi_*X, \pi_*Y) + \bar{g}_2(\sigma_*X, \sigma_*Y),$$

for any $X, Y \in \Gamma(T\bar{M})$. From the definition of g , \bar{M}_1 and \bar{M}_2 are totally-geodesic submanifolds of $\bar{M}_1 \times \bar{M}_2$. Setting $F = \pi_* - \sigma_*$, then we can easily see that $F^2 = I$ and g satisfies

$$g(FX, Y) = g(X, FY), \tag{7}$$

for any $X, Y \in \Gamma(T\bar{M})$. Thus F defines an almost Riemannian product structure on \bar{M} . Furthermore, we denote the Levi-Civita connection on \bar{M} by $\bar{\nabla}$, then we have

$$(\bar{\nabla}_X F)Y = 0 \tag{8}$$

for any $X, Y \in \Gamma(T\bar{M})$ (For the more detail, we refer the readers to [5]).

In the rest of this paper, we denote the Riemannian product manifold $(\bar{M}_1 \times \bar{M}_2, \bar{g}_1 \otimes \bar{g}_2)$ by (\bar{M}, g) .

4. Semi-Invariant Submanifolds of A Riemannian Product Manifold

Definition 4.1 *Let M be an immersed submanifold of a Riemannian product manifold \bar{M} . Let us assume that M has two distributions such as D and D^\perp such that $TM = D \oplus D^\perp$, D is an invariant distribution, i.e., $F(D) = D$ and D^\perp is an anti-invariant distribution, i.e., $F(D^\perp) \subset TM^\perp$. Then we recall that M is semi-invariant submanifold of the Riemannian product manifold \bar{M} [6].*

In the rest of this paper, we suppose that M is a semi-invariant submanifold of a Riemannian product manifold \bar{M} . Now, we denote the orthogonal complement of $F(D^\perp)$

in TM^\perp by V ; then we have direct sum

$$TM^\perp = F(D^\perp) \oplus V. \quad (9)$$

In this case, we infer that V is an invariant vector bundle with respect to F . On the other hand, for each X tangent to M , FX can be written as follows:

$$FX = fX + \omega X, \quad (10)$$

where fX and ωX are the tangential part and normal part of FX , respectively. Also, for each vector field ξ normal to M , $F\xi$ can be written as follows:

$$F\xi = B\xi + C\xi, \quad (11)$$

where $B\xi$ and $C\xi$ are the tangential part and normal part of $F\xi$, respectively.

We denote dimensions of the invariant distribution D and anti-invariant distribution D^\perp by p and q , respectively. Then for $q = 0$ (resp. $p = 0$), semi-invariant submanifold becomes an invariant (resp. an anti-invariant) submanifold. A proper semi-invariant submanifold is a semi-invariant submanifold which is neither an invariant submanifold nor an anti-invariant submanifold.

Now, we give an example for semi-invariant submanifold of Riemannian product manifold to illustrate our results.

Example 4.2 In $\mathbb{R}^7 = \mathbb{R}^3 \times \mathbb{R}^4$ consider the submanifold

$$M : x_1 - x_3 = 0, \quad x_2 + x_5 = 0, \quad x_7 = \frac{1}{2} \log(1 + (x_6 - x_4)^2), \quad x_6 \neq x_4.$$

It is easy to check that M is a semi-invariant submanifold of \mathbb{R}^7 . In fact that

$$\begin{aligned} TM = \text{Span}\{U_1 &= \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, \quad U_2 = \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_5}, \\ U_3 &= (1 + (x_6 - x_4)^2) \frac{\partial}{\partial x_4} - (x_6 - x_4) \frac{\partial}{\partial x_7}, \\ U_4 &= (1 + (x_6 - x_4)^2) \frac{\partial}{\partial x_6} + (x_6 - x_4) \frac{\partial}{\partial x_7}\} \end{aligned}$$

and

$$\begin{aligned} TM^\perp &= \text{Span}\{\xi_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3}, \quad \xi_2 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_5} \\ \xi_3 &= (x_6 - x_4) \frac{\partial}{\partial x_4} - (x_6 - x_4) \frac{\partial}{\partial x_6} + (1 + (x_6 - x_4)^2) \frac{\partial}{\partial x_7}\}. \end{aligned}$$

By direct calculations, we get

$$FU_1 = U_1, \quad FU_3 = -U_3, \quad FU_4 = -U_4, \quad \text{and} \quad FU_2 = \xi_2.$$

Then take as the distributions $D = \text{Span}\{U_1, U_3, U_4\}$ and $D^\perp = \text{Span}\{U_2\}$.

Theorem 4.3 *Let M be a proper semi-invariant submanifold of Riemannian product manifold \bar{M} . Then the anti-invariant distribution D^\perp is integrable if and only if the shape operator of M satisfies*

$$A_{FW}Z = 0$$

for any $Z, W \in \Gamma(D^\perp)$.

Proof. By using (1), (2), (8), (10) and (11), we obtain

$$\begin{aligned} \bar{\nabla}_Z FW &= F\bar{\nabla}_Z W \\ -A_{FW}Z + \nabla_Z^\perp FW &= f\nabla_Z W + \omega\nabla_Z W + Bh(Z, W) + Ch(Z, W), \end{aligned} \quad (12)$$

for any $Z, W \in \Gamma(D^\perp)$. From the tangential part of (12) we have

$$-A_{FW}Z = f\nabla_Z W + Bh(Z, W). \quad (13)$$

Replacing vector fields Z by W in (13), we get

$$-A_{FZ}W = f\nabla_W Z + Bh(W, Z). \quad (14)$$

Taking into account (13), (14) and the symmetry of h , we conclude

$$-A_{FW}Z + A_{FZ}W = f[Z, W].$$

Furthermore, it was proven in [6] that the shape operator of M satisfies

$$A_{FW}Z = -A_{FZ}W$$

for any $Z, W \in \Gamma(D^\perp)$. Thus we have

$$f[Z, W] = 2A_{FZ}W.$$

Then $[Z, W] \in \Gamma(D^\perp)$ if and only if $A_{FZ}W = 0$. This completes the proof of the Theorem. □

Theorem 4.4 *Let M be a proper semi-invariant submanifold of Riemannian product manifold \bar{M} . Then the invariant distribution D is integrable if and only if the shape operator of M satisfies*

$$FA_{FZ}X = A_{FZ}FX$$

for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$.

Proof. By using (1), (2), (7) and the symmetry of h , then we have

$$\begin{aligned} g([X, Y], Z) &= g(\nabla_X Y - \nabla_Y X, Z) = g(\bar{\nabla}_X Y - \bar{\nabla}_Y X, Z) \\ &= g(\bar{\nabla}_Y Z, X) - g(\bar{\nabla}_X Z, Y) = g(\bar{\nabla}_Y FZ, FX) - g(\bar{\nabla}_X FZ, FY) \\ &= -g(A_{FZ}Y, FX) + g(A_{FZ}X, FY) = g(FA_{FZ}X - A_{FZ}FX, Y) \end{aligned}$$

for any $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$. Therefore, we obtain that $[X, Y] \in \Gamma(D)$ if and only if $FA_{FZ}X = A_{FZ}FX$. This completes the proof of the Theorem. \square

Definition 4.5 *Let M be a proper semi-invariant submanifold of Riemannian product manifold \bar{M} . M is said to be D -geodesic (resp. D^\perp -geodesic) submanifold, if the second fundamental form of M satisfies $h(X, Y) = 0$, for any $X, Y \in \Gamma(D)$ (resp. $h(Z, W) = 0$ for any $Z, W \in \Gamma(D^\perp)$).*

Theorem 4.6 *Let M be a proper semi-invariant submanifold of Riemannian product manifold \bar{M} . Then*

i) The invariant distribution D is integrable and its leaves are totally-geodesic in M if and only if

$$g(h(X, Y), FZ) = 0$$

for any $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$.

ii) The invariant distribution D is integrable and its leaves are totally-geodesic in \bar{M} if and only if M is D -geodesic submanifold.

Proof. *i)* From (1), (2) and (10) we have

$$g(h(X, Y), FZ) = g(\bar{\nabla}_X Y, FZ) = g(\bar{\nabla}_X FY, Z) = g(\nabla_X fY, Z)$$

for any $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$. Now, we suppose that the distribution D is integrable and its leaves are totally-geodesic in M , then we have

$$\nabla_X Y \in \Gamma(D)$$

which is equivalent to $g(h(X, Y), FZ) = 0$.

Conversely, if $g(h(X, Y), FZ) = 0$, then we obtain

$$g(\nabla_X fY, Z) = 0$$

for all $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$. It follows that $\nabla_X fY \in \Gamma(D)$.

ii) We assume that the distribution D is integrable and its leaves are totally-geodesic in \bar{M} ; then we have $\bar{\nabla}_X Y \in \Gamma(D)$ for any $X, Y \in \Gamma(D)$. Thus we get

$$g(h(X, Y), \xi) = g(\bar{\nabla}_X Y, \xi) = 0$$

for any $\xi \in \Gamma(TM^\perp)$, that is, M is a D -geodesic semi-invariant submanifold.

Conversely, let M be D -geodesic a proper semi-invariant submanifold of \bar{M} . Then for each $X, Y \in \Gamma(D)$ and $\xi \in \Gamma(TM^\perp)$, we have

$$g(\bar{\nabla}_X Y, \xi) = g(h(X, Y), \xi) = 0$$

which proves our assertion. □

Theorem 4.7 *Let M be a proper semi-invariant submanifold of Riemannian product manifold \bar{M} . Then there exists no proper semi-invariant submanifolds which are curvature-invariant in any positively or negatively curved Riemannian product manifold \bar{M} .*

Proof. We denote the Riemannian curvature tensor of \bar{M} by \bar{R} . Then by a direct calculation, we infer that \bar{R} satisfies

$$\bar{R}(FX, FY)Z = \bar{R}(X, Y)Z \tag{15}$$

for any $X, Y, Z \in \Gamma(T\bar{M})$. On the other hand, because \bar{M} is a real space form, we have

$$\bar{R}(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\}, \tag{16}$$

for any $X, Y, Z \in \Gamma(T\bar{M})$. Thus from (15) and (16) we have

$$\begin{aligned} \bar{R}(X, Y)Z &= c\{g(FY, Z)FX - g(FX, Z)FY\} \\ &= R(X, Y)Z + A_{h(X, Z)}Y - A_{h(Y, Z)}X \\ &+ (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z) \end{aligned} \tag{17}$$

for any $X, Y, Z \in \Gamma(TM)$. Taking $X, Z \in \Gamma(D)$ and $Y \in \Gamma(D^\perp)$ in (17), then we infer

$$-cg(FX, Z)FY = (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z),$$

which proves our assertion. \square

Theorem 4.8 *Let M be a proper semi-invariant submanifold of Riemannian product manifold \bar{M} . Then M is a locally Riemannian product, if M is totally-geodesic submanifold.*

Proof. By using the assertion (i) and (ii) of Theorem 4.6, Theorem 4.3 and (12) we obtain that both distributions D and D^\perp are integrable and their leaves are totally-geodesic in M . Thus M is a locally Riemannian product. \square

Definition 4.9 *Let M be a proper semi-invariant submanifold of Riemannian product manifold \bar{M} . We call that M is mixed-geodesic submanifold, if the second fundamental form of M satisfies $h(X, Z) = 0$ for any $X \in \Gamma(D)$ and $Y \in \Gamma(D^\perp)$.*

Theorem 4.10 *Let M be a proper semi-invariant submanifold of Riemannian product manifold \bar{M} . Then M is mixed-geodesic submanifold if and only if the shape operator of M satisfies*

$$A_\xi FX \in \Gamma(D)$$

for any $X \in \Gamma(D)$ and $\xi \in \Gamma(TM^\perp)$.

Proof. By using (1), (2) and $\bar{\nabla}$ is Levi-Civita connection, we infer

$$\begin{aligned} g(h(FX, Y), \xi) &= g(\bar{\nabla}_Y FX, \xi) = -g(\bar{\nabla}_Y \xi, FX) \\ &= g(A_\xi Y, FX) = g(A_\xi FX, Y) \end{aligned}$$

for any $X \in \Gamma(D)$, $Y \in \Gamma(D^\perp)$ and $\xi \in \Gamma(TM^\perp)$. Thus we get $h(FX, Y) = 0$ if and only if $A_\xi FX \in \Gamma(D)$. \square

Now, we denote the integral manifolds of the distributions D and D^\perp by M_1 and M_2 , respectively.

Theorem 4.11 *Let M be a proper semi-invariant submanifold of Riemannian product manifold \bar{M} . Then M_1 is totally-geodesic in M if and only if the second fundamental form of M satisfies*

$$h(FY, X) \in \Gamma(V)$$

for all $X, Y \in \Gamma(D)$.

Proof. By using (1), (2), (3) and (7), we derive

$$\begin{aligned} g(\nabla_X Y, Z) &= g(\bar{\nabla}_X Y, Z) = g(\bar{\nabla}_X FY, FZ) \\ &= g(h(FY, X), FZ) \end{aligned}$$

for any $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$. Thus we have $\nabla_X Y \in \Gamma(D)$ if and only if $h(FY, X) \in \Gamma(V)$. \square

From Theorem 4.11 we have

Corollary 4.12 *Let M be a proper semi-invariant submanifold of Riemannian product manifold \bar{M} . Then M_1 is totally-geodesic in M , if M is D -geodesic submanifold in \bar{M} .*

Theorem 4.13 *Let M be a proper semi-invariant submanifold of Riemannian product manifold \bar{M} . M_2 is totally-geodesic in M if and only if*

$$h(FX, Z) \in \Gamma(V)$$

for all $Z \in \Gamma(D^\perp)$ and $X \in \Gamma(D)$.

Proof.

$$\begin{aligned} g(\nabla_Z W, X) &= g(\bar{\nabla}_Z W, X) = -g(\bar{\nabla}_Z X, W) = -g(\bar{\nabla}_Z FX, FW) \\ &= -g(h(Z, FX), FW) \end{aligned}$$

for all $Z, W \in \Gamma(D^\perp)$ and $X \in \Gamma(D)$. It follows that M_2 is totally-geodesic in M if and only if $h(Z, FX) \in \Gamma(V)$. \square

From Theorem 4.13 we have the following Corollary.

Corollary 4.14 *Let M be a proper semi-invariant submanifold of Riemannian product manifold \bar{M} . Then each leaf D^\perp is totally-geodesic in M , if M is a mixed-geodesic submanifold of \bar{M} .*

Theorem 4.15 *Let M be a proper semi-invariant submanifold of Riemannian product manifold \bar{M} . Then M_2 is totally-geodesic submanifold in \bar{M} if and only if*

$$h(Z, U) \in \Gamma(V) \text{ for any } Z \in \Gamma(D^\perp) \ U \in \Gamma(TM), \quad (18)$$

and

$$\nabla_Z^\perp FW \in \Gamma(F(D^\perp)) \text{ for any } Z, W \in \Gamma(D^\perp). \quad (19)$$

Proof. Suppose M_2 is totally-geodesic in \bar{M} . Then we have $\bar{\nabla}_Z W \in \Gamma(D^\perp)$, for any $Z, W \in \Gamma(D^\perp)$. By using (1) and (2) and (7), we obtain

$$\begin{aligned} g(h(Z, U), FW) &= g(\bar{\nabla}_Z U, FW) = -g(\bar{\nabla}_Z W, FU) \\ &= -g(\bar{\nabla}_Z W, fU) - g(\bar{\nabla}_Z W, \omega U) = 0, \end{aligned}$$

that is, $h(Z, U) \in \Gamma(V)$ for any $Z, W \in \Gamma(D^\perp)$, $U \in \Gamma(TM)$.

In the same way, we obtain

$$g(\nabla_Z^\perp FW, V) = g(\bar{\nabla}_Z FW, V) = g(\bar{\nabla}_Z W, FV) = 0, \quad (20)$$

that is, $\nabla_Z^\perp FW \in \Gamma(F(D^\perp))$ for all $Z, W \in \Gamma(D^\perp)$, $V \in \Gamma(V)$.

Conversely, (18) and (19) are satisfied. By using (1), (2) and (7), we obtain

$$\begin{aligned} g(\bar{\nabla}_Z W, X) &= -g(\bar{\nabla}_Z X, W) = -g(\bar{\nabla}_Z FX, FW) \\ &= -g(h(Z, FX), FW) = 0 \\ g(\bar{\nabla}_Z W, FU) &= g(h(Z, W), FU) = 0 \end{aligned}$$

and

$$g(\bar{\nabla}_Z W, V) = g(\bar{\nabla}_Z FW, FV) = g(\nabla_Z^\perp FW, FV) = 0$$

for any $Z, W, U \in \Gamma(D^\perp)$, $X \in \Gamma(D)$ and $V \in \Gamma(V)$. This completes the proof of the Theorem. \square

Theorem 4.16 *Let M be a proper semi-invariant submanifold of Riemannian product manifold \bar{M} . Then M_2 is totally-geodesic in \bar{M} if and only if M is D^\perp -geodesic and*

$$g(h(X, Z), FW) = 0$$

for any $Z, W \in \Gamma(D^\perp)$ and $X \in \Gamma(D)$.

Proof. Let us assume that M_2 is totally-geodesic in \bar{M} . Then we have $\bar{\nabla}_Z W \in \Gamma(D^\perp)$. By using (1), (2), (7) and taking account of $\bar{\nabla}$ is Levi-Civita connection, we obtain

$$\begin{aligned} g(h(X, Z), FW) &= g(\bar{\nabla}_Z X, FW) = -g(\bar{\nabla}_Z FW, X) \\ &= g(\bar{\nabla}_Z W, FX) = 0. \end{aligned}$$

Similarly, because M_2 is totally-geodesic in \bar{M} , from Theorem 4.15 we have

$$\begin{aligned} g(h(Z, W), V) &= g(\bar{\nabla}_Z W, V) = g(\bar{\nabla}_Z FW, FV) \\ &= g(\nabla_Z^\perp FW, FV) = 0 \end{aligned}$$

for any $Z, W \in \Gamma(D^\perp)$, $X \in \Gamma(D)$ and $V \in \Gamma(V)$, that is, M is D^\perp -geodesic submanifold.

Conversely, we assume that $g(h(X, Z), FW) = 0$ and M is D^\perp -geodesic submanifold. Then we have

$$\begin{aligned} g(\bar{\nabla}_Z W, FX) &= -g(\bar{\nabla}_Z FX, W) = -g(\bar{\nabla}_Z X, FW) \\ &= -g(h(Z, X), FW) = 0 \end{aligned}$$

and

$$g(\bar{\nabla}_Z W, \xi) = g(h(Z, W), \xi) = 0$$

for any $Z, W \in \Gamma(D^\perp)$, $X \in \Gamma(D)$ and $\xi \in \Gamma(TM^\perp)$. Thus we conclude that M_2 is totally-geodesic in \bar{M} . This completes the proof of the Theorem. \square

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