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On Irregular Semi Strong P-ADIC U Numbers

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Abstract

The concept of the "relation of comparability" was introduced by Maillet in [7], who showed that if α, β are comparable Liouville numbers then each of the numbers $\alpha+\beta$, $\alpha-\beta$, $\alpha\beta$ *and* α/β is either a rational or Lioville number. Moreover those which are Liouville numbers are comparable aamong theem and toα and β. Maillet's proof uses in an essential way the transitivity of the comparability relation. Unfortunately, as the comparability relation is not transitive, his proof is defective. In this paper, without using the comparability relation, we obtain some uncountable subfields of p-adic numbers field, Q_p.

In [1] using a different notion of comparability, Alnıaçık was able to define some uncountable subfields of C.

In this paper, without using comparability relation, we define irregular semi-strong p-adic U_m numbers and obtain some uncountable subfields of p-adic numbers field Q_p.

1. Introduction

For the convenience of the reader we shall briefly recall Koksma's well known classification [2] for the p-adic numbers, which was introduced by Schlikewei [3].

For an algebraic number α, define the height $H(\alpha)$ as the height of the minimal polynomial of α, say $P(x) \in Z[x]$, where the P is supposed to be normalized, such that, its coefficients are relatively prime.

For a p-adic number ξ in Q_p and a natural number n put

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$$w_n^*(H,\xi) = \min_{\substack{\deg\alpha\le n\\ H(\alpha)\le H\\ \alpha\neq\xi}} |\ \xi-\alpha\ |_p,$$

and

$$w_n^*(\xi) = \lim_{H\to\infty}\sup\left(-\frac{\log w_n^*(H,\xi)}{\log H}\right)$$

and

$$w^*(\xi) = \lim_{n\to\infty}\sup\frac{w_n^*(\xi)}{n}.$$

Define $\mu^*(\xi)$ as being the smallest n , such that $w_n^*(\xi)=\infty$, if such an n exists. Otherwise put $\mu^*(\xi)=\infty$. Now call a p-adic number ξ

$$S^*-number \text{ if } 0<w^*(\xi)<\infty \text{ and } \mu^*(\xi)=\infty,$$
$$T^*-number\text{if } w^*(\xi)=\infty \text{ and } \mu^*(\xi)=\infty,$$
$$U^*-number \text{ if } w^*(\xi)=\infty \text{ and } \mu^*(\xi)<\infty.$$

Every $S^*-, T^*, U^*-number$ is a $S-, T-, U-$, number respectively. Moreover in [4] Xin has proved that Mahler's subclasses U_m are equal to the Koksma's subclasses U_m^*.

For the proof of the main results we shall need the following lemmas.

Lemma 1 *Let α,β be two p-adic algebraic numbers with different minimal polynomials. Then, for $|\ \alpha\ |_p= p^{-h}$ and $r=\min\{0,h\}$,*

$$|\ \alpha-\beta\ |_p > \frac{c_1}{H(\alpha)^{M-1}H(\beta)^M},$$

where $M>\max\{\deg\alpha, deg\beta\}$ and $c_1=\frac{p^{(M-1)\,r-M(|h|+1)}}{(2M!)}$ (Schlikewei [3] *).*

Lemma 2 *Let $P(x)=a_nx^n+...+a_0\in Z[x]$. If α is a root of P then*

$$|\ \alpha-\beta\ |_p > \frac{1}{H(P)}\,(Morrison\ [5]).$$

Lemma 3 *Let $\alpha_1, ...\alpha_k$ $(k \geq 1)$ be algebraic numbers in Q_p with $Q[(\alpha_1, ...\alpha_k); Q] = g$ and let $F(y, x_1, ...x_k)$ be a polynomial with integral coefficients, whose degree in y is at least one. If η is an algebraic number such that $F(\eta, \alpha_1, ...\alpha_k) = 0$, then the degree of $\eta \leq dg$ and*

$$h_\eta \leq 3^{2dg+(l_1+...+l_k)g}.h_{\alpha_1}^{l_1 g}...h_{\alpha_k}^{l_k g},$$

where h_η is the height of η, h_{α_i} is the height of α_i $(i = 1, ..., k)$, H is the maximum of the absolute values of the coefficients of F, l_i is the degree of F in x_i $(i = 1, ..., k)$ and d is the degree of F in y (see [6] *)*

Our first main result is the following theorem.

Theorem 4 *Let $(\alpha_i)_{i \in N}$ be a sequence of p-adic algebraic numbers with*

$$(1) \deg \alpha_i = m_i \leq k, \lim_{i \to \infty} H(\alpha_i) = \infty, (k > 0 \text{ constant}), \qquad (1)$$

$$0 < | \alpha_{i+1} - \alpha_i |_p = \frac{1}{H(\alpha_i)^{w_i}}, \text{ where } \lim_{i \to \infty} w_i = \infty, \qquad (2)$$

$$| \alpha_{i+1} - \alpha_i |_p < \frac{1}{H(\alpha_i)^{\delta}} \text{ for some } \delta > 0. \qquad (3)$$

Then $\lim_{i \to \infty} \alpha_i \in U_m^*$, where $m = \liminf_{i \to \infty} m_i$.

Proof. It follows from lemma 2 and hypothesis (2) that consecutive $\alpha_i's$ cannot be conjugates and if i is sufficiently large, $| \alpha_{i+1} |_p = | \alpha_i |_p$. Hence by putting $| \alpha_i |_p = p^{-h}$ $(h \in Z)$ and $t = \min\{0, h\}$ and using lemma 1, we get

$$\frac{c_0}{H(\alpha_i)^k H(\alpha_{i+1})^{k-1}} < | \alpha_{i+1} - \alpha_i |_p, \qquad (4)$$

where $c_0 = \frac{p^{(k-1)\,t - k\,(|h|+1)}}{(2k!)^k}$.

Since $\lim_{i \to \infty} | \alpha_{i+1} - \alpha_i |_p = 0$, the sequence $(\alpha_i)_{i \in N}$ is a Cauchy sequence in Q_p and so $\lim_{i \to \infty} \alpha_i = \xi$ exists. Let's show that $\xi \in U_m^*$. First we shall prove that, for sufficiently large i and $s > i$, $\mid \alpha_s - \alpha_i \mid_p = \mid \alpha_{i+1} - \alpha_i \mid_p$. Indeed, combining (4) and (2) and using both the facts that $H(\alpha_i) \to \infty$ and $w_i \to \infty$ we obtain

$$H(\alpha_i)^{\frac{w_i}{2}} < H(\alpha_{i+1})^k \tag{5}$$

and from this

$$H(\alpha_i)^{w_i} < H(\alpha_i)^{w_{i+1}} \text{ (for } i \text{ large enough).} \tag{6}$$

Combining relations (2) and (6) we get

$$\mid \alpha_{i+1} - \alpha_i \mid_p > \mid \alpha_{i+t} - \alpha_{i+t+1} \mid_p,$$

for each $t = 2, ...s - i$.
Hence

$$\mid \alpha_s - \alpha_i \mid_p = \mid \alpha_{i+1} - \alpha_i \mid_p = \frac{1}{H(\alpha_i)^{w_i}} \qquad (i \text{ large and } s > i) \tag{7}$$

Since $\alpha_i \to \xi$, for sufficiently large i, there is a $s > i$ such that

$$\mid \xi - \alpha_s \mid_p < H(\alpha_i)^{-w_{i+1}} . \tag{8}$$

Therefore a combination of $(6), (7)$ and (8) together with the equalities,

$$\mid \xi - \alpha_i \mid_p = \max\{ \mid \xi - \alpha_s \mid_p, \mid \alpha_s - \alpha_i \mid_p \} = \mid \alpha_i - \alpha_s \mid_p = H(\alpha_i)^{-w_i} ,$$

gives us

$$\mid \xi - \alpha_i \mid_p = H(\alpha_i)^{-w_i} , \qquad \text{(for } i \text{ large).} \tag{9}$$

On the other hand, since $\liminf_{i\to\infty} m_i = m$, we have a subsequence $(\alpha_{i_k})_{k\in N}$ of $(\alpha_i)_{i\in N}$ such that $\liminf_{k\to\infty} \alpha_{i_k} = m$. Hence for sufficiently large k, $\deg \alpha_{i_k} = m$. Hence, using (9) we get $\mid \xi - \alpha_{i_k} \mid_p = H(\alpha_{i_k})^{-w_{i_k}}$, which shows that $\mu^*(\xi) \leq m$.

We shall complete the proof by showing the opposite inequality $\mu^*(\xi) \geq m$. For this we shall distinguish two cases according to $m = 1$ or $m > 1$. In the case $m = 1$, by definition of $\mu^*(\xi)$, we have $\mu^*(\xi) \geq 1$. So together with $\mu^*(\xi) \leq m = 1$, we obtain $\mu^*(\xi) = 1$.

Now suppose that $m > 1$. Let β be a p-adic number of degree $< m$. Since $\liminf_{i\to\infty} m_i = m$, $\deg \alpha_i \geq m$ for sufficiently large i. Applying lemma 1, we get

$$| \beta - \alpha_i |_p > c_1 H(\alpha_i)^{-(k-1)}, H(\beta)^{-k} \qquad (i \text{ large}). \tag{10}$$

On the other hand, as $w_i \to \infty$, for sufficiently large i, we have

$$w_i > \frac{2k(k+\delta)}{\delta} \tag{11}$$

Now suppose that the p-adic algebraic number β satisfies the condition

$$H(\beta) > \max\{H(\alpha_{i_0}), \frac{1}{c_1}\}, \tag{12}$$

where i_0 is a sufficiently large, fixed index. It is clear that there exists a natural number $i \geq i_0$ such that, for every p-adic algebraic number satisfying (12), we have

$$H(\alpha_i) \leq H(\beta) \leq H(\alpha_{i+1}). \tag{13}$$

Taking into account $(5), (11)$ and (13) we can have only one of the following cases:

$$H(\alpha_i) \leq H(\beta) \leq H(\alpha_{i+1})^{\frac{\delta}{2k}}$$

$$\text{or} \tag{14}$$

$$H(\alpha_{i+1})^{\frac{\delta}{2k}} \leq H(\beta) \leq H(\alpha_{i+1}).$$

Suppose that the first relation in (14) holds. Then a combination of (2), (3), (9), (10), (12) and (14) gives us

$$| \beta - \alpha_i |_p > H(\beta)^{-2k} > H(\alpha_{i+1})^{-\delta} > H(\alpha_i)^{-w_i} = | \xi - \alpha_i |_p . \qquad (15)$$

Furthermore, using the equation, $| \xi - \beta |_p = \max\{ | \xi - \alpha_i |_p, | \beta - \alpha_i |_p \}$, we get

$$| \xi - \beta |_p > H(\alpha_i)^{-2k} \qquad \text{(for large } i\text{)}. \qquad (16)$$

If the second relation in (14) holds, then using the relations (9), (10), (11), (12), we get

$$| \xi - \alpha_{i+1} |_p = H(\alpha_{i+1})^{-w_{i+1}} < H(\beta)^{\frac{-2k\,(k+\delta)}{\delta}} < H(\beta)^{\frac{-2k}{\delta} - k^2} < | \alpha_{i+1} - \beta |_p$$

so that

$$| \xi - \beta |_p = | \alpha_{i+1} - \beta |_p > H(\beta)^{\frac{-2k^2}{\delta} - k}, \qquad (17)$$

As the exponent of $H(\beta)$ on the right hand side of (17) is greater than that of (16), (17) is verified for all p-adic algebraic numbers of degree at most $m-1$ and height greater than $\max\{H(\alpha_{i_0}), \frac{1}{c_1}\}$. This shows us that $\mu^*(\xi) \geq m$. This result together with the inequality $\mu^*(\xi) \leq m$ imply that $\mu^*(\xi) = m$ also in case $m > 1$, as well. Hence $\xi \in U_m$, and this completes the proof. □

Definition 5 *Given a U number ξ in Q_p. If there is a sequence (α_i) of p-adic algebraic numbers satisfying the conditions $(1), (2)$ and (3) of Theorem 1, then we say that "$\xi = \lim_{i\to\infty} \alpha_i$ is an irregular semi-strong U-number."*

In Theorem 1 we have seen that if $\liminf_{i\to\infty} m_i = m$, then $\xi \in U_m$.
In the sequel U^{is} and U_m^{is} will denote the set of all irregular semi-strong $U-, U_m-$ numbers.

Example. If p is a prime number and α is a p-adic algebraic number of degree m, then

$$\varkappa = \alpha + \sum_{i=1}^{\infty} p^{n!} \text{ is in } U_m^{is}.$$

By defining $\alpha_n = \alpha + p^{1!} + ... + p^{n!}$, one can show that $(\alpha_n)_{n\in N}$ satisfies the conditions of Theorem 1.

The main result of the paper is the following theorem.

Theorem 6 *The set $F = A \cup U_m^{\hat{i}s}$ is an uncountable subfield of Q_p,where A denotes the set of p-adic algebraic numbers.*

Proof. Let $y_1, y_2 \in F$. Assume that $y_1 \in U_r^{is}, y_2 \in U_t^{is}$. Then there are positive numbers ρ_1, ρ_2 and sequences of algebraic numbers $(\alpha_i)_{i\in N}, (\beta_i)_{i\in N}$ ($\deg \alpha_i \leq l, \deg \beta_i \leq l$ where $l \geq \max\{r, t\}$) such that

$$0 < \mid \alpha_{i+1} - \alpha_i \mid_p = \frac{1}{H(\alpha_i)^{w_i}} < \frac{1}{H(\alpha_{i+1})^{\rho_1}}, \text{ where } \lim_{i\to\infty} w_i = \infty, \lim_{i\to\infty} H(\alpha_i) = \infty \quad \text{(A)}$$

$$0 < \mid \beta_{i+1} - \beta_i \mid_p = \frac{1}{H(\beta_i)^{\upsilon_i}} < \frac{1}{H(\beta_{i+1})^{\rho_2}}, \text{where } \lim_{i\to\infty} \nu_i = \infty, \lim_{i\to\infty} H(\beta_i) = \infty \quad \text{(B)}$$

Let $(x_i)_{i\in N}$ be a monotonic union sequence formed from $\{H(\alpha_i)\}$ and $\{H(\beta_i)\}$. Assume that $x_{i_0} > \max\{H(\alpha_1), H(\beta_1)\}$. We shall introduce positive integers $j(i), t(i)$ and then p-adic algebraic numbers δ_i as follows. For $i \geq i_0$,

$$\begin{aligned} j(i) &= \max\{\nu : H(\alpha_\nu) \leq x_i\} \\ t_i &= \max\{\dot{\nu} : H(\beta_\nu) \leq x_i\} \\ \delta_i &= \alpha_{j(i)} + \beta_{t(i)}. \end{aligned} \quad (18)$$

Now consider the set $B = \{\delta_i : i \geq i_0\}$. If B contains only finitely many p-adic algebraic numbers then $\lim_{i\to\infty} \delta_i = y_1 + y_2 \in B$. Hence $y_1 + y_2$ is a p-adic algebraic numbers i.e. $y_1 + y_2 \in F$.

Hence we suppose that B contains infinitely many p-adic algebraic numbers. In this case, we define a subsequence $(\delta_{i_k})_{k\in N}$ of $(\delta_i)_{i\in N}$ as follows.

$$\text{If } \delta_{i_{k+1}} = \delta_{i_k+s} \text{ then } \delta_{i_k} = \delta_{i_k+1} = ... = \delta_{i_k+s-1} (s = 1, 2..., i_{k+1} - i_k - 1, i_{k+1} - i_k) \quad (19)$$

Now by lemma 3, we) have

$$H(\delta_{i_k}) < 3^{4l^2} H(\alpha_{j(i_k)})^{l^2} H(\beta_{t(i_k)})^{l^2} \qquad (k = 1, 2, ...).$$

Hence using (18), we get

$$H(\delta_{i_k}) < x_{i_k}^{3l^2} \qquad \text{(for large } k). \tag{20}$$

On the other hand, from the definitions of $j\,(i)$ and $t\,(i)$, we see that

$$0 \leq j\,(i_{k+1}) - j\,(i_{k+1} - 1) \leq 1 \text{ and } 0 \leq t\,(i_{k+1}) - t\,(i_{k+1} - 1) \leq 1. \tag{21}$$

Moreover, from the definition of $(\delta_{i_k})_{k \in N}$, one can see easily that the numbers in (21) cannot be zero at the same time. If the numbers are both different from zero, then a combination of $(A)\,, (B)\,, (C)\,,$ (19) and (21) gives us

$$\begin{aligned} \mid \delta_{i_{k+1}} - \delta_{i_k} \mid_p &= \mid \delta_{i_{k+1}} - \delta_{i_{k+1}-1} \mid_p \\ &\leq \max\{\mid \alpha_{j(i_{k+1})} - \alpha_{j(i_{k+1}-1)} \mid_p, \mid \beta_{t(i_{k+1})} - \beta_{t(i_{k+1}-1)} \mid_p\} \\ &\leq \max\{(H(\alpha_{j(i_{k+1}-1)+1}))^{-\rho_1}, (H(\beta_{t(i_{k+1}-1)+1}))^{-\rho_2}\} \\ &= \frac{1}{x_{i_{k+1}}^{\rho}}, \end{aligned} \tag{22}$$

where $\rho = \min\{\rho_1, \rho_2\}$. Hence using the relations $(20)_{k+1}$ and (22) and putting $\nu_{i_k} = \frac{\delta \ \log x_{i_{k+1}}}{3^{l^2} \log x_{i_k}}$, we get

$$\mid \delta_{i_{k+1}} - \delta_{i_k} \mid_p < H(\delta_{i_k})^{-\nu_{i_k}} \qquad (k \text{ large}). \tag{23}$$

Let's show that $v_{i_k} \to \infty$ as $k \to \infty$. It's enough to consider only the case $x_{i_{k+1}} = (H(\alpha_{j\,(\,i_{k+1})})$ and. $x_{i_{k+1}-1} = (H(\beta_{t\,(\,i_{k+1}-1)}))$, since the other three cases are trivial and similar. First using lemma 1 and relation (A) , theen replaacing in (5) (α_i) and w_i, respectively, by β_i and $v_{i,}$ we get,

$H(\alpha_{j(\,i_{k+1})})^{-3l} <\mid \alpha_{j(\,i_{k+1})} - \alpha_{j(\,i_{k+1}-1)} \mid_p < H(\alpha_{j(\,i_{k+1})})^{-\rho_1} < H(\beta_{t(\,i_{k+1})})^{-\rho_1} <$
$H(\beta_{t(\,i_{k+1}-1)})^{-v_{t\ (i_{k+1}-1)}\,\rho_1/\ 2^l}$.

Therefore we have $\lim_{k\to\infty} v_{i_k} = \infty$.
Now let's show that $\lim_{k\to\infty} H(\delta_{i_k}) = \infty$. By lemma1 and (22), we have

$$c_2 H(\delta_{i_k})^{-l^2} H(\delta_{i_{k+1}})^{-l^2} <\mid \delta_{i_{k+1}} - \delta_{i_k} \mid_p < x_{i_{k+1}}^{-\rho}. \qquad (24)$$

Since $\lim_{k\to\infty}(\log x_{i_{k+1}})\ /\ (\log x_{i_k}) = \infty$, we have

$$x_{i_k}^{3l^4} < x_{i_{k+1}}^{\rho\ /4} \text{ and } x_{i_k}^{\rho\ /\ 8} > c_2^{-1} \qquad (\text{for } k \text{ large}). \qquad (25)$$

Therefore using $(20)_{k+1}$,(24) and (25) respectively, we get

$$x_{i_{k+1}}^{\frac{5\rho}{8\,l^2}} < H(\delta_{i_{k+1}}) \qquad (\text{for } k \text{ large}), \qquad (26)$$

which shows that $H(\delta_{i_k}) \to \infty,\ as\ \ k \to \infty$. Furthermore, using $(20)_{k+1}$ in (22), we have

$$\mid \delta_{i_{k+1}} - \delta_{i_k} \mid_p < x_{i_{k+1}}^{-\rho} < H(\delta_{i_{k+1}})^{-3\,l^2\rho}.$$

Now we show that the product $y_1 y_2$ is in F. To show this we shall approximate $y_1 y_2$ by algebraic numbers $\delta_i^{\text{!`}}$ as defined as

$$\delta_i^{\text{!`}} = \alpha_{j(i)}\beta_{t(i)} \qquad (i \geq i_0). \qquad (27)$$

If $B = \{\delta_i^{\text{!`}}\ :\ i \geq i_0\}$contains only finitely many p-adic algebraic numbers, then it is closed. Therefore by (27) we have $y_1 y_2 \in A \subset F$. If $B$ is not finite, we can choose a subsequence $(\ \delta_{i_k}^{\text{!`}})_{k\in N}$ of $(\ \delta_i^{\text{!`}})_{i\in N}$ as follows.

If $\delta_{i_{k+1}}^{\text{!`}} = \delta_{i_k+s}^{\text{!`}}$ then $\delta_{i_k}^{\text{!`}} = \delta_{i_k+1}^{\text{!`}} = ... = \delta_{i_k+s-1}^{\text{!`}}$ $(\ s = 1, 2..., i_{k+1} - i_k - 1, i_{k+1} - i_k)$
Now using (18), (C) and (22) we obtain

$$
\begin{aligned}
&|\ \delta^{!}_{i_{k+1}} - \delta^{!}_{i_k}\ |_p = |\ \delta^{!}_{i_{k+1}} - \delta^{!}_{i_{k+1}-1}\ |_p \leq \max\{|\ \alpha_{j(i_{k+1})} - \alpha_{j(i_{k+1}-1)}\ |_p \\
&\cdot\ |\ \beta_{t(i_{k+1})}\ |_p, |\ \beta_{t(i_{k+1})} - \beta_{t(i_{k+1}-1)}\ |_p\ .\ |\ \alpha_{j(i_{k+1}-1)}\ |_p\}\ g \leq M \max\{|\ \alpha_{j(i_{k+1})} \\
&-\alpha_{j(i_{k+1}-1)}\ |_p, |\ \beta_{t(i_{k+1})} - \beta_{t(i_{k+1}-1)}\ |_p\},
\end{aligned}
\tag{28}
$$

where $M = \max\{1, |\ y_1\ |_p, |\ y_2\ |_p\}$. On the other hand, using an argument similar the one used in the previous steps, we obtain

$$
H(\delta^{!}_{i_k}) < x^{3l^2}_{i_k} (\text{for } k \text{ large}). \tag{29}
$$

Hence a combination of (28) and (29) gives us

$$
|\ \delta^{!}_{i_{k+1}} - \delta^{!}_{i_k}\ |_p < H(\delta^{!}_{i_{k+1}})^{\frac{-\rho}{6l^2}} \qquad (\text{ for } k \text{ large}).
$$

Moreover, using the same arguments that we have used to get (23), we obtain

$$
|\ \delta^{!}_{i_{k+1}} - \delta^{!}_{i_k}\ |_p < H(\delta^{!}_{i_k})^{\frac{-\nu_{i_k}}{2}} \qquad (\text{for } k \text{ large}),
$$

which shows that $y_1 y_2$ is in U^{is}_m for some $m \leq l^2$. Next, as we have shown for $(H(\delta_{i_k}))_{k\in N}$, one can easily show that respectively, $H(\delta^{!}_{i_k}) \to \infty$.

Finally let $\alpha \in A$ and $y_1 \in F - A$. Then using an similar argument to the one used to prove that $\alpha + y_1 \in F$ and approximating $\alpha y_1, \alpha + y_1$ by $(\alpha\alpha_i)_{i\in N}, (\alpha + \alpha_i)_{i\in N}$ respectively, one shows that $\alpha y_1, \alpha + y_1 \in F$. □

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