Turk J Math
30 (2006) , 117 – 127.

On Certain Modified Meyer-König and Zeller Operators

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Abstract

We introduce certain modified Meyer-König and Zeller operators and we study their approximation properties.

The similar results for modified Bernstein polynomials were given in [6].

Key Words: Meyer-König and Zeller operator, degree of approximation, Voronovskaya theorem.

1. Introduction

1.1. In 1960, W. Meyer-König and K. Zeller in [7] introduced the following operators for functions $f \in C_Q$ and $n \in N = \{1, 2, ..., \}$:

$$M_n(f;x) := \begin{cases} \sum\limits_{k=0}^{\infty} p_{nk}(x)\, f\left(\frac{k}{n+k}\right) & \text{if } \ 0 \le x < 1, \\ f(1) & \text{if } \ x = 1, \end{cases} \tag{1.1}$$

where

$$p_{nk}(x) := \begin{pmatrix} n+k \\ k \end{pmatrix} x^k\, (1-x)^{n+1}, \quad k \in N_0 = N \cup \{0\}, \tag{1.2}$$

AMS Mathematics Subject Classification: 41A36.

and C_Q is the space of all real-valued functions f, continuous on the interval $Q = [0, 1]$ and the norm is defined by

$$\|f\| \equiv \|f(\cdot)\| := \max_{x \in Q} |f(x)| . \tag{1.3}$$

$M_n(f)$ is called the n–th Meyer–König and Zeller operator.

Approximation properties of $M_n(f)$ have been examined in many papers (e.g. [1, 3, 5, 7]). Moreover, in many papers were introduced some modifications of operators $M_n(f)$ (e.g. [2, 4, 5]) and were studied their approximation properties.

It is known ([1, 3, 5]) that if $f \in C_Q$, then $M_n(f) \in C_Q$ and $\|M_n(f)\| \leq \|f\|$ for $n \in N$, Moreover,

$$\|M_n(f; \cdot) - f(\cdot)\| \leq A\, \omega_2 \left(f; \frac{1}{\sqrt{n}} \right), \qquad n \in N, \tag{1.4}$$

where A is a suitable positive constant independent on n and x and $\omega_2(f; \cdot)$ is the second modulus of smmothness of f. Obviously (1.4) implies that

$$\lim_{n \to \infty} \|M_n(f; \cdot) - f(\cdot)\| = 0 , \tag{1.5}$$

for every $f \in C_Q$. Moreover for $f \in C_Q^{r+2} = \{f \in C_Q : f^{(r+2)} \in C_Q\}$with a fixed $r \in N_0$, we have

$$\|M_n(f; \cdot) - f(\cdot)\| = O\left(\frac{1}{n}\right), \qquad n \in N, \tag{1.6}$$

and this estimation can not be improved.

1.2. In this paper we will show that the estimations (1.4) and (1.6) can be improved for $f \in C_Q^r$, $r \geq 2$, by certain modification of the operators $M_n(f)$. We introduce the following definition.

Definition 1 *Let $r \in N_0$ be a fixed number. For $f \in C_Q^r$ and $n \in N$ we define the modified Meyer-König and Zeller operators*

$$M_{n;r}(f; x) := \begin{cases} \sum_{k=0}^{\infty} p_{nk}(x) \sum_{j=0}^{r} \frac{f^{(j)}(\xi_{nk})}{j!} (x - \xi_{nk})^j & \text{if} \quad 0 \leq x < 1, \\ f(1) & \text{if} \quad x = 1, \end{cases} \tag{1.7}$$

where

$$\xi_{nk} := \frac{k}{n+k} \qquad \textit{for} \quad k \in N_0, \;\; n \in N, \tag{1.8}$$

and $p_{nk}(x)$ *is defined by* (1.2). *Clearly* $M_{n;0}(f;x) \equiv M_n(f;x)$ *for every* $f \in C_Q$, $x \in Q$, $n \in N$.

In Section 2 we will give some auxiliary results. The main theorems will be given in Section 3.

In this paper we will denote by $A_i(q)$, $i \in N$, a suitable positive constant depending only on parameter q.

2. Lemmas

2.1. It is well known [1, 2, 3, 4, 7, 8] that

$$M_n(1;x) = 1 \qquad M_n(t-x;x) = 0, \tag{2.9}$$

$$M_n((t-x)^2;x) = \frac{x(1-x)^2}{n} + O_x\left(\frac{1}{n^2}\right), \quad \text{for } \; x \in Q \text{ and } n \in N.$$

Moreover in [4] is given the following lemma.

Lemma 1 *For every fixed* $q \in N$ *there exists* $A_1(q) = const. > 0$ *such that*

$$M_n\left((t-x)^{2q};x\right) \;\leq\; A_1(q)\, n^{-q}, \qquad n \in N,$$

uniformly for $x \in Q$.

2.2. Now we will give some elementary properties of operators $M_{n;r}(f)$ defined by (1.7)and (1.8).

From (1.7) it follows that $M_{n;r}(1;x) = 1$ for $x \in Q$, $n \in N$ and $r \in N_0$ and

$$M_{n;r}(f;0) \;=\; f(0), \qquad n \in N, \quad r \in N_0. \tag{2.10}$$

Lemma 2 *Let $n, r \in N$ be fixed numbers. Then for every $f \in C_Q^r$ we have $M_{n;r}(f) \in C_Q$, i.e. $M_{n;r}(f)$ is an operator from the space C_Q^r into C_Q. Moreover there exists $A_2(r) = const. > 0$ such that for every $f \in C_Q^r$ we have*

$$\|M_{n;r}(f;\cdot)\| \le A_2(r) \sum_{j=0}^{r} \left\| f^{(j)} \right\|, \qquad n \in N. \tag{2.11}$$

Proof. We observe that if $f \in C_Q^r$, then for every fixed $n \in N$ and $j, q = 0, 1, ..., r$ the sequence $((\xi_{nk})^q f^{(j)}(\xi_{nk}))_{k=0}^{\infty}$ is convergent to $f^{(j)}(1)$ as $k \to \infty$. Moreover, it is easily verified that the limitability method of sequences, generated by $(p_{nk}(x))_{k=0}^{\infty}$, with a fixed $n \in N$ and $x \to 1-$, is regular. Hence we can write

$$\lim_{x \to 1-} \sum_{k=0}^{\infty} p_{nk}(x)\,(\xi_{nk})^q\, f^{(j)}(\xi_{nk}) \;=\; f^{(j)}(1)$$

for every $j, q = 0, 1, ..., r$ and $n \in N$.

From the above, and by (1.7) and (1.8), we get

$$\begin{aligned} \lim_{x \to 1-} M_{n;r}(f;x) \;&=\; \sum_{j=0}^{r} \frac{1}{j!} \sum_{q=0}^{j} \binom{j}{q} (-1)^q \\ &\quad \times \lim_{x \to 1-} x^{j-q} \sum_{k=0}^{\infty} p_{nk}(x) f^{(j)}(\xi_{nk}) \xi_{nk}^q = \\ &= \sum_{j=0}^{r} \frac{f^{(j)}(1)}{j!} \sum_{q=0}^{j} \binom{j}{q} (-1)^q. \end{aligned}$$

Since

$$\sum_{q=0}^{j} \binom{j}{q} (-1)^q = \begin{cases} 1 & \text{if} \quad j = 0 \\ 0 & \text{if} \quad j \ge 1, \end{cases} \tag{2.12}$$

we have

$$\lim_{x \to 1-} M_{n;r}(f;x) \;=\; f(1), \qquad n \in N, \;\; r \in N_0,$$

which by (1.7) shows that $M_{n;r}(f)$ is a continuous function at $x = 1$. The continuity of $M_{n;r}(f)$ at $x \in [0, 1)$ is obvious by the properties of sum of power series convergent on $[0, 1)$.

From this and (1.7), (1.8) and (1.1), we deduce that

$$M_{n;r}(f;x) = \sum_{j=0}^{r} \frac{(-1)^j}{j!} M_n((t-x)^j f^{(j)}(t);x)$$

for $x \in Q$, $n \in N$ and $r \in N_0$. Further, by (1.1)–(1.3), (1.8) and the Hölder inequality and Lemma 1, we have

$$\begin{aligned} \left| M_n((t-x)^j f^{(j)}(t);x) \right| &\leq \left\| f^{(j)} \right\| M_n(|t-x|^j;x) \\ &\leq \left\| f^{(j)} \right\| (M_n((t-x)^{2j};x))^{1/2} (M_n(1;x))^{1/2} \leq A_1(j) \left\| f^{(j)} \right\| n^{-j/2}, \end{aligned}$$

for $x \in Q$, $n \in N$ and $0 \leq j \leq r$. Consequently

$$\begin{aligned} \|M_{n;r}(f;\cdot)\| &\leq \sum_{j=0}^{r} \frac{1}{j!} \left\| M_n((t-\cdot)^j f^{(j)}(t);\cdot) \right\| \\ &\leq A_2(r) \sum_{j=0}^{r} \left\| f^{(j)} \right\|, \qquad n \in N. \end{aligned}$$

Thus the proof of (2.11) is completed. □

3. Theorems

3.1. First we will prove an analogue of (1.4) for $f \in C_Q^r$ and $M_{n;r}(f)$, but we will use the modulus of continuity of the derivative $f^{(r)}$, i.e.

$$\omega(f^{(r)};t) := \sup \left\{ \left| f^{(r)}(x) - f^{(r)}(y) \right| \, : \, x, y \in Q, \, |x-y| \leq t \right\}$$

for $t \in [0,1]$ ([9]). The application of the second modulus of continuity $\omega_2(f^{(r)};\cdot)$ to approximation theorem for $f \in C_Q^r$ and $M_{n;r}(f)$, $r \in N$, is difficult by derivatives $f^{(j)}$ and factors $(x - \xi_{nk})^j$, $j = 1, ..., r$, in the formula (1.7).

Theorem 1 *Let $r \in N_0$ be a fixed number. Then there exists $A_3(r) = const. > 0$ such* *that for every $f \in C_Q^r$ and $n \in N$ holds the following inequality*

$$\|M_{n;r}(f;\cdot) - f(\cdot)\| \le A_3(r)\, n^{-r/2}\, \omega\left(f^{(r)}; \frac{1}{\sqrt{n}}\right), \tag{3.13}$$

where $\omega(f^{(r)};\cdot)$ is the modulus of continuity of $f^{(r)}$.

Proof. The estimation (3.13) for $r = 0$ follows from (1.4).

Let $r \in N$. Similarly as in [6] we apply the following modified Taylor formula for $f \in C_Q^r$ in a given point $t \in Q$:

$$\begin{aligned} f(x) \;=\; & \sum_{j=0}^{r} \frac{f^{(j)}(t)}{j!}(x-t)^j \\ & + \frac{(x-t)^r}{(r-1)!} \int_0^1 (1-u)^{r-1}[f^{(r)}(t+u(x-t)) - f^{(r)}(t)]\, du, \quad x \in Q. \end{aligned}$$

Choosing $t = \xi_{nk}$ and applying (2.9), we derive from the above Taylor formula and (1.7)

$$f(x) \;=\; \sum_{k=0}^{\infty}(p_{nk}(x)\, f(x) = M_{n;r}(f;x) + \sum_{k=0}^{\infty} p_{nk}(x)\, \frac{(x-\xi_{nk})^r}{(r-1)!} I_r(x,\xi_{nk}), \tag{3.14}$$

where

$$I_r(x,\xi_{nk}) \;=\; \int_0^1 (1-u)^{r-1} \left[f^{(r)}(\xi_{nk} + u(x-\xi_{nk})) - f^{(r)}(\xi_{nk})\right] du.$$

The definition and properties of modulus of continuity of function ([9]) imply that

$$\begin{aligned} \left|f^{(r)}\left(\xi_{nk} + u(x-\xi_{nk})\right) - f^{(r)}\left(\xi_{nk}\right)\right| &\le \omega\left(f^{(r)}; u|x-\xi_{nk}|\right) \\ &\le \omega\left(f^{(r)}; |x-\xi_{nk}|\right) \le \left(\sqrt{n}|x-\xi_{nk}|+1\right) \omega\left(f^{(r)}; 1/\sqrt{n}\right), \end{aligned}$$

for every $0 \le u \le 1$, $0 \le x < 1$, $k \in N_0$ and $n \in N$. From this and (3.14) we get

$$\begin{aligned} & |f(x) - M_{n;r}(f;x)| \le \qquad (3.15) \\ & \le \frac{1}{r!}\, \omega\left(f^{(r)}; 1/\sqrt{n}\right) \sum_{k=0}^{\infty} p_{nk}(x)|x-\xi_{nk}|^{\,r} \left(\sqrt{n}|x-\xi_{nk}|+1\right) \\ & \le \frac{1}{r!}\, \omega\left(f^{(r)}; 1/\sqrt{n}\right) \left(\sqrt{n}\, M_n\left(|t-x|^{\,r+1}; x\right) + M_n\left(|t-x|^{\,r}; x\right)\right), \end{aligned}$$

for $0 \le x < 1$ and $n \in N$. Using the Hölder inequality and (2.9) and Lemma 1, we get

$$\begin{aligned} M_n\left(|t-x|^q;x\right) &\le \left(M_n\left((t-x)^{2q};x\right)\right)^{1/2}\left(M_n(1;x)\right)^{1/2} \\ &\le A_1(q)\,n^{-q/2}, \qquad x\in Q, \quad n,q\in N. \end{aligned}$$

Further from (3.15) results that

$$|f(x)-M_{n;r}(f;x)| \le A_4(r)\,\omega\left(f^{(r)};n^{-1/2}\right)\left(\sqrt{n}\,n^{-(r+1)/2}+n^{-r/2}\right)$$

for all $0 \le x < 1$ and $n \in N$. This inequality and (1.7) for $x = 1$ immediately yield (3.13). □

From Theorem 1 we can derive the following two corollaries.

Corollary 1 *Let $f \in C_Q^r$, $r \in N_0$, then*

$$\lim_{n\to\infty} n^{r/2}\,\|M_{n;r}(f;\cdot)-f(\cdot)\| = 0.$$

Corollary 2 *Let $f \in C_Q^r$, $r \in N_0$, and let $f^{(r)} \in Lip\ \alpha$ with a fixed $0 < \alpha \le 1$, i.e. $\omega(f^{(r)};t) = O(t^\alpha)$ for $t \in (0,1]$. Then*

$$\|M_{n;r}(f;\cdot)-f(\cdot)\| = O\left(n^{-(r+\alpha)/2}\right), \qquad n\in N.$$

Remark. Theorem 1, Corollary 1 and Corollary 2 show that the degree of approximation of function $f \in C_Q^r$ with $r \ge 2$ by operators $M_{n;r}(f)$ is better than (1.4) and (1.6) for $M_n(f)$.

3.2. Now we will prove the Voronovskaya type theorem.

Theorem 2 *Suppose that $f \in C_Q^{r+2}$ with a fixed $r \in N_0$. Then*

$$\begin{aligned} M_{n,r}(f;x)-f(x) &= \frac{(-1)^r f^{(r+1)}(x)M_n((t-x)^{r+1};x)}{(r+1)!} \qquad (3.16)\\ &+\frac{(-1)^r(r+1)f^{(r+2)}(x)M_n((t-x)^{r+2};x)}{(r+2)!} + g_n(x;r) \end{aligned}$$

for every $x \in Q$ and $n \in N$, where

$$g_n(x;r) = o\left(n^{-(r+2)/2}\right) \quad \text{as} \quad n \to \infty \tag{3.17}$$

uniformly for $x \in Q$.

Proof. By (1.1), (1.7) and (2.10) we have (3.16) for $x = 0$ and $x = 1$.

Fix $0 < x < 1$. For $f \in C_Q^{r+2}$ we have $f^{(j)} \in C_Q^{r+2-j}$, $0 \le j \le r$, and by the Taylor formula we can write

$$f^{(j)}(t) = \sum_{i=0}^{r+2-j} \frac{f^{(j+i)}(x)}{i!}(t-x)^i + \varphi_j(t,x)(t-x)^{r+2-j} \tag{3.18}$$

for $t \in Q$, where $\varphi_j(t) \equiv \varphi_j(t,x)$ is function such that $\varphi_j(t)\, t^{r+2-j} \in C_Q^{r+2-j}$ and $\lim_{t\to x} \varphi_j(t) = \varphi_j(x) = 0$. Taking $t = \xi_{nk}$ in (3.18) and applying this formula to $M_{n;r}(f)$, we get

$$\begin{aligned} M_{n;r}(f;x) &= \sum_{k=0}^{\infty} p_{nk}(x) \sum_{j=0}^{r} \frac{(x-\xi_{nk})^j}{j!} \sum_{i=0}^{r+2-j} \frac{f^{(j+i)}(x)}{i!}(\xi_{nk}-x)^i \\ &\quad + \sum_{k=0}^{\infty} p_{nk}(x) \sum_{j=0}^{r} \frac{(x-\xi_{nk})^j}{j!} \varphi_j(\xi_{nk})(\xi_{n,k}-x)^{r+2-j} \\ &:= \sum\nolimits_1 + \sum\nolimits_2, \qquad n \in N. \end{aligned} \tag{3.19}$$

by elementary calculations we get

$$\begin{aligned}\sum\nolimits_1 &= \sum_{k=0}^{\infty} p_{nk}(x) \sum_{j=0}^{r} \frac{(x-\xi_{nk})^j}{j!} \sum_{q=j}^{r+2} \frac{f^{(q)}(x)}{(q-j)!}(\xi_{nk}-x)^{q-j} \\
&= \sum_{k=0}^{\infty} p_{nk}(x) \sum_{j=0}^{r} \frac{(-1)^j}{j!} \left\{ \sum_{q=j}^{r} \frac{f^{(q)}(x)}{(q-j)!}(\xi_{nk}-x)^q \right. \\
&\quad \left. + \frac{f^{(r+1)}(x)}{(r+1-j)!}(\xi_{nk}-x)^{r+1} + \frac{f^{(r+2)}(x)}{(r+2-j)!}(\xi_{nk}-x)^{r+2} \right\} \\
&= \sum_{k=0}^{\infty} p_{nk}(x) \sum_{q=0}^{r} \frac{f^{(q)}(x)}{q!}(\xi_{nk}-x)^q \sum_{j=0}^{q} \binom{q}{j} (-1)^j \\
&\quad + \frac{f^{(r+1)}(x)}{(r+1)!} \sum_{k=0}^{\infty} p_{nk}(x)(\xi_{nk}-x)^{r+1} \sum_{j=0}^{r} \binom{r+1}{j} (-1)^j \\
&\quad + \frac{f^{(r+2)}(x)}{(r+2)!} \sum_{k=0}^{\infty} p_{nk}(x)(\xi_{nk}-x)^{r+2} \sum_{j=0}^{r} \binom{r+2}{j} (-1)^j
\end{aligned}$$

for $n \in N$. Applying (2.12) and equalities

$$\sum_{j=0}^{r} \binom{r+1}{j} (-1)^j = (-1)^r,$$

$$\sum_{j=0}^{r} \binom{r+2}{j} (-1)^j = (r+1)(-1)^r,$$

with $r \in N_0$, and by (1.1) and (2.9), we obtain

$$\begin{aligned}\sum\nolimits_1 &= f(x) + \frac{(-1)^r f^{(r+1)}(x) M_n((t-x)^{r+1};x)}{(r+1)!} \\
&\quad + \frac{(-1)^r (r+1) f^{(r+2)}(x) M_n((t-x)^{r+2};x)}{(r+2)!}, \qquad n \in N.\end{aligned} \tag{3.20}$$

Denoting by

$$\phi_r(t) := \sum_{j=0}^{r} \frac{(-1)^j}{j!}\, \varphi_j(t), \qquad t \in Q,$$

we have $\phi_r \in C_Q$, $\lim_{t \to x} \phi_r(t) = \phi_r(x) = 0$ and

$$\begin{aligned}\sum\nolimits_2 &= \sum_{k=0}^{\infty} p_{nk}(x)(\xi_{nk} - x)^{r+2} \phi_r(\xi_{nk}) \\ &= M_n\left((t-x)^{r+2}\phi_r(t); x\right), \quad n \in N.\end{aligned}$$

Further, by the Hölder inequality, we have

$$\left|\sum\nolimits_2\right| \le \left(M_n\left((t-x)^{2r+4}; x\right)\right)^{1/2} (M_n(\phi_r^2(t); x))^{1/2} := g_n(x; r) \tag{3.21}$$

for $n \in N$. The properties of $\phi_r(\cdot)$ and (1.5) imply that

$$\lim_{n \to \infty} M_n(\phi_r^2(t); x) = \phi_r^2(x) = 0$$

uniformly on Q. From this and (3.21) and Lemma 2 it follows that

$$g_n(x; r) = o\left(n^{-(r+2)/2}\right) \quad \text{as} \quad n \to \infty,$$

uniformly on Q. This result and (3.19)–(3.21) imply the desired assertions (3.15) and (3.16). Thus the proof is completed. □

Theorem 2 implies the following Voronovskaya type theorem for operators $M_n(f)$ ([1], [2]):

Corollary 3 *If $f \in C_Q^2$, then*

$$\lim_{n \to \infty} n(M_n(f; x) - f(x)) = \frac{x(1-x)^2}{2} f''(x)$$

for every $x \in Q$.

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Received 03.09.2003