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The Pitch and the Pseudo Angle of Pitch of a Closed Piece of (k+1)-Dimensional Ruled Surface in R^n_ν

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Abstract

In this paper, we define the closed piece of ruled surface, the pitch and the pseudo angle of pitch of a closed piece of ruled surface and calculate these values in Minkowski space $R^n_\nu = (R^n, -\sum_{i=1}^{\nu} dx_i + \sum_{i=\nu+1}^{n} dx_i)$.

Key Words: Minkowski space, Ruled surface, Closed ruled surface, Pitch, Pseudo Angle of pitch.

1. Introduction

In Euclidean space R^n, a ruled surface which has periodic directrix curves is called closed ruled surface [1]. Since periodic timelike curves do not exist in R^n_ν, there is no closed ruled surface whose directrix curves are timelike. Therefore, in [2] we give similar formulas on a closed piece of a ruled surface, obtained by restricting the directrix curve of a ruled surface to a closed interval $[a, b]$ which is contained in the domain of the directrix curve. In the case the directrix curve is a periodic spacelike, the formulae are the same as in the Euclidean case when the length of the closed interval is equal to the period of the curve. Since the notion of angle is not defined in R^n_ν, in this paper we defined the notion of pseudo angle of pitch and which coincides with the definitions in [1–3] when the vectors are spacelike.

Let $\eta : I \to R^n_\nu$ be a curve, where $I \subset R$, and let $\{e_1(t), e_2(t), ..., e_k(t)\}$ be a given orthonormal subset of $T_{\eta(t)}(R^n_\nu)$ at each point $\eta(t)$.

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The set $\{e_1(t), e_2(t), ..., e_k(t)\}$ spans a k-dimensional subspace of the tangent space $T_{\eta(t)}(R^n_\nu)$ at the point $\eta(t)$ of R^n_ν. Let us denote this space by $E_k(t)$. Consider the set $M = \cup_{t\in I} E_k(t)$. Let us choose a parametrization for this set as

$$\varphi : I \times R^k \to R^n_\nu, \ \ \varphi(t, v_1, ..., v_k) = \eta(t) + \sum_{i=1}^{k} v_i e_i(t). \tag{1}$$

If $rank(\varphi_t, \varphi_{v_1}, ..., \varphi_{v_k}) = k+1$, then M is a $(k+1)$-dimensional submanifold of R^n_ν. This manifold is called $(k+1)$-dimensional ruled surface. The space $E_k(t)$ is called the generator space of the ruled surface at $\eta(t)$ and the curve η is called the directrix (base) curve of the ruled surface [7] . For the ruled surface M, a directrix curve may also be chosen other than η. The line, whose director vector is $e_i(t)$, that passes through $\eta(t)$ is said to be $i-th$ generator line of the surface.

If $rank(\eta'(t), e_1(t), ..., e_k(t), e'_1(t), ..., e'_k(t)) = 2k+1$, then M is said to be the non-developable ruled surface [7]. A curve which intersects each space $E_k(t)$ orthogonally is said to be an orthogonal trajectory of M. Each generator space of a non-developable ruled surface has only one central point. These central points build up a curve which is called the striction line [7]. The surface which is obtained by restricting φ to $[a, b] \times R$, is called $[a, b]$-closed piece of M [2].

Adapting the algorithm of [7] to R^n_ν , if the initial basis

$$\{e_1(t_0), e_2(t_0), ..., e_k(t_0)\}$$

is given, then the basis $\{e_1(t), e_2(t), ..., e_k(t)\}$ satisfying

$$< e_i(t), e_j(t) >= \varepsilon_i \delta_{ij} \ \ and \ \ < e'_i(t), e_j(t) >= 0, \ \ 1 \le i, j \le k \tag{2}$$

is uniquely determined, where we denote the derivative of the vector field e_v along the curve η by e'_v and $<,>$ denotes the scalar product in R^n_ν.

From now on, we assume that the orthonormal set $\{e_1(t), e_2(t), ..., e_k(t)\}$ in the equation (1) satisfies the equation in (2).

2. The Pitch and Pseudo Angle of Pitch

Theorem 2.1 *Let M be a ruled surface of dimension $(k+1)$ in R^n_ν. For each point of M, there exists a unique orthogonal trajectory passes through P* [3] .

Proof. Let $P \in M$ and $P = \varphi(t_0, v_{10}, ..., v_{k0})$. Let us consider the curve

$$\beta : I \to M \ , \ \beta(t) = \eta(t) + \sum_{i=1}^{k} f_i(t) e_i(t), \tag{3}$$

where f_i's are a function from I into R. In order β to be an orthogonal trajectory passing through P, we need to have $\beta(t_0) = P$ and

$$f_i'(t) = -\varepsilon_i < \eta'(t), e_i(t) > . \tag{4}$$

Hence we have

$$f_i(t) = -\int \varepsilon_i < \eta'(t), e_i(t) > dt + c_i.$$

If we denote the integral on the right by $F_i(t)$, i.e,

$$-\int \varepsilon_i < \eta'(t), e_i(t) > dt = F_i(t) \tag{5}$$

then

$$f_i(t) = F_i(t) + c_i \quad 1 \leq i \leq k. \tag{6}$$

Since $\beta(t_0) = P$, then we have

$$\beta(t_0) = \eta(t_0) + \sum_{i=1}^{k} (F_i(t_0) + c_i) e_i(t) = \eta(t_0) + \sum_{i=1}^{k} v_{i0} e_i(t).$$

Therefore,

$$F_i(t_0) + c_i = v_{i0}.$$

Hence

$$c_i = -F_i(t_0) + v_{i0}.$$

As a result we have

$$\beta(t) = \eta(t) + \sum_{i=1}^{k}(F_i(t) - F_i(t_0) + v_{i0})e_i(t). \tag{7}$$

Note also that

$$\beta(t) = \eta(t) + \sum_{i=1}^{k}(-\int_{t_0}^{t} \varepsilon_i < \eta'(u), e_i(u) > du + v_{i0})e_i(t).$$

□

Definition 2.2 *Let α be a regular curve in R_v^n and $\{V_1(t), V_2(t), ..., V_r(t)\}$ be Serre-Frenet frame at the point $\alpha(t)$. The functions $k_i : I \longrightarrow R$ defined by*

$$k_i(t) = \varepsilon_{V_i(t)}\varepsilon_{V_{i+i}(t)} < V_i'(t), V_{i+1}(t) >, \quad 1 \leq i \leq r-1 \tag{8}$$

are called the curvature functions on α, where $\varepsilon_{V_i(t)} =< V_i(t), V_i(t) >= 1$ or -1. The real number $k_i(t)$ is called the i–th curvature at the point $\alpha(t)$ [4].

Being inspired from [5], we may define the pseudo angle of pitch of a closed ruled surface piece.

Definition 2.3 *Let M be a non-developable $(k+1)$-dimensional ruled surface in R_ν^n. Since M is non-developable, we may take directrix curve η to be the striction line of the surface. Let us choose β to be the orthogonal trajectory which passes through the point $\eta(a) = \beta(a)$. The distance between $\eta(b)$ and $\beta(b)$ is called the pitch of $[a, b]$-closed piece of M. Let k_1 be the curvature function on β. The integral $\int_a^b k_1(t)dt$ is called the pseudo angle of pitch of $[a, b]$-closed piece of M.*

From now on, M will be taken as non-developable $(k+1)$-dimensional ruled surface in R_ν^n.

Let as denote by β the orthogonal trajectory passing through $\eta(a)$ then we have $v_{i0} = 0$. From (7), we get

$$\beta(t) = \eta(t) + \sum_{i=1}^{k}(F_i(t) - F_i(a))e_i(t).$$

Setting $t = b$,

$$\beta(b) = \eta(b) + \sum_{i=1}^{k} (F_i(b) - F_i(a)) e_i(b). \tag{9}$$

Theorem 2.4 *Let L be the pitch of $[a, b]$-closed piece of M. Then*

$$L = \sqrt{\mid \varepsilon_1 L_1^2 + \varepsilon_2 L_2^2 + ... + \varepsilon_k L_k^2 \mid}, \tag{10}$$

where $L_i = F_i(b) - F_i(a) = -\int_a^b \varepsilon_i < \eta'(t), e_i(t) > dt$.

Proof. From the definition 2.3,

$$L = \parallel \overrightarrow{\eta(b)\beta(b)} \parallel .$$

From (9)

$$\overrightarrow{\eta(b)\beta(b)} = L_1 e_1 + L_2 e_2 + ... + L_k e_k.$$

The length of this vector renders the pitch of the $[a, b]$-closed piece of M. Thus we obtain (10).

L_i mentionad above is called the pitch on the *i–th* generator of the $[a, b]$-closed piece of M.

□

Remark. If β is the orthogonal trajectory starting from $\eta(a)$ ant $\tilde{\beta}$ is the orthogonal trajectory passing through a point $P = \eta(a) + \sum_{i=1}^{k} v_{i0} e_i(a)$ in $E_k(a)$ then we have $\tilde{\beta}(t) = \beta(t) + \sum_{i=1}^{k} v_{i0} e_i(t)$. Thus the pitch of [a,b]-closed ruled surface is the distance between the point $\tilde{\beta}(b)$ and $\eta(b) + \sum_{i=1}^{k} v_{i0} e_i(b)$.

Theorem 2.5 *If we employ $\theta_{[a,b]}$ to denote the pseudo angle of the pitch of $[a, b]$-closed piece of M, then*

$$\theta_{[a,b]} = \int_a^b \varepsilon_{V_1}(t) \varepsilon_{V_2}(t) \frac{2}{\mid A(t) \mid^{1/2}} (-\frac{(B(t)^2}{A(t)} + C(t)) dt$$

where

$$A(t) =< \beta'(t), \beta'(t) >, \quad B(t) =< \beta'(t), \beta''(t) >, \quad C(t) =< \beta''(t), \beta''(t) > .$$

Proof. From (1) a parametrization of M is

$$\varphi : I \times R^k \to R^n_\nu, \ \ \varphi(t, v_1, ..., v_k) = \eta(t) + \sum_{i=1}^{k} v_i e_i(t).$$

Since η is striction line of the surface we have

$$< \eta'(t), e_i'(t) >= 0, \ , \ \ 1 \leq i \leq k. \tag{11}$$

Let β be the orthogonal trajectory passing through the point $\eta(a)$ and k_1 be curvature function on β. From (8)

$$k_1(t) = \varepsilon_{V_1(t)} \varepsilon_{V_2(t)} < V_1'(t), V_2(t) > . \tag{12}$$

Adapting the algorithm in [6] to R^n_ν, we have

$$V_1(t) = \frac{\beta'(t)}{\parallel \beta'(t) \parallel}, \ \ V_2(t) = -\frac{< \beta'(t), \beta''(t) >}{< \beta'(t), \beta'(t) >} \beta'(t) + \beta''(t). \tag{13}$$

Differentiating (3), using (2), (4) and (11) we get

$$< \beta'(t), \beta'(t) >=< \eta'(t), \eta'(t) > - \sum_{i=1}^{k} \varepsilon_i(t) (f_i'(t))^2 + \sum_{i,j=1}^{k} f_i(t) f_j(t) < e_i'(t), e_j'(t) >= A(t) \tag{14}$$

and

$$< \beta'(t), \beta''(t) >=< \eta'(t), \eta''(t) > - \sum_{i=1}^{k} \varepsilon_i(t) f_i'(t) f_i''(t) + \sum_{i,j=1}^{k} f_i(t) f_j'(t) < e_i'(t), e_j'(t) >$$

$$+ \sum_{i,j=1}^{k} f_i(t) f_j(t) < e_i'(t), e_j''(t) >= B(t). \tag{15}$$

Now putting (14) and (15) in (13), we have

$$V_1(t) = \frac{\beta'(t)}{\mid A(t) \mid^{1/2}}, V_2(t) = -\frac{B(t)}{A(t)} \beta'(t) + \beta''(t)$$

and thus

$$< V_1'(t), V_2(t) >= \frac{2}{\mid A(t) \mid^{1/2}}(-\frac{(B(t))^2}{A(t)} + C(t)).$$

If we replace these in (10) we get

$$k_1 = \varepsilon_{V_1}(t)\varepsilon_{V_2}(t)\frac{2}{\mid A(t) \mid^{1/2}}(-\frac{(B(t))^2}{A(t)} + C(t)).$$

This completes the proof. □

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