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The Homological Theory of Degree of FQL- Mappings

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Abstract

In this article the homological theory is specially worked out, adapted for definition of the degree of mapping from one of the classes of infinite-dimensional mappings, exactly FQL-mappings, introduced in [7].

Key Words: Simplex, complex, cylinder, chain, cycle, homology, dimension, family with the finite dimensional opening, Fredholm Quasi-Linear mapping.

0. Introduction

Different homological theories, which are equivalent to each other for many topological spaces are known. That is why by the h elp of them the definition of homological degree for finite-dimensional mappings creates the same results, but application to infinite-dimensional mappings dont serve the aim. Therefore, it is necessary to include special homological theories adapted to different categories of infinite-dimensional mappings.

In this paper, a homological theory for FQL-mappings, included by A.I.Shnirelman (see [7]) is given; using this theory the degree of the FQL-mapping was defined, and the characteristics which are similar to those of finite-dimensional mappings are proved. Besides, equality of homological degree and degree to that of article [7] is indicated.

As seen, calculating the degree of mapping with homologies makes topological problem an algebraic one, and thus the given problem transforms into a combinatorial one. It is obvious that this homological theory has some other advantages.

1. The Simplex and Singular Theories

The simplex theory. Let H be the real Hilbert space and σ_n be the Euclid simplex from H of dimension n, H^k is the subspace of H of co-dimension k.

Definition 1 *The cylinder* $\sigma_n^k = \sigma_n \times H^k$ *from* $H \times H$ *is called a Hilbert simplex of bi-dimension* (n, k).

The σ_n *is called a basis of* σ_n^k.

Definition 2 *A set* $\sigma_s \times H^k$ *is called a bound simplex of* σ_n^k *of bi-dimension* (s, k), $0 \leq s \leq n$.

Here σ_s *be bound of simplex* σ_n *of dimension* s.

Definition 3 *The set* $K = \{\sigma_s^k\}$, $0 \leq s \leq n$, *is called a simplex complex, if together with each simplex* σ_s^k *in* K *enter all its bounds; two simplexes can be intersected only at their general bound.*

Definition 4 *The simplex* $\sigma_n^k = \sigma_n \times H^k$ *is called oriented, if its basis is oriented. In this case, the orient on* σ_n *is adoption for orient on* σ_n^k.*It is obvious, that* σ_n^k *can be oriented two ways. Denote them by* ${}^+\sigma_n^k$, ${}^-\sigma_n^k$.

Definition 5 *The factor-group of group of the formal linear combinations (finites) of kind* $\sum g_i \cdot \sigma_{n,i}^k$, $g_i \in Z$, *relatively of subgroup elements of kind* $g \cdot^+ \sigma_{n,i}^k + g \cdot^- \sigma_{n,i}^k$ *and their linear combinations is called the group* $C_n^k(K)$ *of chains of bi-dimension* (n, k) *of simplex complex* K.

In other words, we identify elements $g^+ \cdot \sigma_{n,i}^k$ *and* $g^- \cdot \sigma_n^k$ *in group of formal linear combinations of oriented simplexes.*

Definition 6 *The differential* ∂_n^k: $C_n^k(K) \to C_{n-1}^k(K)$, $\forall\ n \geq 1, \forall k$ *is defined by the equality*

$$\partial_n^k \left(g \cdot \left(\left[\alpha^{i_0}, ..., \alpha^{i_n}\right] \times H^k\right)\right) = \sum_0^n (-1)^j \left(\left[\alpha^{i_0}, ..., \alpha^{i_{j-1}}, \alpha^{i_{j+1}}, ..., \alpha^{i_n}\right] \times H^k\right)$$

for each oriented simplex and extended to all group $C_n^k\left(K\right)$ *by linearity.*

Here $\alpha^{i_0}, ..., \alpha^{i_n}$ *are apexes of simplex* σ_n. *In the future* $C_n^k\left(K\right)$ *will be denoted by* C_n^k.

Theorem 7 *The equality*

$$\partial_{n-1}^{k} \circ \partial_{n}^{k} = 0$$

holds for all $n \geq 1$ *and* k.

The proof analogous to finite-dimensional.

The set $\ker \partial_n^k = \left\{ c_n^k \left| \partial_n^k \, c_n^k \ ? \ 0 \right. \right\}$ *is a subgroup in* C_n^k, *which is called the group of cycles of bi-dimension* (n,k)*; its elements are called cycles of bi-dimension* (n,k). *The set* $Im\partial_{n+1}^k = \left\{ c_n^k \mid c_n^k = \partial_{n+1}^k c_{n+1}^k \right\}$ *is also a subgroup in* C_n^k, *which is called the group of boundaries of bi-dimension* (n,k)*; its elements are called boundaries of bi-dimension* (n,k).

Theorem 7 implies that $Im\partial_{n+1}^k \subset \ker \partial_n^k$.

Definition 8 *The factor-group of group of cycles relatively subgroup of boundaries,* $\ker \partial_n^k / Im\partial_{n+1}^k$, *is called a group of* (n,k)*-dimensional homologies of complex* K.

It is denoted by $H_n^k(K)$.

The cycles $^{/}c_n^k$, $^{//}c_n^k$ *from one of the classes of contiguity are called linear (L)-homologous and denote this as* $^{/}c_n^k \ \tilde{} \ ^{//}c_n^k$.

The singular theory. Let σ_n^k is a Hilbert simplex of bi-dimension (n,k), X is a real Hilbert space.

Definition 9 *The continuous mapping* $f_n^k : \sigma_n^k \to X$ *is called* (n,k)*-dimensional singular simplex, if*

a) f_n^k *is a affine invertible mapping on each layer* $H_\alpha^k = \alpha \times H^k$, $\alpha \in \sigma_n$;

b) co-dim $f_n^k(H_\alpha^k) = k$, $\alpha \in \sigma_n$;

c) $f_{n,\alpha}^k = f_n^k \left|_{H_\alpha^k} \right.$ *depends continuously on* α.

Definition 10 *The formal linear combination (finite)* $\sum\limits_i g_i \cdot f_{n,i}^k$ *of* (n,k)*-dimensional singular simplexes of space* X *with coefficients* $g_i \in Z$ *is called a* (n,k)*-dimensional singular chain of* X.

The set of all the (n,k)*-dimensional singular chains of space* X *is denoted by* $\tilde{C}_n^k\ (X)$. *It is an Abelian group relatively of addition of chains, as linear combinations. This group is free, since* $g_i \in Z$, Z *is the ring of integers.*

Definition 11 *We define the differential*

$$\tilde{\partial}_n^k : \tilde{C}_n^k(X) \to \tilde{C}_{n-1}^k(X) \forall n \geq 1, \forall k$$

as follows

$$\tilde{\partial}_n^k f_n^k = \sum (-1)^i \left(f_n^k \Big|_{\sigma_{n-1,i}^k} \right),$$

where $\sigma_{n-1,i}^k$ is $(n-1,k)$-dimensional bound of simplex σ_n^k and extend it on all group $\tilde{C}_n^k(X)$ by addition. For $n = 0$ suppose that $\tilde{\partial}_0^k : \tilde{C}_0^k \to 0 \ \forall k$.

Theorem 12 *The equality*

$$\tilde{\partial}_{n-1}^k \circ \tilde{\partial}_n^k = 0$$

holds for all $n \geq 1$ and k.

The proof is analogous to that in finite dimensional case.

By analogy to simplex case can be defined the groups $\ker \tilde{\partial}_n^k$, $Im\tilde{\partial}_{n+1}^k$, $\tilde{H}_n^k$, *accordingly of (n,k)-dimensional of cycles, boundaries, homologies.*

The theory of relative homologies in X is more interesting which is stated in the next paragraph.

2. The Relative Linear Homologies

Let X be the real Hilbert space, D a bounded domain in X. In this paragraph the definition of linear homologies of pair $(X, X \setminus D)$ is given. For simplicity, we consider the case, when $D = B(R)$ is open ball in X of radius R and with center at zero.

Definition 13 *The (n,k)-dimensional chain from $\tilde{C}_n^k(X)$ is called a relative cycle of bi-dimension (n,k), if its boundary enters $\tilde{C}_{n-1}^k(X \setminus B\ (R))$.*

Now we occupy ourselves with the definition of homology to zero of relative cycle (see definition 17).

Definition 14 $\{Y_\alpha^n\}$ *is called a family with the finite-dimensional opening, if it can be divided to family $\{Y_\beta^m\}$ of parallel planes of some co-dimension $m \geq n$* [1]**.**

[1] A (affine) bundle (Y_1, p_1, B_1) is called a dividing of (affine) bundle (Y_2, p_2, B_2), if $Y_1 = Y_2$ and $\forall \alpha \in B_1 \ \exists \beta \in B_2$, $p_1^{-1}(\alpha) \subset p_2^{-1}(\beta)$.

Definition 15 *f_n^k is called a singular simplex with the finite-dimensional opening, if the family $\{f_n^k(H_\alpha^k)\}$ has a finite-dimensional opening.*

By analogy, $\tilde{\sigma}_n^k = \sum\limits_1^m g_i \cdot f_{n,i}^k$ will call the chain with the finite-dimensional opening, if each simplex $f_{n,i}^k$, $i = 1, ..., m$, has a finite-dimensional opening. In this case, without loss of generality, we will propose that all the families $\{X_{i,\alpha}^k \setminus X_{i,\alpha}^k = f_{n,i}^k(H_\alpha^k)\}$, $i = 1, ..., m$, are divided into one family of parallel plains.

We now provide the supportive material in order to define the most important concept, i.e. homology of relative cycle to zero.

A) *Using the continuity of family of plains and the compactness of basis of each singular simplex $f_{n,i}^k$, entering into the chain $\tilde{\sigma}_n^k$, by sufficient little perturbing (i.e. unimportant motion), this chain can be approximated (in ball of radius R and with center at zero), by chain $'\tilde{\sigma}_n^k = \sum\limits_1^m g_i \cdot' f_{n,i}^k$, satisfying the following conditions:*

1) $'\tilde{\sigma}_n^k$ has the finite-dimensional opening[2]

2) if $'\tilde{\sigma}_n^k$ is relative cycle, then $'\tilde{\sigma}_n^k$ also will be relative cycle.

B) *Let $'f_{n,i}^k$ be some simplex from $'\tilde{\sigma}_n^k$, $\{'X_{i,\alpha}^k \setminus 'X_{i,\alpha}^k =' f_{n,i}^k(H_\alpha^k),\ \alpha \in \sigma_n\}$ is correspondent to the family plains (which has finite dimensional opening) and $\{'X_{i,\alpha,\beta}^{k'}\}$ is the dividing of $\{'X_{i,\alpha}^k\}$ on parallel plains of co-dimension $k', k' \geq k$,*

$$(\propto, \beta) \in \sigma_n \times R_{k'-k},$$

that is

$$'X_{i,\alpha}^k = \bigcup_\beta {'X_{i,\alpha,\beta}^{k'}} \quad \forall i \in \sigma_n;$$

$\forall (i_1, \alpha_1, \beta_1), (i_2, \alpha_2, \beta_2)\ 'X_{i_1,\alpha_1,\beta_1}^{k'} \|' X_{i_2,\alpha_2,\beta_2}^{k'}$ [3].

Transfer it dividing (by mappings $('f_{n,i,\alpha}^k)^{-1}, \alpha \in \sigma_n$) on σ_n^k. Then the plains H_α^k of $\sigma_n^k = \sigma_n \times H^k$ are also divided to parallel plains of co-dimension k'. Let $s : \sigma_n \to \sigma_n^k$ is some continuously section of trivial affine bundle $(\pi_n, \sigma_n^k, \sigma_n)$, $\pi_n : \sigma_n^k \to \sigma_n$. Consider orthogonal supplements (in H_α^k) to one from parallel plains $('f_{n,i,\alpha}^k)^{-1}('X_{i,\alpha,\beta}^{k'}), (\alpha, \beta) \in \sigma_n \times R_{k'-k}$, going by section $s : \sigma_n \to \sigma_n^k$. Then get the affine bundle with basis σ_n

[2] See [1].

[3] See [1].

and layers of dimension $(k'-k)$. Denote that bundle by $\sigma_{n'}$, $n' = n + (k'-k)$ and the layers by $H_{k'-k,\alpha}$. Because basis σ_n (of bundle $\sigma_{n'}$) convex (σ_n is n -dimensional simplex), then $\sigma_{n'}$ is trivial. Now consider the affine bundle, induced by families $\{H^{k'}_{i,\alpha,\beta} \backslash H^{k'}_{i,\alpha,\beta} = ('f^k_{i,n,\alpha})^{-1}('X^{k'}_{\alpha,\beta,i})\}$. Denote it by $\sigma^{k'}_{n'}$. $\sigma_{n'}$ is the basis of $\sigma^{k'}_{n'}$. Because the set $\sigma_{n'}$ is contractible, then $\sigma^{k'}_{n'}$ is trivial. Hence, without restriction of generality, one can suppose that the trivial affine bundle $\sigma^k_n = \sigma_n \times H^k$ divided on the trivial affine bundle $\sigma^{k'}_{n'} = \sigma_{n'} \times H^{k'}$. Divide the basis $\sigma_{n'} = \sigma_n \times H_{k'-k}$ by $\sigma_{n',i}$, where $\sigma_{n',i}, i = 1, 2...$, are parallel prisms of dimension n'. Take the Cartesian product $\sigma_{n',i} \times H^{k'}$. Obviously, the dividing of $\sigma_{n'}$ on to prisms can be made so, that the contraction f^k_n only on one from these prisms have intersection with $B(R)$. It is possible, because of the linearity f^k_n on each H^k_α, uniformly continue $f^k_{n,\alpha}$ at α and bounded ness of $B(R)$; the other contractions in this case will be consisted out of $B(R)$. In this case, the orientation of one of the simplexes $\sigma^{k'}_{n',i} = \sigma_{n',i} \times H^{k'}$ get out arbitrary and the orientations of other simplexes coordinate with it.Therefore each of two neighboring simplexes $\sigma^{k'}_{n',i}$, $\sigma^{k'}_{n',i\div 1}$, $i = 0, \pm 1, \pm 2, ...$, induce on its general bound contrary orientations.

Definition 16 *The contraction of f^k_n on the (n', k') -dimensional cylinder is called its (n', k') -dimensional bearer, if*

a) $n' - n = k' - k$;

b) the image of this contraction consists all the points of intersection $f^k_n(\sigma^k_n)$ with $B(R)$.

By analogy, well call the chain $\tilde{\sigma}^{k'}_{n'} = \sum\limits_1^m g_i \cdot f^{k'}_{n',i}$ the (n', k') -dimensional bearer of the chain $\tilde{\sigma}^k_n = \sum\limits_1^m g_i \cdot f^k_{n,i}$, if $f^{k'}_{n',i}$ is (n', k') -dimensional bearer of $f^k_{n,i}$ for each i.

In this case, without restriction of generality, well suppose that all the plains from all the simplexes, induced by the bearers, entering $\tilde{\sigma}^{k'}_{n'}$, parallel to each other. Let $\tilde{\sigma}^k_n$ be a relative singular cycle. Lets orient each simplex $f^{k'}_{n',i}, i = 1, ..., m$, so that for two simplexes $f^k_{n,i}$ and $f^k_{n,i'}$ from $\tilde{\sigma}^k_n$, having a general bound, according to their simplexes $f^{k'}_{n',i}$ and $f^{k'}_{n',i'}$ induce on that bound opposite orientations. So, the oriented relative cycle $\tilde{\sigma}^{k'}_{n'}$ can be constructed (with two possible ways). So, we have finished preparatory material. We now present our **main** *definition.*

Definition 17 *The relative cycle σ_n^k is called homological to zero, if for some number $l \geq 0$, its $(n+l, k+l)$-dimensional bearer $\tilde{\sigma}_{n+l}^{k+l}$ homological to zero.*

From this definition it is easy to get that sum of (n,k)-dimensional relative cycles, homologized to zero, is also (n,k)-dimensional relative cycle, homologized to zero. Therefore, (n,k)-dimensional relative cycles, homologized to zero, make subgroup of group (n,k)-dimensional relative cycles.

Lemma 18 *Relative cycles $\tilde{\sigma}_n^k$, $'\tilde{\sigma}_n^k$ from point* **A)***, are homologized to each other in sense of definition 17 (for this $l = 0$), at sufficiently little ε >0, right*

$$2\varepsilon < dist(B(R), \bigcup_i \bigcup_\alpha X_{i,\alpha}^k),$$

where $\alpha \in \partial\sigma_n, i = 1, ..., m, X_{i,\alpha}^k$ is arbitrary plain from $\tilde{\partial}_n^k \tilde{\sigma}_n^k$.

3. Calculation $\tilde{H}_n^k(X, X \setminus B(R))$

Theorem 19

$$\tilde{H}_n^k(X, XnB(R)) = \begin{cases} 0, n \neq k, \\ Z, n = k. \end{cases}$$

The proof reduces to calculation of the group $\tilde{H}_{n+l}(X_{k+l}, X_{k+l} \setminus B_{k+l}(R))$, where X_{k+l} is $(k+l)$-dimensional subspace $X, B_{k+l}(R)$ is open bull in X_{k+l} of radius R with center at zero. More precisely, taking into consideration the isomorphism

$$\tilde{H}_{n+l_1}(X_{k+l_1}, X_{k+l_1} \backslash B_{k+l_1}(R)) = \tilde{H}_{n+l_2}(X_{k+l_2}, X_{k+l_2} \backslash B_{k+l_2}(R)),$$

proves that

$$\tilde{H}_n^k(X, X \backslash B(R)) = \tilde{H}_n(X_k, X_k \backslash B_k(R))$$

and

$$\tilde{H}_n(X_k, X_k \backslash B_k(R)) = \begin{cases} 0, n \neq k, \\ Z, n = k. \end{cases}$$

Here n, k, l, l_1, l_2 are arbitrary natural numbers.

4. The Homological degree of an FQL-mapping

Let X, Y be the real Hilbert spaces, $F : X \to Y$ be an FQL-mapping. Suppose, the a priori estimate

$$\|x\|_x \leq \Phi(\|F(x)\|_y) \tag{1}$$

where Φ is some positive monotonous function. Let $B_x(R)$ be a ball in X of radius R with center at zero, $x \in X \setminus B_x(R)$, that is $\|x\|_x \geq R$.

Lemma 20 *At above said conditions,*

$$\exists R' > 0, F(x) \in Y \setminus B_y(R'),$$

where $B_Y(R')$ is a ball in Y of radius R' with center at zero.

Here for simple, it is supposed that Φ is an identical mapping. From lemma 20 follows, that F is mapping of pairs $(X, X \setminus B_x(R)), (Y, Y \setminus B_y(R'))$.

Lemma 21 *Let $F_m : X \to Y, m = 1, 2, 3, ...,$ be the sequence of FL-mappings, uniformly converging F in each bounded ball and F satisfy estimate (1).Then at sufficiently large m, FL-mapping F_m will be mapping of pairs.*

Obviously, at sufficiently large m, FL-mapping F_m induced the homomorphism

$$F_{m,*} : \tilde{H}_n^n(X, X \setminus B_x(R)) \to \tilde{H}_n^n(Y, Y \setminus B_y(R' - \varepsilon)),$$

where $\varepsilon > 0$ is the arbitrary positive number. Let $[\tilde{\sigma}_n^n]$ be a generator of group $\tilde{H}_n^n(X, X \setminus B_x(R))$ and $[\tilde{\omega}_n^n] = F_{m,*}[\tilde{\sigma}_n^n]$. As $\tilde{H}_n^n(Y, Y \setminus B_y(R' - \varepsilon)) = Z$, then $[\tilde{\omega}_n^n]$ corresponds some of the numbers from Z. Let denote that number by $\deg_H(F_m)$.

Definition 22 *The number $\deg_H(F_m)$ is called of homological degree of FL- mapping F_m.*

The signum $\deg_H(F_m)$ depended at selected generators in groups $\tilde{H}_n^n(X, X \setminus B_x(R))$ and $\tilde{H}_n^n(Y, Y \setminus B_y(R' - \varepsilon))$. The correction of definition 22 is easily proved. It is easy to prove that at sufficiently large m, $\deg_H(F_m)$ is stabilized. Because of this we can give the following.

Definition 23

$$\deg_H(F) = \lim_{m\to\infty} \deg_H(F_m).$$

Theorem 24 *Let $\{F_t\}$ be the family of FQL-mappings, which depend continuously (but in each sphere is uniformly continuous) on parameter $t \in [0,1]$ and for all $t \in [0,1]$ the a priori estimate (1), where the function Φ is independent on t, is satisfied. Then*

$$\deg_H(F_1) = \deg_H(F_0).$$

Let $F : X \to Y$ be an FQL-mapping, satisfying condition (1), $\deg_1(F)$ *is degree of F as FQL-mapping, defined by A.I.Shnirelman (see [7]).*

Theorem 25

$$\deg_H(F) = \deg_1(F)$$

Consequence 26 *If* $\deg_H(F_m) \neq 0$*, then $\forall y \in B_y(R' - \varepsilon)$ the equation $F_m(x) = y$ has solution in $B_x(R)$.*

Theorem 27 *Let $F : X \to Y$ is FQL-mapping, satisfying of a priori estimate (1) and* $\deg_H(F) \neq 0$*. Then the equation $F(x) = y$ has solution $\forall y \in Y$.*

Proof. Consider $y_0 \in F$. Since at sufficiently large m

$$\forall x \in B_x(R) \ \ ||F(x) - F_m(x)||_y < \varepsilon,$$

then from condition (1) follows that solutions of equation $F_m(x) = y_0$ belong to ball $B_x(R_0)$, where $R_0 = \Phi(||y_0||_y + 2\varepsilon)$. Then mappings F_m will transfer $X \setminus B_x(R_0)$ in $Y \setminus B_y(||y_0||_y + \varepsilon)$. As $\deg_H(F) = \lim_{m\to\infty} \deg_H(F_m)$, then at sufficiently large m $\deg_H(F_m) = \deg_H(F)$. Hence $\forall m \gg \deg_H(F) \neq 0$. Therefore from consequence 26 follows that $F_m(x) = y_0$ has solution. The proof of existence of solution of equation $F(x) = y_0$ is conducted analogous to as given in [7]. $\square$

5. Appendix

Let X, Y be real Banach spaces, let Ω be abound domain in X and suppose that $\pi_n : X \to X_n$ is a linear mapping from X to a n -dimensional space X_n and $X^n_\alpha = \pi^{-1}(\alpha)$, $\alpha \in X_n$.

Definition 28 *Continuous mapping $f^n : \Omega \to Y$ is called a Fredholm Linear (FL), if*

a) some linear mapping $\pi_n : X \to X_n$ is fixed;

b) on each plane X^n_α, $\alpha \in X_n$, passing through Ω, $f^n_\alpha \equiv f^n \mid_{X^n_\alpha}$ is an affine invertible mapping from X^n_α on to its image $Y^n_\alpha = f(X^n_\alpha)$ that is, closed in Y and has co-dimension n and f^n_α depends continuously on α.

Definition 29 *inuous mapping $f : X \to Y$ is called Fredholm Quasi-Linear (FQL), if there exists a sequence FL-mappings $\{ f^{n_k} \}$, uniformly approximating f on each bounded domain $\Omega \subset X$, such that*

$$\|f^{n_k}_\alpha\| < C(\Omega), \left\|(f^{n_k}_\alpha)^{-1}\right\| < C(\Omega),$$

with $k > k_0(\Omega)$, if $\alpha \in \pi_{n_k}(\Omega)$ and $C(\Omega)$ does not depend on k, if $k > k_0(\Omega)$.

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