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Inequality for Ricci Curvature of Slant Submanifolds in Cosymplectic Space Forms

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Abstract

In this article, we establish inequalities between the Ricci curvature and the squared mean curvature, and also between the k-Ricci curvature and the scalar curvature for a slant, semi-slant and bi-slant submanifold in a cosymplectic space form of constant φ- sectional curvature with arbitrary codimension.

Key Words: Mean curvature, sectional curvature, k-Ricci curvature, slant submanifold, semi-slant submanifold, bi-slant submanifold, cosymplectic space form.

1. Introduction

Let $\tilde{M}$ be a $(2m+1)$-dimensional almost contact manifold endowed with an almost contact structure (φ, ξ, η), that is, φ is a (1,1) tensor field, ξ is a vector field and η is a 1-form such that

$$\varphi^2 = -I + \eta \otimes \xi \quad \text{and} \quad \eta(\xi) = 1.$$

Then, $\varphi(\xi) = 0$ and $\eta \circ \varphi = 0$. Let g be a compatible Riemannian metric with (φ, ξ, η), that is, $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$ or equivalent, $g(X, \varphi Y) = -g(\varphi X, Y)$ and $g(X, \xi) = \eta(X)$ for all $X, Y \in \tilde{M}$. Then, $\tilde{M}$ becomes an almost contact metric manifold equipped with an almost contact metric structure (φ, ξ, η, g). An almost contact metric manifold is *cosymplectic* ([1]) if $\tilde{\nabla}_X \varphi = 0$, where $\tilde{\nabla}$ is the Levi-Civita connection of the Riemannian metric g. From the formula $\tilde{\nabla}_X \varphi = 0$ it follows that $\tilde{\nabla}_X \xi = 0$.

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A plane section π in $T_p\tilde{M}$ of an almost contact metric manifold $\tilde{M}$ is called a *φ-section* if $\pi \perp \xi$ and $\varphi(\pi) = \pi$. $\tilde{M}$ is of *constant φ-sectional curvature* if sectional curvature $\tilde{K}(\pi)$ does not depend on the choice of the φ-section π of $T_p\tilde{M}$ and the choice of a point $p \in \tilde{M}$. A cosymplectic manifold $\tilde{M}$ is said to be a *cosymplectic space form* if the φ-sectional curvature is constant c along $\tilde{M}$. A cosymplectic space form will be denoted by $\tilde{M}(c)$. Then the Riemannian curvature tensor $\tilde{R}$ on $\tilde{M}(c)$ is given by ([9])

$$\begin{aligned} 4\tilde{R}(X,Y,Z,W) =& c\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W) + g(X,\varphi W)g(Y,\varphi Z) \\ & - g(X,\varphi Z)g(Y,\varphi W) - 2g(X,\varphi Y)g(Z,\varphi W) - g(X,W)\eta(Y)\eta(Z) \\ & + g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W) + g(Y,W)\eta(X)\eta(Z)\}. \end{aligned} \tag{1.1}$$

Let M be an n-dimensional submanifold of a cosymplectic space form $\tilde{M}(c)$ equipped with a Riemannian metric g. The Gauss and Wiengarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad \text{and} \quad \tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N$$

for all $X, Y \in TM$ and $N \in T^{\perp}M$, where $\tilde{\nabla}, \nabla$ and $\nabla^{\perp}$ are the Riemannian, induced Riemannian and induced normal connections in $\tilde{M}(c), M$ and the normal bundle $T^{\perp}M$ of M respectively, and h is the second fundamental form related to the shape operator A by $g(h(X,Y),N) = g(A_N X, Y)$. Then the equation of Gauss is given by

$$\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)), \tag{1.2}$$

for any vectors X, Y, Z, W tangent to M.

For any vector X tangent to M we put $\varphi X = PX + FX$, where PX and FX are the tangential and the normal components of φX, respectively. Given an orthonormal basis $\{e_1, \ldots, e_n\}$ of M, we define the squared norm of P by

$$||P||^2 = \sum_{i,j=1}^{n} g^2(\varphi e_i, e_j)$$

and the mean curvature vector $H(p)$ at $p \in M$ is given by $H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$.
We put

$$h_{ij}^r = g(h(e_i,e_j),e_r) \quad \text{and} \quad ||h||^2 = \sum_{i,j=1}^{n} g(h(e_i,e_j),h(e_i,e_j)),$$

where $\{e_{n+1},\ldots,e_{2m+1}\}$ is an orthonormal basis of $T_p^{\perp}M$ and $r=n+1,\ldots,2m+1$. A submanifold M in $\tilde{M}(c)$ is called *totally geodesic* if the second fundamental form vanishes identically and *totally umbilical* if there is a real number λ such that $h(X,Y)=\lambda g(X,Y)H$ for any tangent vectors X,Y on M.

For an n-dimensional Riemannian manifold M, we denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi\subset T_pM, p\in M$. For an orthonormal basis $\{e_1,\ldots,e_n\}$ of the tangent space T_pM, the scalar curvature τ is defined by

$$\tau=\sum_{i<j}K_{ij}, \tag{1.3}$$

where K_{ij} denotes the sectional curvature of the 2-plane section spanned by e_i and e_j.

Suppose L is a k-plane section of T_pM and X a unit vector in L. We choose an orthonormal basis $\{e_1,\ldots,e_k\}$ of L such that $e_1=X$. Define the Ricci curvature Ric_L of L at X by

$$Ric_L(X)=K_{12}+\cdots+K_{1k}. \tag{1.4}$$

We simply call such a curvature a *k-Ricci curvature.* The scalar curvature τ of the k-plane section L is given by

$$\tau(L)=\sum_{1\le i<j\le k}K_{ij}. \tag{1.5}$$

For each integer $k, 2\le k\le n$, the Riemannain invariant Θ_k on an n-dimensional Riemannian manifold M is defined by

$$\Theta_k(p)=\frac{1}{k-1}\inf_{L,X}Ric_L(X),\quad p\in M, \tag{1.6}$$

where L runs over all k-plane sections in T_pM and X runs over all unit vectors in L.

Recall that for a submanifold M in a Riemannain manifold, the relative null space of M at a point $p\in M$ is defined by

$$N_p=\{X\in T_pM|h(X,Y)=0\quad\text{for all}\quad Y\in T_pM\}.$$

In [8], A. Lotta has introduced the following notion of slant submanifolds into almost contact metric manifolds. A submanifold M tangent to ξ is said to be *slant* if for any

$p \in M$ and any $X \in T_pM$, linearly independent of ξ, the angle between φX and T_pM is a constant $\theta \in [0, \pi/2]$, called the *slant angle* of M in $\tilde{M}(c)$. Invariant and anti-invariant submanifolds of $\tilde{M}(c)$ are slant submanifolds with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively.

We say that a submanifold M tangent to ξ is a *bi-slant* submanifolf in $\tilde{M}(c)$ if there exist two orthogonal distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ on M such that

(1) TM admits the orthogonal direct decomposition $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \{\xi\}$

(2) For any $i = 1, 2$, $\mathcal{D}_i$ is slant distribution with slant angle θ_i.

On the other hand, CR-submanifolds of $\tilde{M}(c)$ are bi-slant submanifolds with $\theta_1 = 0,\ \theta_2 = \pi/2$.

Let $2d_1 = \dim\mathcal{D}_1$ and $2d_2 = \dim\mathcal{D}_2$.

Remark. If either d_1 or d_2 vanishes, the bi-slant submanifold is a slant submanifold. Thus, slant submanifolds are particular cases of bi-slant submanifolds.

A submanifold M tangent to ξ is called a *semi-slant* submanifold in $\tilde{M}(c)$ if there exist two orthogonal distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ on M such that

(1) TM admits the orthogonal direct decomposition $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \{\xi\}$.

(2) The distribution $\mathcal{D}_1$ is an invariant distribution, i.e., $\varphi(\mathcal{D}_1) = \mathcal{D}_1$.

(3) The distribution $\mathcal{D}_2$ is slant with angle $\theta \neq 0$.

Remark. The invariant distribution of a semi-slant submanifold is a slant distribution with zero angle. Thus, it is obvious that in fact, semi-slant submanifolds are particular cases of bi-slant submanifolds.

(1) If $d_2 = 0$, then M is an invariant submanifold.

(2) If $d_1 = 0$ and $\theta = \pi/2$, then M is an anti-invariant submanifold.

For the other properties and examples of slant, bi-slant and semi-slant submanifolds in almost contact metric manifold, we refer to the reader [2], [3].

2. Ricci Curvature and Squared Mean Curvature

B.Y. Chen established a sharp relationship between the Ricci curvature and the squared mean curvature for submanifolds in real space forms (see [6]). We prove similar inequalities for slant, bi-slant and semi-slant submanifolds in a cosymplectic space form

$\tilde{M}(c)$. We consider submanifolds M tangent to the vector field ξ.

Theorem 2.1 *Let M be an n-dimensional θ-slant submanifold tangent to ξ into a $(2m+1)$-dimensional cosymplectic space form $\tilde{M}(c)$. Then, we have*

(1) *For each unit vector $X \in T_pM$ orthogonal to ξ*

$$\mathrm{Ric}(X) \leq \frac{1}{4}\left\{(n-1)c + \frac{1}{2}(3\cos^2\theta - 2)c + n^2||H||^2\right\}. \tag{2.1}$$

(2) *If $H(p) = 0$, then a unit tangent vector X orthogonal to ξ at p satisfies the equality case of* (2.1) *if and only if $X \in N_p$.*

(3) *The equality case of* (2.1) *holds identically for all unit tangent vectors orthogonal to ξ at p if and only if p is a totally geodesic point.*

Proof. Let $X \in T_pM$ be a unit tangent vector at p orthogonal to ξ. We choose an othonormal basis $e_1, \cdots, e_n = \xi, e_{n+1}, \cdots, e_{2m+1}$ such that $e_1, \cdots, e_n$ are tangent to M at p with $e_1 = X$. Then, from the equation of Gauss, we have

$$n^2||H||^2 = 2\tau + ||h||^2 - \{n(n-1) + 3(n-1)\cos^2\theta - 2n + 2\}\frac{c}{4}. \tag{2.2}$$

From (2.2) we get

$$\begin{aligned} n^2||H||^2 &= 2\tau + \sum_{r=n+1}^{2m+1}[(h_{11}^r)^2 + (h_{22}^r + \cdots + h_{nn}^r)^2 + 2\sum_{1\leq i<j\leq n}(h_{ij}^r)^2] \\ &\quad - 2\sum_{r=n+1}^{2m+1}\sum_{2\leq i<j\leq n} h_{ii}^r h_{jj}^r - \frac{c}{4}[n(n-1) + 3(n-1)\cos^2\theta - 2n + 2] \\ &= 2\tau + \frac{1}{2}\sum_{r=n+1}^{2m+1}\left[(h_{11}^r + h_{22}^r + \cdots + h_{nn}^r)^2 + (h_{11}^r - h_{22}^r - \cdots - h_{nn}^r)^2\right] \\ &\quad + 2\sum_{r=n+1}^{2m+1}\sum_{1\leq i<j\leq n}(h_{ij}^r)^2 - 2\sum_{r=n+1}^{2m+1}\sum_{2\leq i<j\leq n} h_{ii}^r h_{jj}^r \\ &\quad - \frac{c}{4}[n(n-1) + 3(n-1)\cos^2\theta - 2n + 2]. \end{aligned} \tag{2.3}$$

By using the equation of Gauss, we have

$$\sum_{2\le i<j\le n} K_{ij} = \sum_{r=n+1}^{2m+1} \sum_{2\le i<j\le n} [h^r_{ii}h^r_{jj} - (h^r_{ij})^2] + (n-1)(n-2)\frac{c}{8} \\ + [3(n-2)\cos^2\theta - 2n + 4]\frac{c}{8}. \tag{2.4}$$

Substituting (2.4) in (2.3), we get

$$\frac{1}{2}n^2||H||^2 \ge 2\mathrm{Ric(X)} - 2(n-1)\frac{c}{4} - (3\cos^2\theta - 2)\frac{c}{4},$$

or equivalently (2.1).
(2) Assume $H(P) = 0$. Equality holds in (2.1) if and only if

$$\begin{cases} h^r_{12} = \cdots = h^r_{1n} = 0, \\ h^r_{11} = h^r_{22} + \cdots + h^r_{nn}, \quad r \in \{n+1, \cdots, 2m+1\}. \end{cases}$$

Then $h^r_{1j} = 0$ for all $j \in \{1, \cdots, n\}, r \in \{n+1, \cdots, 2m+1\}$, that is, $X \in N_p$.
(3) Then equality case of (2.1) holds for all unit tangent vectors orthogonal to ξ at p if and only if

$$\begin{cases} h^r_{ij} = 0, \quad i \ne j, \quad r \in \{n+1, \cdots, 2m+1\}, \\ h^r_{11} + \cdots + h^r_{nn} - 2h^r_{ii} = 0, \quad i \in \{1, \cdots, n\}, \quad r \in \{n+1, \cdots, 2m+1\}. \end{cases}$$

In this case, it follows that p is a totally geodesic point. The converse is trivial. □

Theorem 2.2 *Let M be an n-dimensional bi-slant submanifold satisfying $g(X, \varphi Y) = 0$, for any $X \in \mathcal{D}_1$ and any $Y \in \mathcal{D}_2$, tangent to ξ in a $(2m+1)$-dimensional cosymplectic space form $\tilde{M}(c)$. Then,*

(1) *For each unit vector $X \in T_pM$ orthogonal to ξ and if*

(i) *X is tangent to $\mathcal{D}_1$ we have*

$$\mathrm{Ric}(X) \le \frac{1}{4}\left\{(n-1)c + \frac{1}{2}(3\cos^2\theta_1 - 2)c + n^2||H||^2\right\}; \tag{2.5}$$

and if

(ii) *X is tangent to $\mathcal{D}_2$, we have*

$$\mathrm{Ric}(X) \le \frac{1}{4}\left\{(n-1)c + \frac{1}{2}(3\cos^2\theta_2 - 2)c + n^2||H||^2\right\}. \tag{2.6}$$

(2) *If $H(p) = 0$, then a unit tangent vector X orthogonal to ξ at p satisfies the equality case of* (2.5) *and* (2.6) *if and only if $X \in N_p$.*

(3) *The equality case of* (2.5) *and* (2.6) *holds identically for all unit tangent vectors orthogonal to ξ at p if and only if p is a totally geodesic point.*

Proof. Let $X \in T_pM$ be a unit tangent vector at p orthogonal to ξ. We choose an othonormal basis $e_1, \cdots, e_n = \xi, e_{n+1}, \cdots, e_{2m+1}$ such that $e_1, \cdots, e_n$ are tangent to M at p with $e_1 = X$. Then, from the equation of Gauss, we have

$$n^2||H||^2 = 2\tau + ||h||^2 - \{n(n-1) + 6(d_1\cos^2\theta_1 + d_2\cos^2\theta_2) - 2n + 2\}\frac{c}{4}, \tag{2.7}$$

where $2d_1 = \dim\mathcal{D}_1$ and $2d_2 = \dim\mathcal{D}_2$.
From (2.7) we get

$$\begin{aligned} n^2||H||^2 = 2\tau &+ \sum_{r=n+1}^{2m+1}[(h_{11}^r)^2 + (h_{22}^r + \cdots + h_{nn}^r)^2 + 2\sum_{1\le i<j\le n}(h_{ij}^r)^2] \\ &- 2\sum_{r=n+1}^{2m+1}\sum_{2\le i<j\le n} h_{ii}^r h_{jj}^r - \frac{c}{4}[n(n-1) + 6(d_1\cos^2\theta_1 + d_2\cos^2\theta_2) - 2n + 2] \\ = 2\tau &+ \frac{1}{2}\sum_{r=n+1}^{2m+1}\left[(h_{11}^r + h_{22}^r + \cdots + h_{nn}^r)^2 + (h_{11}^r - h_{22}^r - \cdots - h_{nn}^r)^2\right] \\ &+ 2\sum_{r=n+1}^{2m+1}\sum_{1\le i<j\le n}(h_{ij}^r)^2 - 2\sum_{r=n+1}^{2m+1}\sum_{2\le i<j\le n} h_{ii}^r h_{jj}^r \\ &- \frac{c}{4}[n(n-1) + 6(d_1\cos^2\theta_1 + d_2\cos^2\theta_2) - 2n + 2]. \end{aligned} \tag{2.8}$$

We distinguish two cases:

(i) if X is tangent to $\mathcal{D}_1$, then we have

$$\sum_{2\le i<j\le n} K_{ij} = \sum_{r=n+1}^{2m+1} \sum_{2\le i<j\le n} [h^r_{ii}h^r_{jj} - (h^r_{ij})^2] + (n-1)(n-2)\frac{c}{8} \\ + [6(d_1\cos^2\theta_1 + d_2\cos^2\theta_2) - 3\cos^2\theta_1 - 2n + 4]\frac{c}{8}. \tag{2.9}$$

Substituting (2.9) in (2.8), one gets

$$\frac{1}{2}n^2||H||^2 \ge 2\mathrm{Ric(X)} - 2(n-1)\frac{c}{4} - (3\cos^2\theta_1 - 2)\frac{c}{4},$$

which is equivalent to (2.5).

(ii) if X is tangent to $\mathcal{D}_2$, then we have

$$\sum_{2\le i<j\le n} K_{ij} = \sum_{r=n+1}^{2m+1} \sum_{2\le i<j\le n} [h^r_{ii}h^r_{jj} - (h^r_{ij})^2] + (n-1)(n-2)\frac{c}{8} \\ + [6(d_1\cos^2\theta_1 + d_2\cos^2\theta_2) - 3\cos^2\theta_2 - 2n + 4]\frac{c}{8}. \tag{2.10}$$

Substituting (2.10) in (2.8), one gets

$$\frac{1}{2}n^2||H||^2 \ge 2\mathrm{Ric(X)} - 2(n-1)\frac{c}{4} - (3\cos^2\theta_2 - 2)\frac{c}{4},$$

which is equivalent to (2.6).

(2) Assume $H(p) = 0$. Equality holds in (2.5) and (2.6) if and only if

$$\begin{cases} h^r_{12} = \cdots = h^r_{1n} = 0, \\ h^r_{11} = h^r_{22} + \cdots + h^r_{nn}, \quad r \in \{n+1, \cdots, 2m+1\}. \end{cases}$$

Then $h^r_{1j} = 0$ for all $j \in \{1, \cdots, n\}, r \in \{n+1, \cdots, 2m+1\}$, that is, $X \in N_p$.

(3) Then equality case of (2.5) and (2.6) holds for all unit tangent vectors orthogonal to ξ at p if and only if

$$\begin{cases} h^r_{ij} = 0, \quad i \ne j, \quad r \in \{n+1, \ldots, 2m+1\}, \\ h^r_{11} + \cdots + h^r_{nn} - 2h^r_{ii} = 0, \quad i \in \{1, \cdots, n\}, \quad r \in \{n+1, \cdots, 2m+1\}. \end{cases}$$

In this case, it follows that p is a totally geodesic point. The converse is trivial. □

Corollary 2.3 *Let M be an n-dimensional semi-slant submanifold in a $(2m+1)$-dimensional cosymplectic space form $\tilde{M}(c)$. Then,*

(1) *For each unit vector $X \in T_pM$ orthogonal to ξ and if*

(i) *X is tangent to $\mathcal{D}_1$ we have*

$$\mathrm{Ric}(X) \leq \frac{1}{4}\left\{(n-2)c + n^2||H||^2\right\} \tag{2.11}$$

and if

(ii) *X is tangent to $\mathcal{D}_2$ we have*

$$\mathrm{Ric}(X) \leq \frac{1}{4}\left\{(n-1)c + \frac{1}{2}(3\cos^2\theta - 2)c + n^2||H||^2\right\}. \tag{2.12}$$

(2) *If $H(p) = 0$, then a unit tangent vector X orthogonal to ξ at p satisfies the equality case of* (2.11) *and* (2.12) *if and only if $X \in N_p$.*

(3) *The equality case of* (2.11) *and* (2.12) *holds identically for all unit tangent vectors orthogonal to ξ at p if and only if p is a totally geodesic point.*

Corollary 2.4 *Let M be an n-dimensional invariant submanifold in a $(2m+1)$-dimensional cosymplectic space form $\tilde{M}(c)$. Then,*

(1) *For each unit vector $X \in T_pM$ orthogonal to ξ*

$$\mathrm{Ric}(X) \leq \frac{1}{4}\left\{(n-\frac{1}{2})c + n^2||H||^2\right\}. \tag{2.13}$$

(2) *If $H(p) = 0$, then a unit tangent vector X orthogonal to ξ at p satisfies the equality case of* (2.13) *if and only if $X \in N_p$.*

(3) *The equality case of* (2.13) *holds identically for all unit tangent vectors orthogonal to ξ at p if and only if p is a totally geodesic point.*

Corollary 2.5 *Let M be an n-dimensional anti-invariant submanifold in a $(2m+1)$-dimensional cosymplectic space form $\tilde{M}(c)$. Then,*

(1) *For each unit vector $X \in T_pM$ orthogonal to ξ*

$$\mathrm{Ric}(X) \leq \frac{1}{4}\left\{(n-2)c + n^2||H||^2\right\}. \tag{2.14}$$

(2) *If* $H(p) = 0$*, then a unit tangent vector* X *orthogonal to* ξ *at* p *satisfies the equality case of* (2.14) *if and only if* $X \in N_p$.

(3) *The equality case of* (2.14) *holds identically for all unit tangent vectors orthogonal to* ξ *at* p *if and only if* p *is a totally geodesic point.*

3. k-Ricci Curvature and Squared Mean Curvature

In this section, we prove the relationship between the k-Ricci curvature and the squared mean curvature for slant, bi-slant and semi-slant submanifolds in a cosymplectic space form $\tilde{M}(c)$. We state an inequality between the scalar curvature and the squared mean curvature for submanifolds M tangent to the vector field ξ.

Theorem 3.1 *Let* M *be an* n*-dimensional* θ*-slant submanifold tangent to* ξ *into a* $(2m+1)$*-dimensional cosymplectic space form* $\tilde{M}(c)$*. Then we have*

$$||H||^2 \geq \frac{2\tau}{n(n-1)} - \frac{[n(n-1) + 3(n-1)\cos^2\theta - 2n + 2]c}{4n(n-1)}, \tag{3.1}$$

equality holding at a point $p \in M$ *if and only if* p *is a totally umbilical point.*

Proof. Let p be a point of M. We choose an orthonormal basis $\{e_1, e_2, \cdots, e_n = \xi\}$ for the tangent space T_pM and $\{e_{n+1}, \cdots, e_{2m+1}\}$ for the normal space $T_p^{\perp}M$ at p such that the normal vector e_{n+1} is in the direction of the mean curvature vector and $e_1, e_2, \cdots, e_n$ diagonalize the shape operator A_{n+1}. Then we have

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_n \end{pmatrix}, \tag{3.2}$$

$$A_r = (h_{ij}^r), \qquad \sum_{i=1}^{n} h_{ii}^r = 0, \quad n+2 \leq r \leq 2m+1.$$

¿From the equation of Gauss

$$n^2||H||^2 = 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - [n(n-1) + 3(n-1)\cos^2\theta - 2n + 2]\frac{c}{4}. \tag{3.3}$$

On the other hand,

$$\sum_{i<j}(a_i - a_j)^2 = (n-1)\sum_{i=1}^{n} a_i^2 - 2\sum_{i<j} a_i a_j. \tag{3.4}$$

Therefore, from the above equation we have

$$n^2||H||^2 = (\sum_{i=1}^{n} a_i)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{i<j} a_i a_j \leq n\sum_{i=1}^{n} a_i^2. \tag{3.5}$$

Combining (3.3) and (3.5), we get

$$n(n-1)||H||^2 \geq 2\tau + \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - [n(n-1) + 3(n-1)\cos^2\theta - 2n + 2]\frac{c}{4}, \tag{3.6}$$

which implies inequality (3.1). If the equality sign of (3.1) holds at a point $p \in M$ then from (3.4) and (3.6), we get $A_r = 0$ $(r = n+2, \cdots, 2m+1)$ and $a_1 = \cdots = a_n$. Consequently, p is a totally umbilical point. The converse is trivial. □

Theorem 3.2 *Let M be an n-dimensional bi-slant submanifold satisfying $g(X, \varphi Y) = 0$, for any $X \in \mathcal{D}_1$ and any $Y \in \mathcal{D}_2$, tangent to ξ into a $(2m+1)$-dimensional cosymplectic space form $\tilde{M}(c)$. Then we have*

$$||H||^2 \geq \frac{2\tau}{n(n-1)} - \frac{[n(n-1) + 6(d_1\cos^2\theta_1 + d_2\cos^2\theta_2) - 2n + 2]c}{4n(n-1)},$$

where $2d_1 = \dim\mathcal{D}_1$ and $2d_2 = \dim\mathcal{D}_2$.

Theorem 3.3 *Let M be an n-dimensional semi-slant submanifold tangent to ξ into a $(2m+1)$-dimensional cosymplectic space form $\tilde{M}(c)$. Then we have*

$$||H||^2 \geq \frac{2\tau}{n(n-1)} - \frac{[n(n-1) + 6(d_1 + d_2\cos^2\theta) - 2n + 2]c}{4n(n-1)},$$

where $2d_1 = \dim\mathcal{D}_1$ *and* $2d_2 = \dim\mathcal{D}_2$.

Theorem 3.4 *Let* M *be an* n*-dimensional* θ*-slant submanifold tangent to* ξ *into a* $(2m+1)$*-dimensional cosymplectic space form* $\tilde{M}(c)$. *Then, for any integer* k $(2 \leq k \leq n)$ *and any point* $p \in M$, *we have*

$$||H||^2 \geq \Theta_k(p) - \frac{[n(n-1) + 3(n-1)\cos^2\theta - 2n + 2]c}{4n(n-1)}.$$

Proof. Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of T_pM. Denote by $L_{i_1\cdots i_k}$ the k-plane section spanned by $e_{i_1}, \cdots, e_{i_k}$. It follows from (1.4) and (1.5) that

$$\tau(L_{i_1\cdots i_k}) = \frac{1}{2} \sum_{i\in\{i_1,\cdots,i_k\}} Ric_{L_{i_1\cdots i_k}}(e_i), \tag{3.7}$$

$$\tau(p) = \frac{1}{\binom{n-2}{k-2}} \sum_{1\leq i_1<\cdots<i_k\leq n} \tau(L_{i_1\cdots i_k}). \tag{3.8}$$

Combining (1.6), (3.7) and (3.8), we obtain

$$\tau(p) \geq \frac{n(n-1)}{2}\Theta_k(p). \tag{3.9}$$

Therefore, by using (3.1) and (3.9) we can obtain the inequality in Theorem 3.4. □

Theorem 3.5 *Let* M *be an* n*-dimensional bi-slant submanifold tangent to* ξ *into a* $(2m+1)$*-dimensional cosymplectic space form* $\tilde{M}(c)$. *Then, for any integer* k $(2 \leq k \leq n)$ *and any point* $p \in M$, *we have*

$$||H||^2 \geq \Theta_k(p) - \frac{[n(n-1) + 6(d_1\cos^2\theta_1 + d_2\cos^2\theta_2) - 2n + 2]c}{4n(n-1)},$$

where $2d_1 = \dim\mathcal{D}_1$ *and* $2d_2 = \dim\mathcal{D}_2$.

Theorem 3.6 *Let* M *be an* n*-dimensional semi-slant submanifold tangent to* ξ *into a* $(2m+1)$*-dimensional cosymplectic space form* $\tilde{M}(c)$. *Then, for any integer* k $(2 \leq k \leq n)$ *and any point* $p \in M$, *we have*

$$||H||^2 \geq \Theta_k(p) - \frac{[n(n-1) + 6(d_1 + d_2\cos^2\theta) - 2n + 2]c}{4n(n-1)},$$

where $2d_1 = \dim\mathcal{D}_1$ *and* $2d_2 = \dim\mathcal{D}_2$*.*

Corollary 3.7 *Let* M *be an* n*-dimensional invariant submanifold tangent to* ξ *into a* $(2m+1)$*-dimensional cosymplectic space form* $\tilde{M}(c)$*. Then, for any integer* k $(2 \leq k \leq n)$ *and any point* $p \in M$*, we have*

$$||H||^2 \geq \Theta_k(p) - \frac{(n+1)c}{4n}.$$

Corollary 3.8 *Let* M *be an* n*-dimensional anti-invariant submanifold tangent to* ξ *into a* $(2m+1)$*-dimensional cosymplectic space form* $\tilde{M}(c)$*. Then, for any integer* k $(2 \leq k \leq n)$ *and any point* $p \in M$*, we have*

$$||H||^2 \geq \Theta_k(p) - \frac{(n-2)c}{4n}.$$

Corollary 3.9 *Let* M *be an* n*-dimensional contact* CR*-submanifold tangent to* ξ *into a* $(2m+1)$*-dimensional cosymplectic space form* $\tilde{M}(c)$*. Then, for any integer* k $(2 \leq k \leq n)$ *and any point* $p \in M$*, we have*

$$||H||^2 \geq \Theta_k(p) - \frac{[n(n-1) + 6d_1 - 2n + 2]c}{4n(n-1)}.$$

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