Turk J Math
30 (2006) , 33 – 41.

p-Elastica in the 3-Dimensional Lorentzian Space Forms

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Abstract

R Huang worked the p-elastic in a Riemannian manifold with constant sectional curvature [1]. In this work, we solve the Euler-Lagrange equation by quadrature and study the Frenet equation of the p-elastica by using the Killing field in the three dimensional Lorentzian space forms.

1. Introduction

Definition 1.1 *Let L be a 3-dimensional Lorentzian space. If (x_1, x_2, x_3) and (y_1, y_2, y_3) are the components of X and Y with respect to an allowable coordinate system, then*

$$\langle X, Y\rangle\,|_L = x_1y_1 + x_2y_2 - x_3y_3$$

which is called a Lorentzian inner product. Furthermore, a Lorentz exterior product $X \times Y$ is given by

$$X \times Y = (x_2y_3 - x_3y_2,\ x_3y_1 - x_1y_3, x_2y_1 - x_1y_2).$$

Then, for any $x \in L^3$ it holds ([2])

$$\langle X \times Y, X \times Y\rangle = \langle X, Y\rangle^2 - \langle X, X\rangle \langle Y, Y\rangle .$$

Mathematics Subject Classification: 53A04, 35Q51, 53B99, 53A35.

Definition 1.2 *A semi-Riemannian manifold M has constant curvature if its sectional curvature function is constant. If M constant curvature C , then ([3])*

$$R_{xy}z = C\{\langle z, x\rangle\, y - \langle z, y\rangle\, x\}.$$

Definition 1.3 *The norm of $\overrightarrow{X} \in R_1^3$ is denoted by $\overrightarrow{\|X\|}$ and defined as ([3])*

$$\overrightarrow{\|X\|} = \sqrt{\left|\left\langle \overrightarrow{X}, \overrightarrow{X} \right\rangle\right|}.$$

Theorem 1.1 *Let $\gamma(s)$ be a unit speed curve in R_1^3, s the arclength parameter. Consider the Frenet frame $\{T = \gamma', N, B\}$ attached to the curve $\gamma = \gamma(s)$ such that is T is the unit tangent vector field, N is the principal normal vector field and $B = T \times N$ is the binormal vector field. The Frenet -Serret formulas is given by*

$$\begin{bmatrix} \nabla_T T = T' \\ \nabla_T N = N' \\ \nabla_T B = B' \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2 \kappa & 0 \\ -\varepsilon_1 \kappa & 0 & -\varepsilon_3 \tau \\ 0 & \varepsilon_2 \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{1.1}$$

where $\langle T, T\rangle = \varepsilon_1, \langle N, N\rangle = \varepsilon_2, \langle B, B\rangle = \varepsilon_3$. ∇ is the semi Riemannian connection on M and $\kappa = \kappa(s)$ and $\tau = \tau(s)$ are the curvature and the torsion functions of γ, respectively.

2. Killing Fields

This section is taken from [4], [5]. Let $\gamma(t)$ be a nonnull immersed curve in three dimensional Lorentzian space form M with sectional curvature C. Let us consider a variation of γ, $\gamma = \gamma(w, t) : (-\varepsilon, \varepsilon) \times I \longrightarrow M$ with $\gamma(0, t) = \gamma(t)$. Associated with γ are two vector fields along γ, $W(w, t) = (\partial\gamma/\partial w)(w, t)$ and $V(w, t) = (\partial\gamma/\partial t)(w, t)$. $W = (\partial\gamma/\partial w)(0, t)$ is the variation vector field along γ and $V = (\partial\gamma/\partial t)$ velocity vector field. $T = T(t)$ will be denoted the unit tangent vector field and the speed of γ will be $v = |\langle V, V\rangle|^{1/2}$. The curvature of γ is defined by $\kappa(t) = |\langle \nabla_T T, \nabla_T T\rangle|^{1/2}$. We will use the notation $W = W(w, t)$, $v = v(w, t)$. If s denote the arclength parameter of the t-curves, we write $v(w, s)$, $\kappa(w, s)$, etc., for the corresponding reparametrizations. Then $s \in [0, L]$, where L is the length of γ. With a direct computation is given following lemma.

Lemma 2.1 *By the above notation, the following formulas holds:*

$$\begin{array}{ll}
(1) & [W,V]=[W,vT]=0 \\
(2) & (\partial v/\partial w)(0,t)=-\varepsilon_1 gv \quad \text{with } g=\langle \nabla_T W, T\rangle \\
(3) & (\partial \kappa^2/\partial w)(0,t)=2\varepsilon_2\left\langle \nabla_T^2 W, \nabla_T T\right\rangle+4\varepsilon_1 g\kappa^2+2\varepsilon_2\left\langle R(W,T)T,\nabla_T T\right\rangle=0. \\
(4) & (\partial \tau^2/\partial w)(0,t)=-2\varepsilon_2\left\langle \begin{array}{c} (1/\kappa)\nabla_T^3 W-(\kappa'/\kappa^2)\nabla_T^2 W+\varepsilon_1(\varepsilon_2\kappa+(C/\kappa))\nabla_T W \\ -\varepsilon_1(C\kappa'/\kappa^2)W,\tau B \end{array}\right\rangle
\end{array} \tag{2.1}$$

where $\langle , \rangle$ *denotes the Lorentzian metric of* M *and* $\kappa'=(\partial\kappa/\partial w)(0,t)$.

Let M be a complete, simply connected, Lorentzian space form and γ a nonnull immersed curve in M. Killing vector fields along γ are charecterized by the equations

$$\frac{\partial v}{\partial w}(0,t)=\frac{\partial \kappa^2}{\partial w}(0,t)=\frac{\partial \tau^2}{\partial w}(0,t)=0. \tag{2.2}$$

W is a Killing vector field along γ if and only if it satisfies the following conditions:

$$\begin{array}{ll}
(1) & \langle \nabla_T W, T\rangle=0 \\
(2) & \left\langle \nabla_T^2 W, N\right\rangle+\varepsilon_1 C\langle W,N\rangle=0 \\
(3) & \left\langle (1/\kappa)\nabla_T^3 W-(\kappa'/\kappa^2)\nabla_T^2 W+\varepsilon_1(\varepsilon_2\kappa+(C/\kappa))\nabla_T W-\varepsilon_1 C(\kappa'/\kappa^2)W,\ \tau B\right\rangle=0.
\end{array} \tag{2.3}$$

3. Equilibrium Equations and the p-elastica

We take $P(\kappa)$ a polynomial of κ with degree $\geqslant 2$ and consider the following curvature energy functional in Minkowski space:

$$\int_0^{L(w)} p(\kappa)ds\big|_{w=0}. \tag{3.1}$$

Now, we want to compute the first derivative restriction of this curvature energy functional:

$$\frac{\partial}{\partial w}\int\limits_0^{L(w)} p(\kappa)ds|_{w=0} = \frac{\partial}{\partial w}\int\limits_0^{L(w)} p(\kappa)vdt|_{w=0}$$
$$= \int\limits_I [p'(\kappa)W(\kappa)v + p(\kappa)\frac{\partial v}{\partial w}]dt\Big|_{w=0}$$
$$= \int\limits_0^L [p'(\kappa)(2\varepsilon_2 \left\langle \nabla_T^2 W, \nabla_T T\right\rangle + 4\varepsilon_1 g\kappa^2 + 2\varepsilon_2 \left\langle R(W,T)T, \nabla_T T\right\rangle) - p(\kappa)\varepsilon_1 g]ds,$$

where $W(0,0) = W(0,L) = 0, \nabla_T W(0,0) = \nabla_T W(0,L) = 0$ and $L(w)$ is the arclength of $\gamma(w,t)$. Then we can give the first variational formula:

$$\frac{\partial}{\partial w}\int\limits_0^{L(w)} p(\kappa)ds|_{w=0} = \int\limits_0^L [\varepsilon_2 \left\langle R(W,T)T + \nabla_T^2 W, \frac{p'(\kappa)}{\kappa}\nabla_T T\right\rangle + \varepsilon_1(2\kappa p'(\kappa) - p(\kappa))g]ds$$
$$= \int\limits_0^L \left\langle \varepsilon_2 \nabla_T^2(\frac{p'(\kappa)}{\kappa}\nabla_T T) + \varepsilon_2 \frac{p'(\kappa)}{\kappa} C\nabla_T T + \varepsilon_1 \nabla_T[(2\kappa p'(\kappa) - p(\kappa))T], W\right\rangle ds. \tag{3.2}$$

So, we obtain Euler-Lagrange equations:

$$E = \varepsilon_2 \nabla_T^2(\frac{p'(\kappa)}{\kappa}\nabla_T T) + \varepsilon_2 \frac{p'(\kappa)}{\kappa} C\nabla_T T + \varepsilon_1 \nabla_T[(2\kappa p'(\kappa) - p(\kappa))T] = 0. \tag{3.3}$$

Definition 3.1 *A regular unit-speed curve is called a p-elastica if it satisfies the above Euler- Lagrange equation (3.3).*

We give the variational formulas by using Theorem 1.1:

$$\begin{aligned} E \quad = \quad & [\varepsilon_2 p'''(\kappa)\kappa'^2 + \varepsilon_2 p''(\kappa)\kappa'' + p'(\kappa)(-\varepsilon_3\tau^2 + \varepsilon_1\kappa^2 + C) \\ & -\varepsilon_1\varepsilon_2 p(\kappa)\kappa]N - \varepsilon_2\varepsilon_3(2p''(\kappa)\kappa'\tau + p'(\kappa)\tau')B \end{aligned} \tag{3.4}$$

Then, we obtain the Euler-Lagrange equation

$$
\begin{aligned}
&\varepsilon_2 p'''(\kappa)\kappa'^2 + \varepsilon_2 p''(\kappa)\kappa'' \\
&+\varepsilon_2 p'(\kappa)(-\varepsilon_2\varepsilon_3\tau^2 + \varepsilon_1\varepsilon_2\kappa^2 + C) - \varepsilon_1\varepsilon_2 p(\kappa)\kappa = 0, \\
&-\varepsilon_2\varepsilon_3(2p''(\kappa)\kappa'\tau + p'(\kappa)\tau') = 0.
\end{aligned} \tag{3.5}
$$

For κ and τ is constant, from Eq. (3.5),

$$
\varepsilon_2 p'(\kappa)(\varepsilon_1\varepsilon_2\kappa^2 - \varepsilon_2\varepsilon_3\tau^2 + \varepsilon_2 C) - \varepsilon_1\varepsilon_2 p(\kappa)\kappa = 0. \tag{3.6}
$$

In this situation, we can give the formula without intermediaries.

If κ is not constant, from the second of Eq. (3.5),

$$
p'(\kappa)^2\tau = k_1. \tag{3.7}
$$

Here, k_1 is a constant. From the integral of the first of Eq. (3.5), we have,

$$
\varepsilon_1(\kappa p'(\kappa) - p(\kappa))^2 + \varepsilon_2(p''(\kappa)\kappa')^2 + Cp'(\kappa)^2 - \varepsilon_3\frac{k_1^2}{p'(\kappa)^2} = k_2. \tag{3.8}
$$

Here k_2 constant. Then, we can give the curvature $\kappa(s)$ by quadratures

$$
\pm\sqrt{\frac{\varepsilon_2 p'(\kappa)^2 p''(\kappa)^2}{p'(\kappa)^2(k_2 - Cp'(\kappa)^2 - \varepsilon_1(\kappa p'(\kappa) - p(\kappa))^2\varepsilon_3 k_1^2}}\,d\kappa = \int ds. \tag{3.9}
$$

We construct the Killing field along the p-elastica $\gamma(s)$. Let this Killing field be of the form

$$
W = h_1(s)T(s) + h_2(s)N(s) + h_3(s)B(s),
$$

where the functions h_1, h_2 and h_3 must satisfy the following equations:

$$\begin{aligned} \varepsilon_1 h_1' - \kappa h_2 &= 0 \\ h_1\kappa' + \varepsilon_2 h_1'\kappa + h_1'\kappa + \varepsilon_2 h_2'' + h_2(-\varepsilon_1\kappa^2 - \varepsilon_3\tau^2 + \varepsilon_1\varepsilon_2 C) + 2h_3'\tau + h_3\tau' &= 0 \end{aligned} \quad (3.10)$$

$$\begin{aligned} &-\varepsilon_2 h_1(\kappa'\kappa\tau^2 + \kappa^2\tau'\tau) + h_2'(-2\varepsilon_2\varepsilon_3\kappa\tau^2 - \tau\kappa\tau' - \tau^2\kappa' - \varepsilon_2\varepsilon_3\tau'\kappa') + \\ &h_2(\varepsilon_1\varepsilon_2\tau^2\kappa^3 + \varepsilon_2\varepsilon_3\tau^4\kappa - \varepsilon_2\varepsilon_3\tau''\tau\kappa - \varepsilon_1(\varepsilon_2\kappa^2 + C)\kappa\tau^2 - \varepsilon_2\varepsilon_3\tau'\kappa'\tau) - \\ &2\tau^2\kappa h_2'' - 3\varepsilon_2\kappa^2\tau h_1' + \varepsilon_3\tau\kappa h_3' + \varepsilon_3\kappa'\tau h_3'' + \\ &h_3(-\varepsilon_2\tau'\tau^2\kappa - 2\tau^2\tau'\kappa - \varepsilon_2\kappa'\tau^3 - \varepsilon_1\varepsilon_3 C\kappa'\tau) + \\ &h_3'(-3\varepsilon_2\tau^3\kappa + \varepsilon_1\varepsilon_3(\varepsilon_2\kappa^3 + C\kappa)\tau) + \\ &h_3'(-3\varepsilon_2\tau^3\kappa + \varepsilon_1\varepsilon_3(\varepsilon_2\kappa^3 + C\kappa)\tau) \quad = 0 \end{aligned}$$

With aid these equations and the Euler-lagrange Eq.(3.5), we obtained that the vector fields

$$J_\gamma = \varepsilon_1(p'(k)k - p(k))T + p''(k)k'N - \varepsilon_3 p'(k)\tau B \quad (3.11)$$

and

$$H_\gamma = -\varepsilon_2 p'(k)B \quad (3.12)$$

are Killing along the p-elastica γ in $3-$dimensional.Lorentzian space form. The solutions of Eq. (3.10) constitute a six dimensional linear space. So we give the following theorem.

Theorem 3.1 *Let M a simply connected manifold with constant sectional curvature C in Lorentzian space form, and let γ be a p-elastica in M. Vector fields $J_\gamma = \varepsilon_1(p'(k)k - p(k))T + p''(k)k'N - \varepsilon_3 p'(k)\tau B$ and $H_\gamma = -\varepsilon_2 p'(k)B$ can expanded to Killing fields J_γ' and H_γ' on M.*

In 3-dimensional Lorentzian space form, we construct a system of cylindrical coordinates using the Killing fields J_γ and H_γ. The Euler Lagrange equation and its first integral intimate are written

$$\nabla_T J_\gamma = -Cp'(k)N \quad (3.13)$$

and

$$\begin{aligned} \|J_\gamma\|^2 &= |\langle J_\gamma, J_\gamma\rangle| \qquad (3.14)\\ &= \left|\varepsilon_1(p'(k)k - p(k))^2 + \varepsilon_2(p''(k)k')^2 - \varepsilon_3(p'(k)\tau)^2\right| = \left|k_2 - Cp'(k)^2\right|. \end{aligned}$$

In R_1^3, $\nabla_T J_\gamma = 0$. Then the Killing field J_γ is a translation field. We can find one coordinate field $\dfrac{\partial}{\partial z} = \dfrac{J_\gamma}{|\langle J_\gamma, J_\gamma\rangle|^{1/2}}$. Due to

$$\langle J_\gamma, H_\gamma\rangle = \varepsilon_2 p'(k)^2\tau = \varepsilon_2 k_1, \qquad (3.15)$$

H_γ shows a rotation along z direction.

$$J_1 = \varepsilon_2 J_\gamma - \frac{1}{k_1}\langle J_\gamma, J_\gamma\rangle H_\gamma \qquad (3.16)$$

is a rotation field perpendicular to J_γ. So, for some normalization factor, we have $\dfrac{\partial}{\partial\varphi} = QJ_1$. Hence

$$\frac{\partial}{\partial r} = \frac{J_\gamma \times B}{|\langle J_\gamma \times B, J_\gamma \times B\rangle|^{1/2}} \qquad (3.17)$$

is given. We can write the unit tangent vector as

$$T = r_s(\partial/\partial r) + \varphi_s(\partial/\partial\varphi) + z_s(\partial/\partial z),$$

with

$$r_s = \left\langle T, \frac{\partial}{\partial r}\right\rangle = \frac{\langle T, J_\gamma \times B\rangle}{|\langle J_\gamma \times B, J_\gamma \times B\rangle|^{1/2}} = -\frac{\varepsilon_1 p'(k)p''(k)k'}{|\varepsilon_3 k_2 p'(k)^2 + k_1^2|^{1/2}}, \qquad (3.18)$$

$$z_s = \left\langle T, \frac{\partial}{\partial z}\right\rangle = \frac{p'(k)k - p(k)}{|\varepsilon_3 k_2 p'(k)^2 + k_1^2|^{1/2}}. \qquad (3.19)$$

QJ_1 has the proper length at the maxima of $\kappa(s)$. Then the length of $\partial/\partial\varphi$ at such a point $\gamma(s)$ is $r = r(s_0)$, the reciprocal of the curvature κ_0 of the circle $r = r(s_0)$, $z = z(s_0)$. At this point, T has vertical component

$$\left\langle T, \frac{J_\gamma}{\left|\langle J_\gamma, J_\gamma\rangle\right|^{1/2}} \right\rangle = \frac{p'(k)k - p(k)}{\left|\langle J_\gamma, J_\gamma\rangle\right|^{1/2}}$$

and the horizontal component is

$$-\frac{p'(k)\tau}{\left|\langle J_\gamma, J_\gamma\rangle\right|^{1/2}}.$$

Note that

$$\begin{aligned} \kappa_0 &= -\frac{\left|\langle J_\gamma, J_\gamma\rangle\right|^{1/2}}{p'(k)\tau} \left| \nabla_T \left(\frac{\partial/\partial\varphi}{\left|\langle \partial/\partial\varphi, \partial/\partial\varphi\rangle\right|^{1/2}} \right) \right| \\ &= -\frac{\left|\langle J_\gamma, J_\gamma\rangle\right|^{3/2}}{k_1 \left|\langle J_1, J_1\rangle\right|}. \end{aligned} \tag{3.20}$$

Hence,

$$Q = \frac{k_1}{\left|k_2\right|^{3/2}}$$

and

$$\varphi_s = \frac{\langle T, (\partial/\partial\varphi)\rangle}{\left|\langle \partial/\partial\varphi, \partial/\partial\varphi\rangle\right|} = \frac{\varepsilon_2 \left|k_2\right|^{3/2} (p'(k)k - p(k))}{k_1 \left|\langle J_1, J_1\rangle\right|}. \tag{3.21}$$

So, we give the following theorem.

Theorem 3.2 *Let (r, φ, z) be cylindrical coordinates given above in R_1^3, and $\gamma(s) = (r(s), \varphi(s), z(s))$. Then we give*

$$r_s = -\frac{\varepsilon_1 p'(k) p''(k) k'}{\left|\varepsilon_3 k_2 p'(k)^2 + k_1^2\right|^{1/2}}, \quad z_s = \frac{p'(k)k - p(k)}{\left|\varepsilon_3 k_2 p'(k)^2 + k_1^2\right|^{1/2}}, \quad \varphi_s = \frac{\varepsilon_2 \left|k_2\right|^{3/2} (p'(k)k - p(k))}{k_1 \left|\langle J_1, J_1\rangle\right|}.$$

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Received 30.07.2004