

Finite Groups all of Whose Abelian Subgroups of Equal Order are Conjugate

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Abstract

In this paper we classify the finite groups whose abelian subgroups of equal order (B^* -groups) are conjugate. The classification has been achieved by means of a lot of general structure properties of B^* -groups, provided in the course of the paper.

Key Words: Finite solvable groups, finite non-solvable groups, conjugacy classes of abelian subgroups, projective special linear groups over a finite field, simple first group of Janko, alternating groups, Sylow subgroups, Hall subgroups, Fitting subgroups, transitive action of groups on groups.

1. Introduction

There are quite a few papers relating the order of subgroups or elements of a finite group to questions on the existence of inner automorphisms, isomorphisms, and conjugacy. In the following, let us consider a list of statements all dealing with that idea; the symbol G stands universally for a finite group. As a matter of fact, all groups regarded in this paper are assumed to be finite. So let us introduce the following classes of groups G .

- A - Elements of G of equal order are conjugate;
- I - Each isomorphism between two subgroups of G is induced by an automorphism of G ;
- S - Each pair of isomorphic subgroups of G is a pair of conjugate subgroups in G ;

- Z_1 - Each isomorphism between cyclic subgroups of G is induced by an automorphism of G ;
- B - Subgroups of G of equal order are conjugate in G ;
- B^* - Abelian subgroups of G of equal order are conjugate in G ;
- Π_p - For a given prime number p , all p -subgroups of G of equal order are conjugate in G ;
- C - Cyclic p -subgroups of G of equal order are conjugate in G for any prime p ;
- Z_2 - Subgroups of G of equal order are isomorphic.

Next we will describe, where find as much information as possible on the structure of those finite groups mentioned in the list of the above statements.

A - There are only three groups here, to wit: $\{1\}$, the cyclic group of order 2, and the symmetric group on three symbols. See [22] for a self contained proof of this assertion based on methods occurring in papers before 1985; look also at the paper of Zhang Jiping [24] from 1988, which we have not seen. The classification follows also as a corollary to more involved investigations of Feit and Seitz published in 1988 ; see [6].

I - Consult the work done by Cherlin and Felgner [3] (it comprises the observation of P.M. Neumann which had earlier classified the solvable groups featuring in this statement). It was Li who treated the non-solvable case (to be found in [15]), whereas Zhang Jiping [25] claims to have done all of it in an unpublished paper; view the very recent paper [4].

S - View Stroth's paper [18] from 1996.

Z_1 - Here we refer to the paper of Zhang Jiping [25] from 1992.

B - The papers of Bensaid, Van der Waall and Lindenbergh are conclusive: see ([2], [16], [21]).

Π_p - It was Gross, who dealt with the groups featured in statement Π_p ; see [9].

Z_2 - View Zhang Jiping's paper [26] from 1995.

C - The classification of groups occurring here has almost been carried out; see a future paper by Sezer with some help by Van der Waall.

B^* - The full classification of the groups in this rubric is the topic of the underlying paper!

The obvious convention of notation suggests that the reader might observe the following “inclusions”:

$$A \subset I \subset Z_1, \quad B \subset B^* \subset C, \quad S \subset C, \quad B \subset Z_2.$$

In general, notation will be standard or self-explanatory; see [8], [10], [11], [12] and [17], of which Huppert’s books and Suzuki’s books are the main standard. For the sake of convenience, we mention the following:

S_n - the *symmetric group* on n symbols;

A_n - the *alternating group* on n symbols;

$|U|$ - the *cardinality* of the subset U of G ;

$H \leq G$ - H is a *subgroup* of G ;

$H < G$ - $H \leq G$ and $H \neq G$;

$H \trianglelefteq G$ - H is a *normal subgroup* of G ;

$H \triangleleft G$ - $H \trianglelefteq G$ and $H \neq G$;

G' , $[G, G]$ - the *commutator subgroup* of G ;

$[H, K]$ - for subsets H and K of G , it means the group generated by all commutators $h^{-1}k^{-1}hk$, where $h \in H$ and $k \in K$;

$[H, K, L]$ - for subsets H, K and L of G , it means that it is equal to $[[H, K], L]$;

$Z(G)$ - the *center* of G ;

$\Phi(G)$ - the *Frattini subgroup* of G ;

$F(G)$ - the *Fitting subgroup* of G ;

$F_2(G)$ - $F_2(G)/F(G) := F(G/F(G))$;

$O_p(G)$ - the subgroup of G generated by all the normal p -subgroups of G where p is a given prime;

$C_G(M)$ - $\{u \in G \mid um = mu \text{ for all } m \in M\}$, the *centralizer* of M in G ;

$N_G(M)$ - $\{t \in G \mid t^{-1}mt \in M \text{ for all } m \in M\}$, the *normalizer* of M in G

$Aut(G)$ - the group of all *automorphisms* of G ;

$Inn(G)$ - $\{\alpha \in Aut(G) \mid \exists a \in G \text{ such that } \alpha(g) = a^{-1}ga, \text{ for all } g \in G\}$, the group of all *inner automorphisms* of G [the former $a \in G$ does not depend on the $g \in G$ in $\alpha(g) = a^{-1}ga$];

$Syl_p(G)$ - the set of all *Sylow p -subgroups* of G for a given prime p ;

$Hall_\pi(G)$ - for a given set of primes π , $H \in Hall_\pi(G)$ if and only if all primes t dividing the order of the subgroup H of G satisfy $t \in \pi$, but $s \notin \pi$ for all primes s dividing the index of H in G ;

$P\Gamma L(n, q)$ - the *automorphism group* of $PSL(n, q)$;

$PSL(n, q)$ - $SL(n, q)/Z(SL(n, q))$;

\mathbb{F}_q - the *finite field* consisting of q elements;

$SL(n, q)$ -the group of all the $(n \times n)$ -matrices with coefficients in \mathbb{F}_q , each matrix being of determinant equal to 1;

$PGL(n, q)$ - $GL(n, q)/Z(GL(n, q))$;

$GL(n, q)$ - the group of all the $(n \times n)$ - matrices with coefficients in \mathbb{F}_q , each matrix being of determinant unequal to zero;

$\Gamma L(1, p^n)$ - the group of all the *semi-linear maps* $x \mapsto ax^\sigma$, with $x \in \mathbb{F}_{p^n}$, p prime , $\sigma \in Gal(\mathbb{F}_{p^n}/\mathbb{F}_p)$, $a \in \mathbb{F}_{p^n} \setminus \{0\}$;

$Gal(\mathbb{F}_q/\mathbb{F}_t)$ - the unique subgroup U of the group H of all field automorphisms of \mathbb{F}_q fixing a given subfield \mathbb{F}_t , i.e.,

$$U = \{h \in H \mid h(f) = f, \text{ for all } f \in \mathbb{F}_t \};$$

but notice then that *here* also \mathbb{F}_t is characterized by

$$\mathbb{F}_t = \{f \in \mathbb{F}_q \mid h(f) = f, \text{ for all } h \in U \};$$

$Exp(G)$ - minimum of $\{t \in \mathbb{N} \mid g^t = 1, \text{ for all } g \in G \}$;

$p^a \parallel n$ -the prime p divides the positive integer n precisely a times, i.e. $p^a \mid n$ and $p^{a+1} \nmid n$.

If X is a group and $H_1 \leq X, H_2 \leq X$, then we say that H_1 and H_2 are conjugate in X (notation $H_1 \sim_X H_2$) if there exists an element $g \in X$ with $H_2 = \{g^{-1}hg \mid h \in H_1\}$. Of course, we write alternatively $H_2 = g^{-1}H_1g$ (or $H_2 = H_1^g$).

In this paper the structure of those groups R will be determined in which any two abelian subgroups of R of equal order are conjugate. Such a group G , apparently belonging to the class B^* , is called a B^* -group; we also write $G \in B^*$. If some group K satisfies statement B , then we say that K is a B -group (or $K \in B$).

It will turn out (see §3) that solvable B^* -groups are B -groups. On the other hand, there are non-solvable B^* -groups that are not B -groups. Subgroups and quotient groups of B^* -groups are not always B^* -groups. Therefore, straightforward induction arguments cannot be applied to the study of the B^* -groups. But in some cases one can exploit the observation that some special type of quotients of B^* -groups are B^* -groups; see Lemma 1.3.

In §1 we deal with the elementary or general properties of B^* -groups. Among them are also properties of Sylow subgroups of B^* -groups. In §2 the solvability and insolvability

properties of B^* -groups are considered, such as structures on the chief factor groups. And then, in §3, we show that solvable B^* -groups are B -groups. In §4 the classification of the non-solvable B^* -groups is presented. The compilation of the structure of the Theorems 3.5, 4.2, 4.3, 4.4, 4.5, and 4.6 provides the Main Theorem of this paper:

Main Theorem *The following statements hold.*

a) *The class consisting of all solvable B^* -groups coincides with the class consisting of all solvable B -groups.*

b) *Every non-solvable B^* -group either is a non-solvable B -group or else is isomorphic to a direct product of the groups M and H , where H is any solvable B -group whose order is relatively prime to the order of M , and where M is either isomorphic*

to the B^ -group J_1 (Janko's first simple group of order 175560),*

or to any of the simple B^ -groups $PSL(2, q)$ with $q = p^f$, p odd prime $f = 1$ or 3 , $q \geq 11$, $q \equiv 3$ or $5 \pmod{8}$,*

or isomorphic to any of the quasi-simple B^ -groups $SL(2, u)$ with $u = p^f$, p odd prime, $f = 1$ or $f = 3$; $u \geq 7$.*

In [2] it is shown that a non-solvable B -group contains either elementary abelian non-cyclic 2-subgroups or else quaternion subgroups of order 8 as Sylow 2-subgroups. As such, the papers [2] and [16] provide the following portemanteau theorem either explicitly or implicitly.

Theorem P₁. *Let G be a non-solvable B -group. Then G is one of the following types of groups.*

a) *$G = N \times T$ for any solvable B -group T with $(|N|, |T|) = 1$; here, N is isomorphic to one of the groups $A_5, SL(2, 8), SL(2, 5)$, or to the semidirect product of U by $(C_p \times C_p)$ for each $p \in \{11, 19, 29, 59\}$ with $U \cong SL(2, 5)$ and $C_p \times C_p$ being a minimal normal subgroup of the semidirect product of U by $(C_p \times C_p)$;*

b) *$G = (N \times T) \langle \alpha \rangle$; here $T \langle \alpha \rangle$ is any solvable B -group satisfying*

$$(|T \langle \alpha \rangle|, 2046) = 1, \alpha^5 \in T, N \cong SL(2, 32), N \langle \alpha \rangle / \langle \alpha^5 \rangle \cong \Gamma L(2, 32);$$

c) *G is the semidirect product of U by $(C_p \times C_p) \times H \langle c \rangle$; here $H \langle c \rangle$ is a solvable B -group satisfying $(|H \langle c \rangle|, 30p) = 1$, $U \cong SL(2, 5)$, $[U, H \langle c \rangle] = \{1\}$, $c^{-1}Hc = H$, $c^{-1}tc = t^i$ for all $t \in C_p \times C_p$, $|c| = \beta^\alpha$ $\alpha \geq 0$, $c^\beta \in H$, and $\{\beta, p, i\}$ is one of the ordered*

triples $\{7, 29, 16\}$, $\{29, 59, 4\}$; $C_p \times C_p$ is a minimal normal subgroup of the semidirect product of U by $(U_p \times U_p)$.

Conversely, any of the groups featuring in the list of this Theorem is indeed a non-solvable B -group.

From paper [21] we provide the following portemanteau results; see Theorems 10 and 11 of [22].

Theorem P₂ *Let G be a solvable group. Assume the following statements hold.*

a) *Any non-cyclic Sylow subgroup of G is normal in G ,*

and

b) *Any two subgroups of G of equal order that are contained in the Fitting subgroup $F(G)$ of G are conjugate in G .*

Then G is a B -group.

Conversely, if H is a solvable B -group, then condition b) holds trivially. But condition a) is denied when non-cyclic Sylow 2-subgroups are not contained in $F(H)$. In that exceptional case it holds that Sylow 2-subgroups of H are quaternion of order 8, that $F(H) = L \times C$, where C is abelian satisfying $(|C|, |L|) = 1$ and with either $L \cong C_5 \times C_5$ and $H/C_H(L) \cong SL(2, 3)$ or else $L \cong C_{11} \times C_{11}$ and $H/C_H(L) \cong SL(2, 3) \times C_u$ with $u = 1$ or $u = 5$. In the exceptional case, there exists $K \trianglelefteq H$ satisfying $K \geq L$ with K/L isomorphic to a quaternion group of order 8.

Theorem P₃ *Suppose T is a solvable B -group for which $S \in \text{Syl}_p(T)$ is not cyclic. If $p \geq 3$, then $S \leq F(T)$ holds (hence $S \trianglelefteq T$) and S is elementary abelian of order p^2 or p^3 . When $p = 2$, then S is either isomorphic to a quaternion group of order 8 or else S is elementary abelian of order 4, 8, or 32, satisfying $S \trianglelefteq T$.*

It is possible to refine the structure of the Fitting subgroup of a B -group analogously to the lines presented in [2] and [21]; it is not the issue to carry that out here, and we postpone it to a later occasion. See also Theorems 6, 7 and 8 of [21], handling exhaustive results regarding the group $V/C_V(M)$, where M is a non-cyclic minimal normal p -subgroup of the solvable B -group V .

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2. Generalities on B^* -groups

As mentioned in the Introduction, a finite group G is called a B^* -group if any two abelian subgroups of equal order are conjugate in G . We unravel the internal structure of a B^* -group. An overview will be given of the structure of the Sylow subgroups of a B^* -group, followed by some elementary and (non-)closure properties of the class of B^* -groups. Finally we end this section with some observations about chief factors of B^* -groups.

The Sylow subgroups of a B^* -group will turn out to be of a very restricted nature. Let G be a B^* -group and let p be a prime. Then any two subgroups of G of order p^2 are conjugate (as they are abelian), whence isomorphic. Therefore either all subgroups of G of order p^2 are elementary abelian, or else all these subgroups are cyclic. If G does not contain an element of order p^2 , then $\text{Exp}(P) = p$ for any $P \in \text{Syl}_p(G)$ when $P > \{1\}$. Thus, according to ([11],III.8.2 Satz), that the following Proposition holds.

Proposition 1.1 *Let $P \in \text{Syl}_p(G)$ with $G \in B^*$. Then*

- (i) *If $p = 2$, then either P is cyclic, or elementary abelian, or generalized quaternion;*
- (ii) *If p is odd, then $\text{Exp}(P) = p$ or else P is cyclic.*

Another elementary property of B^* -groups is given in the next Proposition.

Proposition 1.2 *Let G be a B^* -group.*

- a) *Let $M \trianglelefteq G$ and $a \in G$. Then $a \in M$, if there exists an element in M of order $|a|$.*
- b) *Let $M \trianglelefteq G$ with $|M| = p^t$ (p prime, $t \geq 2$). Suppose that M is elementary abelian. Then M is a (in fact the unique) Sylow p -subgroup of G .*
- c) *Let $M \trianglelefteq G$, $A \leq M$, $C \leq M$, $|A| = |C|$. If A and C are abelian, then $\#\{mAm^{-1} \mid m \in M\} = \#\{mCm^{-1} \mid m \in M\}$.*

Proof. a) Suppose $s \in M$ with $|s| = |a|$. Then $\langle s \rangle$ and $\langle a \rangle$ are conjugate within G . So, as $M \trianglelefteq G$, $a \in M$ follows immediately.

b) Since $|M| = p^t \geq p^2$, $Exp(P) = p$ will hold for any Sylow p -subgroup P of the B^* -group G as soon as M is elementary abelian. Now apply a) and the result follows.

c) As the centralizer $C_M(A)$ of A in M is conjugate to $C_M(C)$ (due to the fact that A and C are conjugate in G), it follows that $|M : C_M(A)| = |M : C_M(C)|$, yielding the result. \square

Now let us look at the (non-)closure properties of B^* -groups. The alternating group A_4 on four symbols satisfies $A_4 \in B \subseteq B^*$, but $P \in Syl_2(A_4)$ does *not* fulfill $P \in B^*$. It is almost clear, (see ([2], Theorem 2), that the class of all B -groups is closed under taking homomorphic images. On the other hand, the class of B^* -groups is NOT closed under taking homomorphic images! As an example, we mention here that $G = SL(2, 7) \in B^*$. Take $c = \begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix} \in SL(2, 7)$ and $d = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \in SL(2, 7)$. Then $|cZ(SL(2, 7))| = 4$ within $PSL(2, 7)$ whereas $\langle c^2Z(SL(2, 7)), dZ(SL(2, 7)) \rangle$ is elementary abelian of order 4 within $PSL(2, 7)$. Thus $PSL(2, 7) \notin B^*$. Not everything is lost however, as the following Lemma shows.

Lemma 1.3 *Let $G \in B^*$, $M \trianglelefteq G$, $|M| \equiv 1 \pmod{2}$. Then $G/M \in B^*$.*

Proof. Suppose G is a counterexample of minimal order to the statement of the Lemma. Now, within G , choose M of smallest order for which $G/M \notin B^*$; so $M \neq \{1\}$. Suppose $L \trianglelefteq G$, $L \leq M$, $L \neq \{1\}$, where M/L is a minimal normal subgroup of G/L . Then we get $G/L \in B^*$. Since M/L is a normal subgroup of odd order of G/L and as $G/M \cong (G/L)/(M/L)$ with $G/M \notin B^*$, the choice of G together with M implies that $L = \{1\}$. Since M is of odd order, it is solvable (by Feit and Thompson; see [7]). Thus M is an elementary abelian p -group for some prime p ; remember that $p \neq 2$.

Suppose X, Y are subgroups of G containing M , for which X/M and Y/M are abelian of equal order. If $p \nmid |X/M|$, then $X = MX_1$ for some $X_1 \leq X$ with $M \cap X_1 = \{1\}$ (by Schur and Zassenhaus; see ([11], I.§18) Likewise $Y = MY_1$, $Y_1 \cap M = \{1\}$ for some $Y_1 \leq Y$. Thus X_1 and Y_1 are abelian subgroups of G of equal order, whence conjugate within G . So X/M and Y/M are conjugate within G/M . Hence assume that p divides $|X/M|$ ($= |Y/M|$). This implies by Propositions 1.1 and 1.2, that M is cyclic. It is

now Proposition 1.2 yielding the cyclicity of any Sylow p -subgroup of G . Since X/M is abelian, it follows easily that there exist some cyclic $P \in \text{Syl}_p(X)$ and Hall p' -subgroup X_1 of X such that $X = PX_1$. [Notice here, that $P \trianglelefteq X$ as X/M is abelian]. Since $[X, X] \leq M \leq P$, we now see that $[P, X_1] \leq M \leq \Phi(P)$; remember P is cyclic. Thus X_1 centralizes $P/\Phi(P)$. Therefore $[P, X_1] = \{1\}$ by ([11], III 3.18 Satz). In other words, $X = PX_1$ is an abelian group; notice $|X| = |Y|$. Hence X and Y are conjugate in G , inducing that X/M and Y/M are conjugate in G/M . So $G/M \in B^*$ after all. This contradiction completes the proof of Lemma 1.3. \square

As for direct products, the following holds.

Lemma 1.4 *Let $G = G_1 \times G_2$.*

- (i) *Suppose $G \in B^*$. Then G_1 and G_2 are both B^* -groups; moreover $(|G_1|, |G_2|) = 1$;*
- (ii) *Suppose $G_1 \in B^*, G_2 \in B^*, (|G_1|, |G_2|) = 1$. Then $G \in B^*$.*

Proof. (i) If $G = G_1 \times G_2 \in B^*$, it is clear that G_1 and G_2 are of relatively prime order. Consider an abelian $A_1 \leq G_1$ and an abelian $B_1 \leq G_1$ of equal order. Then there exists $g = g_1g_2 \in G$ with $g_i \in G_i$ ($i = 1, 2$) such that $gA_1g^{-1} = B_1$. Therefore, $B_1 = gA_1g^{-1} = g_1g_2A_1g_2^{-1}g_1^{-1} = g_1A_1g_1^{-1}$. Therefore, $G_1 \in B^*$ and likewise $G_2 \in B^*$.

(ii) Any subgroup of G is of the form H_1H_2 , where $H_1 \leq G_1$ and $H_2 \leq G_2$. See ([19], Corollary to Ch. 2, Th (4.19)); it is used here, that $(|G_1|, |G_2|) = 1$. Suppose H and K are abelian subgroups of G of equal order. So there are $H_i \leq G_i$ and $K_i \leq G_i$ ($i = 1, 2$), such that $H = H_1H_2$ and $K = K_1K_2$. Furthermore, there exist $g_i \in G_i$ ($i = 1, 2$) satisfying $g_iH_ig_i^{-1} = K_i$. Altogether we find $(g_1g_2)H(g_1g_2)^{-1} = (g_1g_2)H_1(g_1g_2)^{-1}(g_1g_2)H_2(g_1g_2)^{-1} = g_1H_1g_1^{-1}.g_2H_2g_2^{-1} = K_1K_2 = K$; of course it is used here that $[H_1, \langle g_2 \rangle] = \{1\} = [H_2, \langle g_1 \rangle]$. This proves the claim that $G \in B^*$. \square

The forgoing Lemma 1.4 tells us that in order to classify all B^* -groups, it is sufficient to know all so-called *indecomposable* B^* -groups, i.e. those that cannot be written as a direct product of two proper non-trivial subgroups. In this respect the following is nice to observe.

Theorem 1.5 *Every nilpotent B^* -group is cyclic.*

Proof. Any nilpotent group is the direct product of its Sylow subgroups; see ([11], III.2.3 Hauptsatz). Therefore, by Lemma 4, all these Sylow subgroups are B^* -groups. If

such a Sylow subgroup S is of odd order, then S must be cyclic. Otherwise, by Proposition 1.2, all subgroups of S of prime order would be conjugate to each other within S , yielding that $Z(S) \neq \{1\}$ is cyclic, and all subgroups of order p would be absorbed by $Z(S)$, so that $|S| = p$, a contradiction! Likewise, such a Sylow 2-subgroup is cyclic or generalized quaternion by Proposition 1.1. But a generalized quaternion 2-group is never a B^* -group, as a direct verification regarding cyclic subgroups of order 4 shows. Thus our nilpotent group is a direct product of cyclic groups of relatively prime order. Hence, it is itself cyclic. The Theorem has been proved. \square

Lemma 1.6 *Let M/N be a chief factor of the B^* -group G . Let p be a prime dividing $|M/N|$ and let $P \in \text{Syl}_p(G)$. If $\text{Exp}(P) = p$, then $P \cap M = P$ and $P \cap N = \{1\}$. If $\text{Exp}(P) > p$, then either P is cyclic or else $p = 2$ with $P \cap M = P$, and $|P \cap N| \leq 2$. If $p = 2$ and $P \in \text{Syl}_2(G)$ is cyclic, then $|M/N| = 2$ follows.*

Proof. Assume $\text{Exp}(P) = p$. Since $P \cap M \in \text{Syl}_p(M)$, it follows that $(P \cap M)N/N \in \text{Syl}_p(M/N)$ is not trivial. All subgroups of order p are conjugate in G and each of them is contained in M . So $P \leq M$. If $P \cap N \neq \{1\}$, then there exists a subgroup of N of order p , but this subgroup cannot be conjugate to a subgroup of N (of order p). Hence $P \cap N = \{1\}$.

So, let us assume that $p = 2$, $\text{Exp}(P) > 2$, and P is generalized quaternion. If $|P \cap M| \geq 4$, then M contains a subgroup of order 4, necessarily being cyclic. It is true that P is generated by its elements of order 4. So $P \leq M$, by Proposition 1.2. In the same way, $|P \cap N| \leq 2$ follows. Assume now that P is a cyclic 2-group. Then Burnside's theorem ([11], IV.2.8 Satz) tells us that G is solvable. In particular, as $2 \mid |M/N|$, M/N is an elementary abelian 2-group. Hence, in fact, M/N is cyclic of order 2. \square

3. Solvability and non-solvability properties around B^* -groups

Let us look at the following situation. Suppose $G \in B^*$. Assume M is an elementary abelian normal subgroup of G of order p^n , with $n \geq 2$, p some prime number. By Proposition 1.2 M is a minimal normal subgroup of G and all subgroups of order p of G (all of them are contained in M) are conjugate in G ; furthermore, p does not divide $|G/M|$, hence also $(p, |G/C_G(M)|) = 1$.

Let $H \cong G/C_G(M)$ be a subgroup of $GL(n, p)$ with $p \nmid |H|$ permuting transitively

all d -dimensional subspaces of the n -dimensional vector space $V(n, p)$ over \mathbb{F}_p whenever $d \in \{0, 1, \dots, n\}$. We will prove that the following Proposition 2.1 holds regarding the structure of H .

Proposition 2.1 *The group $H \cong G/C_G(M)$ as defined in the beginning of §2 satisfies precisely one of the three statements of the following list F .*

$$F = \left\{ \begin{array}{l} (a) \ H \text{ is isomorphic to a subgroup of } \Gamma L(1, p^n); \\ (b) \ H \text{ contains a central subgroup } C, \text{ admitting a 2 - group } N/C \\ \text{as minimal normal subgroup of } H/C \text{ of order 4, where} \\ \quad n = 2 \text{ and } p \in \{3, 5, 7, 11, 23\}; \\ (c) \ \text{The last member } H^\infty \text{ of the derived series of } H \text{ is isomorphic} \\ \quad \text{to } SL(2, 5), \text{ where } n = 2, p \in \{11, 19, 29, 59\}. \end{array} \right.$$

Proof. Consider the group $\overline{H} = HU$, where by definition $U = \{\alpha I \mid \alpha \in \mathbb{F}_p \setminus \{0\}\}$, and I is the $n \times n$ -identity matrix in $GL(n, p)$. So $U \leq Z(\overline{H})$ and $|U| = p - 1$. Then it is possible to apply results as worked out by Hering in §5 of [10]. We now assume, until first notice, that \overline{H} is not solvable. This implies that p is odd [Indeed, by Proposition 1.2 and the Feit-Thompson Theorem we get for $|M| = 2^n$ and $n \geq 2$, that G is solvable, hence H is solvable too].

By Proposition 1.1 it follows that Sylow 2-subgroups of the B^* -group G are either generalized quaternion, cyclic, or elementary abelian. Suppose firstly that the Sylow 2-subgroups of G are generalized quaternion. Then the Brauer-Suzuki-Wall-Glauberman Theorem ([8], Ch. 12, Th. 11 and page 462) implies that H admits a chief factor C that is isomorphic to the simple group $PSL(2, t)$, where t is an odd prime power greater than 3 or it is isomorphic to the alternating group A_7 ; notice that the same holds for a non-abelian (simple) chief factor of \overline{H} . The group A_7 is ruled out by ([10], Theorem 5.12) as it appears then that $p = 3$, so that $3 \nmid |C| = |A_7|$ yields a contradiction. Thus $C \cong PSL(2, t)$ remains. Then ([10], Theorem 5.13) together with $p \nmid |\overline{H}|$ yields the following. Either the last member \overline{H}^∞ of the derived series of \overline{H} is isomorphic to $SL(2, 5)$ with $p = 11, 19, 29, 59$, or $\overline{H} \cong SL(2, 13)$ with $p = 3$ and $n = 6$, or $SL(n, p) \leq \overline{H} \leq \Gamma L(n, p)$. Note that $\overline{H}^\infty = (HU)^\infty = H^\infty$. The case $\overline{H} \cong SL(2, 13)$ with $n = 6$ and $p = 3$ is ruled out by the fact that the 360 1-dimensional subspaces of $V(6, 3)$ cannot be transitively permuted by \overline{H} , as $9 \nmid |SL(2, 13)|$. Now by ([11], II, 6.10 Satz), $SL(n, p)$ is equal to its own commutator subgroup unless $n = 2$ and $p \leq 3$. So if $SL(n, p) \leq \overline{H} \leq \Gamma L(n, p)$,

either p divides $|SL(n, p)|$ (hence $p \mid |[\overline{H}, \overline{H}]| = |[H, H]|$ producing a contradiction to $p \nmid |H|$) whenever $n \geq 3$ or $n = 2$ together with $p \geq 5$, or else \overline{H} is a subgroup of one of the solvable groups $\Gamma L(2, 2)$, $\Gamma L(2, 3)$ or $\Gamma L(1, p)$, violating the nonsolvability of H . *Suppose now that the Sylow 2-subgroups of G are elementary abelian.* Then Walter's theorem, as mentioned in [23] implies that a non-abelian simple group whose Sylow 2-subgroups are elementary abelian, is either isomorphic to $SL(2, 2^m)$ with $m \geq 2$, or to $PSL(2, t)$ with t a prime power satisfying $t \equiv 3$ or $5 \pmod{8}$, but $t \neq 3$, or to a simple group R of Ree type in the sense of Thompson and Ree (see also the comments in the proof of Theorem 4.6 regarding simple groups of Ree type). Therefore, consider a non-solvable chief section C of \overline{H} . By Theorems 2.3 and 4.2, C is simple. If $C \cong PSL(2, t)$, we argue similarly as in the quaternionic case above. If $C \cong SL(2, 2^m)$ with $m \geq 2$, then from ([10], Theorem 5.13) we get that either $H^\infty = \overline{H}^\infty \cong SL(2, 5)$ with $p = 3$ and $n = 4$ or $p = 11, 19, 29, 59$ with $n = 2$ for these four primes, or $\overline{H} \cong SL(2, 13)$ with $p = 3$ and $n = 6$ (but this case is ruled out in the same way as we did above), or $SL(n, p) \leq \overline{H} \leq \Gamma L(n, p)$ holds. In the last case, consider \overline{H}^∞ . If $n \geq 3$ or if $n \geq 2$ and $p \geq 5$, then $SL(n, p) = [SL(n, p), SL(n, p)] \leq \overline{H}^\infty = H^\infty$. In this case p divides $|SL(n, p)|$ so $p \mid |H|$, contrary to $p \nmid |H|$ by assumption. On the other hand, if $n = 2$ and $p \leq 3$, then \overline{H} is solvable, contrary to our assumption that \overline{H} is not solvable. Next, consider $C \cong J_1$. This case does not occur, due to ([10], Theorem 5.14). Finally, let $C \cong R$. In Proposition 1.1 it was shown that a Sylow t -subgroup T of a B^* -group equals either a cyclic group or else $Exp(T) = t$, whenever t is an odd prime. Any Sylow 3-subgroup of R , however, is not abelian and its Exponent is at least 9; see [20]. Hence $C \cong R$ does not occur. *Suppose finally that a Sylow 2-subgroup of G is cyclic.* Then ([11], IV, 2.8 Satz) in conjunction with the Theorem of Feit and Thompson yields the solvability of G , contrary to the assumption that \overline{H} is not solvable. *Now we recall our hypothesis that \overline{H} is non-solvable.* Henceforth, assume \overline{H} is solvable. *Assume also from now on, that p is an arbitrary prime number.* Then we are able to invoke [10], Corollary 5.6, due to Huppert, resulting in the fact that either $SL(n, p) \leq \overline{H} \leq \Gamma L(n, p)$, or else that in Hering's notation that \overline{H} satisfies property (IV), meaning that \overline{H} contains a normal subgroup E isomorphic to an extraspecial group of order 2^{n+1} with $C_{\overline{H}}(E) = Z(\overline{H})$, where $\overline{H}/Z(\overline{H})$ is faithfully represented on $E/Z(E)$ and either $n = 2$ with $p \in \{3, 5, 7, 11, 23\}$ or else $n = 4$ with $p = 3$. In our B^* -group G situation, both these possibilities for H can be sharpened as follows. Because \overline{H} is solvable, the assumption $SL(n, p) \leq \overline{H} \leq \Gamma L(n, p)$ implies that $n = 2$ with $p \leq 3$,

or else $n = 1$, yielding $H \leq \overline{H} \leq \Gamma L(1, p)$. If $n = 2$ and $p = 2$ should hold, then $SL(2, 2) \leq \overline{H} \leq H$, contrary to $6 = |SL(2, 2)|$ and $2 \nmid |\overline{H}|$. If $n = 2$ and $p = 3$ would be the case, we have $SL(2, 3) \leq \overline{H} = HU$; observe $|U| = 2$ holds here and that $3 \nmid |H|$ now by assumption. So, as $|SL(2, 3)| = 24$, we have a contradiction. Hence indeed $n = 1$ follows. Finally, as Sylow 2-subgroups of \overline{H} are isomorphic to factor group of Sylow 2-subgroups of the B^* -group G , we observe E is isomorphic to a quaternion group of order 8 (i.e. $n = 2$) in the case where H satisfies property (IV) of Hering; see above.

In the proof of this theorem we began by considering $H \cong G/C_G(M)$ and we have now shown that the structure of $\overline{H} = HU$ is very restricted for any p . Thus, the question arises: what can be said about a preliminary structure for H itself. When \overline{H} is not solvable, then we have shown above that all structures that has been derived for \overline{H} hold for H , too. When \overline{H} is solvable and p is an arbitrary prime, then we have proved above, that either $H \leq \overline{H} \leq \Gamma L(1, p)$ or else that \overline{H} satisfies Hering's property (IV) with E quaternion of order 8 and with $p \in \{3, 5, 7, 11, 23\}$. As E and H are normalized by \overline{H} , it follows that $E \cap H$ is normalized by \overline{H} . In particular, $E \cap H$ is a normal subgroup of E . As $H \leq HE \leq HU$, it follows now that $HE/H \cong E/E \cap H$ is cyclic. So, $E \cap H$ must contain $[E, E] = Z(E)$. Thus $|E/E \cap H| = 1$ or 2 . Therefore, let us assume that the order of $E \cap H$ equals 4; thus observe that $E \cap H$ as a subgroup of a quaternion group is cyclic. As $\overline{H} = HU$, $E \cap H$ is a normal subgroup of H . Thus there would exist a chain $G \triangleright B \triangleright C_G(M)$ with $E \cap H \cong B/C_G(M)$. We know that non-abelian Sylow 2-subgroups of a B^* -group are generalized quaternion and that each of them is generated by elements of order 4. Hence G/B would be of odd order, so that we have arrived at a contradiction. Therefore it follows that $E = E \cap H$, i.e. $E \leq H$. In fact, $E \triangleleft H$ holds because H normalizes E , as we saw above. As said before, a quaternion Sylow 2-subgroup of G is generated by its elements of order 4. This implies that $E/Z(E)$ is a chief section of H , whence of G , of order 4. The proof of Proposition 2.1 is complete, writing in (b) C instead of $Z(E)$ and N instead of E . \square

Conversely, if $G/C_G(M)$ is a group from (b) in that list F occurring in Proposition 2.1, then $C_G(M)$ is *solvable*. [Indeed, assume $C_G(M)$ admits a non-solvable chief factor E/R with $E \trianglelefteq G, R \trianglelefteq G$. Because no Sylow 2-subgroup of E is cyclic (by [11], IV.2.8 Satz, and the theorem of Feit and Thompson [7]) and as E does not contain elements of order 4 (otherwise any Sylow 2-subgroup of G , being generalized quaternion now, would

be contained in $C_G(M)$; see Proposition 1.3 and $2 \nmid |G/C_G(M)|$ which is not the case), any Sylow 2-subgroup of E and of E/R has to be elementary abelian of order at least 4. Then, however, any Sylow 2-subgroup of G would be elementary abelian, contradicting $G/C_G(M) \in F(b)$. Now observe, that if $G/C_G(M) \in F(b)$ that there exists an abelian subgroup $U/C_G(M)$ with $U \trianglelefteq G$ in such a way that $U/C_G(M)$ contains an elementary abelian 2-subgroup of order at least 4 for which G/U is solvable. In total, it follows now, that a B^* -group G with $G/C_G(M) \in F(b)$ and $|M| = p^2$ ($p \in \{3, 5, 7, 11, 23\}$) must be solvable.

Next, let us once look again at the following situation. Suppose that G is a non-solvable B^* -group. Suppose that there exists an elementary abelian minimal normal subgroup M of G of order p^t with $t \geq 2$, p prime. Then by Proposition 1.3 and the Feit-Thompson theorem, p is odd. Assume $G/C_G(M)$ does not belong to $F(c)$. Then, by the reasoning provided above, it must be that $G/C_G(M)$ belongs to $F(a)$, whence $G/C_G(M)$ is solvable in that case. Therefore $C_G(M)$ is *not* solvable. It holds too that $C_G(M) = KM$ with $K \leq C_G(M)$, $(|K|, |M|) = 1$, $[K, M] = \{1\}$; remember $M \in \text{Syl}_p(G)$ by Proposition 1.3.

We collect some of the above properties in the following portmanteau theorem.

Theorem 2.2 *Let M be an elementary abelian normal subgroup of a B^* -group G . Assume $|M| = p^t$ with $t \geq 2$ and p prime. Then precisely one of the following statements for the group G is fulfilled.*

1. $G/C_G(M)$ is isomorphic to a (solvable) subgroup of $\Gamma L(1, p^t)$;
2. $t = 2$, $p \in \{3, 5, 7, 11, 23\}$, G is solvable, and $G/C_G(M)$ admits a 2-group of order 4 as chief factor;
3. The last member $(G/C_G(M))^\infty$ of the derived series of $G/C_G(M)$ is isomorphic to $SL(2, 5)$, where $t = 2$ and $p \in \{11, 19, 29, 59\}$.

Remark As we shall prove in §3, any solvable B^* -group is B -group. Therefore, in a B^* -group G satisfying case 2) of the Theorem 2.2, it also holds that $p = 5$ or that $p = 11$; see Theorems 8 and 10 of [21].

Let us look at non-solvable chief factors of the B^* -groups.

Theorem 2.3 *Let $\{1\} = N_0 \trianglelefteq N_1 \trianglelefteq \dots \trianglelefteq N_k \trianglelefteq N_{k+1} \trianglelefteq \dots \trianglelefteq N_m = G$ be a chief series of the B^* -group G . Then G is solvable or there exists precisely one index $k \in \{0, 1, \dots, m-1\}$ such that N_{k+1}/N_k is non-solvable.*

Proof. Assume the existence of at least one index $t \in \{0, 1, \dots, m-1\}$ such that N_{t+1}/N_t is non-solvable [otherwise G is solvable]. So G is not solvable. We have $4 \mid |N_{t+1}/N_t|$, by Burnside's theorem ([11], IV.2.8 Satz) and the Feit-Thompson theorem [7]. Let $P \in \text{Syl}_2(G)$. Since P is not cyclic, we have $P = \langle a \in P \mid |a| = 4 \rangle$ or P is elementary abelian; see ([11], I.14.9 Satz). In both these cases $P \leq N_{t+1}$ follows. Thus $2 \nmid |G/N_{t+1}|$, yielding (by Feit-Thompson) the solvability of all the factors N_{j+1}/N_j whenever $j \geq t+1$. From Lemma 1.6, we learned that $|P \cap N_t| \leq 2$. Hence Burnside's theorem ([11], IV.2.8 Satz) gives that N_t , whence also each factor N_{r+1}/N_r for $r \in \{0, 1, \dots, t-1\}$ in case $t \geq 1$, is solvable. Hence the proof of the Theorem is done, also if $t = 0$ features. \square

The following Theorem will turn out to be very useful.

Theorem 2.4 *Suppose M/N is a non-solvable chief factor of the B^* -group G . Then there exists a subgroup L of G such that $M = NL$, $L = L'$, $L/Z(L) \cong M/N$, $N \cap L = Z(L)$.*

Proof. Let $\{1\} = N_0 \trianglelefteq N_1 \trianglelefteq \dots \trianglelefteq N_r = G$ be a chief series of G passing through N and M . Then there exists $k \in \{0, 1, \dots, r-1\}$ such that $N = N_k$ and $M = N_{k+1}$. Consider the set $\tau = \{K \leq G \mid KN = M, K' = K\}$. Since M/N is not solvable, $(M/N)^u \neq \{1\}$, where M^u (for some specific integer u) represents the last term in the derived series of M ; note that $(M/N)^u = M^u N/N$. Since M/N is a chief factor, $M^u N = M$ follows. Hence τ is not empty as $\{1\} \neq M^u \in \tau$. Let now L be an element of τ of smallest order; so indeed $L \neq \{1\}$. We will show that $L \cap N_k = Z(L)$. Since N_k is solvable by Lemma 1.6, we see already that $L \neq L \cap N_k$.

Assume there exists an index $i \leq k$ such that $(L \cap N_i)/(L \cap N_{i-1})$ in the series $L \supset L \cap N_k \supseteq \dots \supseteq L \cap N_1 \supseteq \{1\}$ is a non-cyclic elementary abelian p -subgroup for some prime p . [Note that $(L \cap N_i)/(L \cap N_{i-1})$ is isomorphic to a subgroup of the solvable (elementary abelian) chief factor N_i/N_{i-1}]. As $p^2 \mid |N_i/N_{i-1}|$, it follows from Lemma 1.6 that N_i/N_{i-1} is isomorphic to a Sylow p -subgroup of G . So $p \nmid |N_{k+1}/N_k|$. Therefore any Sylow p -subgroup of L is in fact, contained N_i whence contained in $L \cap N_i$. Thus an application of the Schur-Zassenhaus theorem yields that, due to $(|L/(L \cap N_i)|, |(L \cap N_i)/(L \cap N_{i-1})|) = 1$, there exists $S/(L \cap N_{i-1}) \leq L/(L \cap N_{i-1})$ such that $L/(L \cap N_{i-1}) = (S/(L \cap N_{i-1}))(L \cap N_i)/(L \cap N_{i-1})$ and $S \cap L \cap N_i = L \cap N_i$.

N_{i-1} . So $|S| < |L|$ and $S(L \cap N_i) = L$. It follows that $N_{k+1} = N_k L = N_k(L \cap N_i)S = N_k S$. Furthermore, the last member S^v in the derived series of S satisfies $S^v N_k = N_{k+1}$ (with $(S^v)' = S^v$); note that $M/N = (M/N)' = M'N/N = (SN)'N/N = S'N/N$, yielding $S'N = M$, etc. Thus $S^v \in \tau$, a contradiction to the choice of $L \in \tau$ with $|S^v| < |L|$.

Therefore, it must be that $(L \cap N_i)/(L \cap N_{i-1})$ is cyclic for any $i \in \{1, \dots, k\}$. Then it follows from Lemma 1.6 that every Sylow subgroup of $L \cap N_k$ is cyclic. [Indeed, 4 divides the order of the non-solvable group M/N . Suppose $P \in \text{Syl}_2(M)$ is not abelian. Then P is generalized quaternion, generated by its elements of order 4. So N does not contain elements of order 4, due to $G \in B^*$. But then any $P \in \text{Syl}_2(N)$ is cyclic of order 2. Again, as $G \in B^*$, $4 \mid |M/N|$, no Sylow 2-subgroup of N can be elementary abelian of order at least 4. Now let $Q \in \text{Syl}_p(M)$ with p odd prime. Assume $\text{Exp}(Q) = p$. Suppose there exists $a \in (L \cap N_i)/(L \cap N_{i-1})$ of order p and assume $b \in L \cap N_{i-1}$ of order p . As $G \in B^*$, $a \in L \cap N_{i-1}$ follows, a contradiction. Hence by Proposition 1.1, we are done]. Now let $P \in \text{Syl}_p(L \cap N_k)$. The Frattini argument ([11], I.7.8 Satz) gives $L = N_L(P)(L \cap N_k)$. Hence $N_{k+1} = LN_k = N_L(P)(L \cap N_k)N_k = N_L(P)N_k$. Furthermore, $N_{k+1}/N_k = (N_{k+1}/N_k)' = (N_L(P)N_k/N_k)' = (N_L(P)N_k)'N_k/N_k = N_L(P)'N_k/N_k$, yielding $N_{k+1} = N_L(P)'N_k$. This construction provides $N_{k+1} = \overline{N_L(P)}N_k$, where $\overline{N_L(P)}$ is the last term in the derived series of $N_L(P)$. Since $L \in \tau$, $L \supseteq \overline{N_L(P)}$, and $\overline{N_L(P)} \in \tau$, we conclude that $L = \overline{N_L(P)}$, yielding $[L, P] \leq P$. Since now $P \trianglelefteq L$, $C_L(P) \trianglelefteq L$ follows with $L/C_L(P)$ isomorphic to a subgroup of $\text{Aut}(P)$. As P is cyclic, $\text{Aut}(P)$ is abelian. Hence we get $L = C_L(P)$ from $L = L'$. Therefore L centralizes each Sylow subgroup of $L \cap N_k$. So $L \cap N_k \leq Z(L)$. As $L/(L \cap N_k) \cong LN_k/N_k = N_{k+1}/N_k$, N_{k+1}/N_k does not contain an abelian non-trivial characteristic proper subgroup. Thus we see that $Z(L)N_k = N_k$, providing at last $L \cap N_k = Z(L)$. The Theorem has been proved. \square

Theorem 2.5 *Let G be a non-solvable B^* -group and let $L \leq G$ be as in (the proof of) the Theorem 2.4. Then $|Z(L)| \leq 2$.*

Proof. Suppose the prime p divides $|Z(L)|$. We have $N \cap L = Z(L) = Z(L) \cap L = Z(L) \cap L' \leq \Phi(L)$; see ([11], III.3.12 Satz). Hence ([11], III.3.8 Satz) gives, that p divides $|L/Z(L)|$. Now ([13], (5.6) Theorem) maintains that (any) $P \in \text{Syl}_p(L)$ is not abelian. Assume p is odd. Then by Proposition 1.1, $\text{Exp}(P) = p$ holds. Then Proposition 1.2 yields that any Sylow p -subgroup of G is contained in N , contradicting that p divides

$|L/Z(L)| = |L/L \cap N|$. Hence $Z(L)$ is a 2-group. Any $P \in \text{Syl}_2(L)$ here is not abelian, whence P , being a subgroup of a non-abelian Sylow 2-subgroup S of G , is generalized quaternion due to Proposition 1.1. If $|Z(L)| \geq 4$, then, as S is generated by its elements of order 4, $G \in B^*$ reveals that $S \leq N$, i.e. $2 \nmid |L/L \cap N| = |L/Z(L)|$, a contradiction to the fact that $2 \mid |L/Z(L)|$, as we saw above. Hence indeed, $|Z(L)| \leq 2$ holds. \square

Theorem 2.6 *Let M/N be a non-solvable chief factor of the B^* -group G . Then there exists $L \leq M$ with $L/Z(L) \cong M/N$, $L = L'$ and $L \cap N = Z(L)$. Moreover, if $Z(L) \leq Z(G)$, then $L \trianglelefteq G$.*

Proof. The first part of the Theorem has been shown in the proof of Theorem 2.4. So assume $Z(L) \leq Z(G)$. Consider a chief series of G going through M and N . If $N = \{1\}$, we are done. Thus, assume $N > \{1\}$. Let $N \geq S \geq T$ be such that S/T is a chief factor of G . We will show that $L \leq C_G(S/T)$. The group S/T is solvable (whence an elementary abelian p -group) by Feit-Thompson. Consider S/T as an additive t -dimensional vector space V over \mathbb{F}_p . If $t = 1$, then $L/C_L(V)$, as subgroup of $\text{Aut}(C_p)$, is abelian. Hence $L = C_L(V)$ by $L = L'$. Assume $t \geq 2$. Then $G/C_G(V)$ acts transitively on the set of all d -dimensional \mathbb{F}_p -subspaces of V , for any $1 \leq d < t$; note here that S/T is isomorphic to an elementary abelian Sylow p -subgroup of G . Now observe, that $L/C_L(V) = L/(L \cap C_G(V)) \cong LC_G(V)/C_G(V) \leq G/C_G(V)$. If $L/C_L(V)$ is solvable, then $L = C_L(V)$ by $L = L'$. Thus assume that $L/C_L(V)$ is not solvable. Then the previous observations about the list F in Proposition 2.1 yields $G/C_G(V) \hookrightarrow GL(2, p)$, with p odd prime, where $t = 2$ holds now. Even better, as $L = L'$, $L/C_L(V) = (L/C_L(V))'$ implies that $L/C_L(V) \hookrightarrow SL(2, p)$. Assume $L \neq C_L(V)$. Then, as $C_L(V) \geq Z(L)$ (by $Z(L) \leq Z(G)$), $L/C_L(V)$ is isomorphic to a direct product of isomorphic non-abelian simple groups, due to $L/Z(L) \cong M/N$. On the other hand, the Sylow 2-subgroups of $SL(2, p)$ are generalized quaternion. Any non-abelian simple group contains Klein four groups as subgroups. Therefore we have a contradictory embedding $L/C_L(V) \hookrightarrow SL(2, p)$.

Hence indeed, $L \leq C_G(S/T)$ for all chief sections S/T of G underneath N . Next we are going to show that $[L, N] = \{1\}$. If done, we need a little additional argument to conclude that $L \trianglelefteq G$.

Let $T \in \text{Syl}_2(L)$. Thus $T \cap N \in \text{Syl}_2(N)$. Then it follows from Lemma 1.6 that $|T \cap N| \leq 2$. So Burnside's theorem gives $N = O_{2'}(N)(T \cap N)$. It follows from the above

arguments that T centralizes all the factors $(O_{2'}(N) \cap S)/(O_{2'}(N) \cap T)$ for any chief factor S/T of G underneath N . Since

$$(|T|, |O_{2'}(N)|) = 1,$$

it therefore follows that $[T, O_{2'}(N)] = \{1\}$. Now, as M/N is a chief factor of G , M/N is generated by the Sylow 2-subgroups of M/N ; likewise $L/Z(L)$ is generated by its Sylow 2-subgroups. Hence L itself is generated by its Sylow 2-subgroups, as $|Z(L)| \leq 2$. We have seen above that $Z(L) \in \text{Syl}_2(N)$. So, as $Z(L) \leq Z(G)$, the Schur-Zassenhaus theorem gives that $N = Z(L)O_{2'}(N)$ with $[Z(L), O_{2'}(N)] = \{1\}$. Since any Sylow 2-subgroup of L centralizes $O_{2'}(N)$, we now get $[L, N] = \{1\}$.

Thus we see that $M = LO_{2'}(N)$ with $[L, O_{2'}(N)] = \{1\}$ and $L \cap O_{2'}(N) = \{1\}$ (as $L \cap O_{2'}(N) \leq L \cap N = Z(L)$ with $|Z(L)| = 2$).

Now look at the derived series of M . We obtain $M' = (LO_{2'}(N))' = L'O_{2'}(N)'$ (where the second equality -sign happens to be the crux), whence, as $O_{2'}(N)$ is solvable, we get after f suitable steps, by invoking $L = L'$, that the f^{th} - derived group M^f of M is equal to L itself! Hence indeed $L = M^f \trianglelefteq G$, as was to be shown. \square

4. Solvable B^* -groups are B -groups

In this section we will show that the set of the solvable B^* -groups coincides with the set of the solvable B -groups. The theory and the properties of the solvable B -groups have been determined in [2], [16] and [21].

The structure of the Sylow subgroups of the solvable B^* -groups can be sharpened in comparison to the general case as stated in Proposition 1.1.

Lemma 3.1 *Let G be a solvable B^* -group.*

a) *If p is an odd prime, then any Sylow p -subgroup of G is abelian. In particular, such a Sylow group is cyclic or elementary abelian.*

b) *If G contains a generalized quaternion Sylow 2-subgroup Q , then $|Q| = 8$.*

Proof. a) Let $P \neq \{1\}$ be a Sylow p -subgroup of G , where $p \neq 2$. Suppose P is not cyclic. Then $\text{Exp}(P) = p$ by Proposition 1.1. There exists a chief factor M/N of G such that p does not divide $|N|$, whereas M/N is an elementary abelian p -group. As $G \in B^*$, $P \leq M$ follows. So $P \cong M/N$.

b) Suppose Q is a generalized quaternion Sylow 2-subgroup of G of order at least 16. Then $O_{2'}(G)Q \trianglelefteq G$, due to [8], Ch.12, Theorem 1.1 and Proposition 1.2. Hence $O_{2'}(G)[Q, Q] \trianglelefteq G$. The group $[Q, Q]$ is cyclic. We have $|[Q, Q]| \geq 4$, so, as $G \in B^*$ and $Q = \langle a \in Q \mid |a| = 4 \rangle$, we get $Q \leq O_{2'}(G)[Q, Q]$, which is absurd. Thus $|Q| = 8$ follows. \square

Contrary to the general case, quotient groups of solvable B^* -groups are also B^* -groups.

Theorem 3.2 *Any quotient group of a solvable B^* -group is a B^* -group.*

Proof. Let G be a solvable B^* -group and assume G is a counterexample of minimal order to the Theorem. Thus there exists $\{1\} \neq N \trianglelefteq G$ of smallest order for which G/N is not a B^* -group. Let N/K be a chief factor of G . Then, if $K \neq \{1\}$, we have $G/K \in B^*$ by the choice of N . Furthermore, if $K \neq \{1\}$, the solvable group G/K is not a counterexample of minimal order to the theorem, whence $(G/K)/(N/K) \in B^*$. As $G/N \cong (G/K)/(N/K)$ we get a contradiction for the choice of N . Thus we must have $K = \{1\}$, i.e. N is an elementary abelian p -group. By Lemma 1.3 we must have $p = 2$ as $G/N \notin B^*$.

Now let X, Y be subgroups of G satisfying $N \leq X \cap Y$, X/N and Y/N both abelian of the same order. If 2 does not divide $|X/N|$ (and $|Y/N|$), then the Schur-Zassenhaus theorem reveals the existence of the subgroups X_1, Y_1 of G satisfying $X = NX_1, Y = NY_1, N \cap X_1 = \{1\} = N \cap Y_1$. The groups X_1 and Y_1 , being abelian are conjugate in G , as $|X_1| = |Y_1|$. So X/N and Y/N are conjugate in G . So we assume $2 \mid |X/N|$. Then Lemma 1.3 together with the propositions 1.1 and 1.2 yield $|N| = 2$. So in the same vein a Sylow 2-subgroup of X is either cyclic of order at least 4, or quaternion of order 8. Let $P \in Syl_2(X)$. Since G is solvable, there exists $Q \in Hall_{2'}(X)$. The structure of X reveals easily that $X = PQ$ with $[P, Q] = \{1\}$. Likewise, $Y = P_1Q_1$ with $[P_1, Q_1] = \{1\}$ where $P_1 \in Syl_2(Y)$ and $Q_1 \in Hall_{2'}(Y)$. Hence Q and Q_1 are both abelian. If both P and P_1 are abelian, then X and Y are abelian, whence X and Y are conjugate in G , yielding the conjugacy of X/N and Y/N within G . Thus it remains to assume that $P \in Syl_2(X)$ is quaternion of order 8. Hence $P \in Syl_2(G)$, whence a Sylow 2-subgroup P_1 of Y is also quaternion of order 8. The groups Q and Q_1 are both abelian of equal order. Then there exists $g \in G$ with $Q^g = Q_1$. Now $P^g \in Syl_2(C_G(Q_1))$ holds. Thus there exists $h \in C_G(Q_1)$ with $(P^g)^h = P_1$. Hence $X^{(gh)} = (PQ)^{(gh)} = (P^g)^h (Q^g)^h = P_1Q_1^h = P_1Q_1 = Y$. Therefore X and Y are conju-

gate in G , thereby establishing the conjugacy of the abelian groups X/N and Y/N . So $G/N \in B^*$ anyway. Hence G is not a counterexample to the Theorem. The statement of the theorem is true. \square

Lemma 3.3 *Let G be a solvable B^* -group. If P is a non-cyclic elementary abelian Sylow subgroup of G , then P is contained in the Fitting subgroup $F(G)$ of G .*

Proof. Let M be a minimal normal subgroup of G . Hence $M \leq F(G)$. If $P \cap M \neq \{1\}$, then $P \leq M \leq F(G)$ by Proposition 1.2. So assume $P \cap M = \{1\}$. Then, as $G/M \in B^*$ by Theorem 3.2, we have by induction that the Sylow subgroup PM/M of G/M is contained in $F(G/M)$. Also, we can assume $P \cap F(G) = \{1\}$, as otherwise $P \leq F(G)$ follows immediately from Proposition 1.2. Hence $(|P|, |F(G)|) = 1$. Since $F(G)/M \leq F(G/M)$, $[P, F(G)] \leq M$ now follows with $F(G)P \trianglelefteq G$. If $C_P(M) \neq \{1\}$, then as $C_G(M) \trianglelefteq G$, Proposition 1.2 yields $P \leq C_G(M)$. In that case, $[F(G), P, P] = \{1\}$ together with $(|F(G)|, |M|) = 1$ yields $[F(G), P] = \{1\}$. So $P \leq C_G(F(G))$. But G is solvable, so ([11], III.4.2 Satz) tells us that $C_G(F(G)) \leq F(G)$, so that we get $P \leq F(G)$, a contradiction to $P \cap F(G) = \{1\}$. Thus $C_P(M) = \{1\}$ holds. Now $G/C_G(M)$ acts irreducibly and faithfully on M . Since $PM/M \trianglelefteq G/M$ (and $PM/M \in \text{Syl}_p(G/M)$ and as $C_G(M) \geq M$, we see that $PC_G(M)/C_G(M) \trianglelefteq G/C_G(M)$, that $PC_G(M)/C_G(M) \in \text{Syl}_2(G/C_G(M))$ and that $PC_G(M)/C_G(M) \cong P$ (by $C_P(M) = \{1\}$). Hence M can be regarded as a homogeneous $\mathbb{F}_p[PC_G(M)/C_G(M)]$ -module; note that G acts transitively on the set of all the t -dimensional subspaces of M , for any $t \in \{1, \dots, \dim_{\mathbb{F}_p}(M)\}$. Then ([17], 0.5 Lemma) forces $PC_G(M)/C_G(M)$ to be cyclic, a contradiction. The Lemma has been proved. \square

Lemma 3.4 *Let G be a solvable B^* -group. Let $P \in \text{Syl}_2(G)$. If P is quaternion group of order 8, then $P \leq F(G)$ or else $P \cap F(G) = \{1\}$ with $P \leq F_2(G)$.*

Proof. Since P is quaternion of order 8, it holds that $F_2(G) > F(G)$; otherwise we get $F(G) = G$, so that G is a nilpotent B^* -group. This is not possible by Theorem 1.5. By Theorem 3.3, $O_q(F_2(G)/F(G))$ is cyclic for any odd prime q dividing $|F(G)/F(G)|$. When $X \leq G$, put $\bar{X} = XF(G)/F(G)$. It follows that $\bar{G}/C_{\bar{G}}(O_q(F(\bar{G})))$ is cyclic. Hence $\bar{P}C_{\bar{G}}(O_q(F(\bar{G}))) / C_{\bar{G}}(O_q(F(\bar{G})))$ is cyclic. Hence $C_{\bar{P}}(O_q(F(\bar{G})))$ is of

order at least 4. Since $\overline{G}/O_q(F(\overline{G}))$ is a solvable B^* -group, it now follows, that $P \leq H$ where $H/F(G)$ represents $C_{\overline{G}}(O_q(F(\overline{G})))$. So $PF(G)/F(G)$ centralizes $O_q(F(\overline{G}))$ for all q dividing $|F(G/F(G))|$. If $2 \nmid |F(G/F(G))|$, we get $PF(G) \leq F_2(G)$ from induction. So, if 4 divides $|F(G/F(G))|$, then $P \leq F_2(G)$ as $G/F(G) \in B^*$. Thus assume $2 \parallel |F(G/F(G))|$. Then $PF(G)/F(G)$ centralizes $O_2(F(\overline{G}))$ as well, yielding $PF(G) \leq F_2(G)$, i.e. $P \leq F_2(G)$. Thus $P \leq F_2(G)$ holds always. If $4 \mid |F(G)|$, then $P \leq F(G)$ follows from $G \in B^*$. Thus assume $2 \parallel |F(G)|$. Now consider an elementary abelian non-trivial subgroup $O_r(F(G))$ for some odd prime r , when it exists. [Indeed, if $F(G)$ is a 2-group with $P^* := P \setminus \{1\}$, such that $P^* \in F_2(G) \setminus F(G)$, then, by Proposition 1.1, $|F(G)| = 2$, which is not the case.] So $O_r(F(G))$ is either cyclic of order r (and then $|P/C_P(M)| \leq 2$, implying $P \leq C_G(M)$ as P is generated by its elements of order 4), or otherwise $M \in Syl_r(F(G))$, meaning that $F(G) \leq C_G(M)$. So as $Z(P) \leq C_G(M)$, $PC_G(M)/C_G(M)$ is an abelian normal subgroup of $G/C_G(M)$; remember $F(G)P \trianglelefteq G$ as $P \leq F_2(G)$. So M can be viewed as an irreducible $\mathbb{F}_p[G/C_G(M)]$ -module. As $G \in B^*$, it is a *homogeneous* $\mathbb{F}_p[PC_G(M)/C_G(M)]$ -module, whence $PC_G(M)/C_G(M)$ is cyclic; see again ([17], 0.5 Lemma). Thus $|P \cap C_G(M)| \geq 4$, i.e. $P \leq C_G(M)$. Hence P centralizes $O_{2'}(F(G))Z(P) = F(G)$. So, by ([11], III.4.2 Satz), $P \leq F(G)$. Resuming: if $|F_2(G)/F(G)|$ is divisible by an odd prime number, then $P \leq F(G)$ or $P \leq (F_2(G) \setminus F(G)) \cup \{1\}$. The case $\{\overline{1}\} \neq F_2(G)/F(G) \leq O_2(G/F(G))$ runs in a like way as 4 always divides $|F(G/F(G))|$ now. We always have $P \leq F_2(G)$, and the hypothetical case $|P \cap F(G)| = 2$ leads to a contradiction in the same way as described before. The proof of the Lemma is complete. \square

Now we will prove the main result of this section.

Theorem 3.5 *Every solvable B^* -group is a B -group.*

Proof. Suppose there exists a non-trivial normal subgroup N of G with $N \leq X \cap Y$, where X and Y are subgroups of G of equal order. Then X/N and Y/N , being subgroups of equal order of the B^* -group G/N , are conjugate in G/N by mathematical induction. Hence X and Y are conjugate in G .

Now let M be a non-trivial normal subgroup of G . So M is an elementary abelian p -group for some prime p . If M is cyclic and p divides $|X|$, then $M \leq X \cap Y$ as M is the unique subgroup of order p in G ; remember $G \in B^*$, where Proposition 1.1 holds. Thus, in this case we also get that X and Y are conjugate in G . Assume again M is cyclic, but

this time assume $p \nmid |X|$ and $p \nmid |Y|$. Then XM/M and YM/M are subgroups of G/M of equal order; hence by induction $X^g M = YM$ for a certain $g \in G$. Then X^g and Y are all *Hall* p' -subgroups of the solvable group YM , hence conjugate in YM . Hence X and Y are conjugate in G here too.

Therefore, X and Y are always conjugate in G , *unless* for each non-cyclic elementary abelian minimal normal subgroup N of G it holds that $|X \cap N| = |Y \cap N| \neq 1$; note that $N \in \text{Syl}_p(G)$ for a certain prime p , by Propositions 1.1 and 1.2. In the latter case we see that $F(G)$ is an abelian *Hall* π -subgroup of G for some set π of prime numbers. Hence $F(G) \cap X$ and $F(G) \cap Y$ are normal *Hall* π -subgroups of X and Y respectively. So $|F(G) \cap X| = |F(G) \cap Y|$. Now $G \in B^*$, so by replacing X by a conjugate in G if necessary, we may assume that $F(G) \cap X = F(G) \cap Y = T$, say. Now $N_G(T) = F(G)L$ for some $L \leq N_G(T)$ with $(|L|, |F(G)|) = 1$, by the Schur-Zassenhaus theorem. Note that $X \leq N_G(T)$ and that $Y \leq N_G(T)$. Again by the Schur-Zassenhaus theorem $(F(G) \cap X)X_1 = X$ and $(F(G) \cap Y)Y_1 = Y$ for certain $X_1, Y_1 \leq N_G(T)$. Now assume that Sylow 2-subgroups of L are cyclic. The other Sylow p -subgroups of L are indeed cyclic for all $p \neq 2$, by the foregoing arguments combined with Lemma 1.6. Hence the *Hall* π' -subgroup L of G is a subgroup closed B -group, by ([9], Corollary to Theorem 3). Applying Hall's generalization of the Sylow theorems for solvable groups, we get that X_1 and Y_1 , being of equal order, are conjugate in $N_G(T)$. Hence X and Y are conjugate in $N_G(T)$.

Thus we may assume by Lemmas 3.1 and 3.3, that a Sylow 2-subgroup of L is quaternion of order 8. Hence $F(G)$ is abelian of odd order and also $F(G)P \trianglelefteq G$, where $P \in \text{Syl}_2(G)$; remember Lemma 3.4. Furthermore $F(G)Z(P) \trianglelefteq G$ follows. Remember that $C_G(F(G)) \leq F(G)$. Therefore Zassenhaus's theorem, see ([11], III,13.4 Satz) gives $F(G) = [F(G), F(G)Z(Q)]U$, with $U \cap [F(G), F(G)Z(Q)] = \{1\}$, where U represents

$$\{a \in F(G) \mid tat^{-1} = a, \text{ for all } t \in F(G)Z(Q)\}.$$

As $[F(G), F(G)Z(Q)] \trianglelefteq G$, we see, that $F(G)Z(Q)/F(G)$ centralizes

$$F(G)/[F(G), F(G)Z(Q)] = F(G)/[F(G), F(G)Z(Q)],$$

which is not possible when $U \neq \{1\}$; see ([11], III.4.2 Satz). Therefore $U = \{1\}$. Thus the unique involution of $G/F(G)$ acts on $F(G)$ by inverting each element of $F(G)$. Now,

analogous reasoning as in the proof of ([21], Theorem 3) shows that $O_r(G)$ is elementary abelian of order r^t , where $2 \leq r \leq 3$, whenever $r \mid |F(G)|$; remember that now $|F(G)|$ is odd, $p^2 \mid |F(G)|$ for any $p \mid |F(G)|$, and all Sylow subgroups of $F(G)$ are *de facto* Sylow subgroups of G . Furthermore, $\{1\} \neq O_r(G) \cap X = O_r(G) \cap Y \neq O_r(G)$ as we saw earlier. We know that $L \leq N_G(T)$ contains a quaternion subgroup Q (of order 8) and that $(|L|, |F(G)|) = 1$. Thus we can view $O_r(G)$ as a completely reducible $F_p[LF(G)/F(G)]$ -module. Therefore in each possibility $|O_r(G)| = r^2$ or $|O_r(G)| = r^3$, we see that there exists $A < O_r(G)$ with $|A| = r$ satisfying $[Q, A] \leq [L, A] \leq A$. Now $L/C_L(A)$ embeds in the cyclic group $\text{Aut}(A)$. Hence $Q/C_Q(A)$ is cyclic, that is, the involution of Q acts trivially on $O_r(F(G))$. This is a contradiction to our knowledge obtained above, namely that it should invert each element of $O_r(F(G))$. The Theorem has been proved. \square

5. The classification of the non-solvable B^* -groups

In this section we shall determine the structure of the non-solvable B^* -groups.

Theorem 4.1 *Let G be a non-solvable B^* -group. Then each chief series of G admits precisely one non-solvable chief factor, say M/N . Moreover, the group M/N is simple.*

Proof. The first assertion is proved in Theorem 2.3. It holds that N is a group of odd order by Propositions 1.1 and 1.2. So, by Lemma 1.3, we can assume that $N = \{1\}$ without loss of generality. Assume M is not simple. The group M is equal to a direct product of isomorphic copies S_1, \dots, S_t of a non-abelian simple group S , i.e. $M = S_1 S_2 \dots S_t$ with $[S_i, S_j] = \{1\}$ for any $1 \leq i < j \leq t$. Furthermore, by ([11], I.9.12 Satz), it holds that G acts transitively on the set of subgroups $\{S_1, S_2, \dots, S_t\}$ by conjugation. Let $y \in S_1$ and $w \in S_2$ be elements of equal order. So the groups $\langle y \rangle$ and $\langle yw \rangle$, being of equal order, are not conjugate in G . This is in conflict with $G \in B^*$. Therefore M is a simple group. \square

Theorem 4.2 *Let G be a B^* -group. Assume that G admits a non-solvable chief factor isomorphic to the Janko group J_1 or to the group $PSL(2, q)$, q odd, $q \neq 5$.*

Then $G = MH$ with $[M, H] = \{1\}$, where either

$M = J_1$; or

$M = PSL(2, q)$ with $q = p^f$, p prime, $f = 1$ or $f = 3$, $q \equiv 3$ or $5 \pmod{8}$, $q \nmid 15$; or
 $M = SL(2, q)$ with $q = p^f$, p prime, $f = 1$ or $f = 3$, $q \nmid 15$.

In each of these three types for M , H is a solvable B -group, whose order is relatively prime to the order of the corresponding M . Conversely, any of the groups $M \times H$ of the types mentioned above, constitutes a non-solvable B^* -group.

Proof. Let K/U be a non-abelian chief factor of G of the type indicated in the hypothesis of the theorem. Then by Theorem 2.5, there exists $L \leq G$ with $L' = L$, $L/Z(L) \cong K/U$, $UL = K$, $|Z(L)| \leq 2$. We assume firstly that G contains a solvable minimal normal subgroup $\neq \{1\}$; see next a) and b).

a) Assume $Z(L) \leq Z(G)$. Then $L \trianglelefteq G$ by Theorem 2.6. Thus $LC_G(L) \trianglelefteq G$ and so $L/Z(L) = L/(C_G(L) \cap L) \cong LC_G(L)/C_G(L)$; hence $LC_G(L)/C_G(L)$ is non-abelian simple and isomorphic to K/U . The group L does not contain cyclic Sylow 2-subgroups, by Burnside's theorem ([11], IV.2.8 Satz) and $L/Z(L)$ does not contain generalized quaternion nor cyclic Sylow 2-subgroups. Suppose G has a minimal normal subgroup M which is an elementary abelian p -group of order p^t with $t \geq 2$. Then $p \neq 2$. [If $p = 2$ would hold, then we would have $M \in Syl_2(G)$ with $|G/M|$ is not 0 (mod 2), whence G is solvable which is not true.] Then $G/M \in B^*$, by Lemma 1.3. Thus by mathematical induction, $G/M = (ML/M)(K/M)$, with $ML/M \cap K/M = M/M$, where both ML/M and K/M are normal in G/M ; note that now $(|ML/M|, |K/M|) = 1$, where K/M is a solvable B^* -group. It follows from Theorem 2.2 that $G/C_G(M) \notin F(b)$ and note that $G/C_G(M) \notin F(c)$, whence $M \leq Z(ML)$; see Theorem 2.3 and note that $p \nmid |G/M|$. So ML does contain, by the Schur-Zassenhaus theorem, a characteristic subgroup \bar{L} isomorphic to L . Therefore $G = LK$ with $(|L|, |K|) = 1$, $L \cap K = \{1\}$, $L \trianglelefteq G$, $K \trianglelefteq G$, K a solvable B -group, where L is of the shape as stated, by mathematical induction. So perhaps G does contain precisely one solvable minimal normal subgroup M . By the foregoing argument, it follows that $|M| = p$, where p is prime. Suppose firstly that $p \geq 3$. So any $S \in Syl_p(G)$ is cyclic, as $G \in B^*$. Then, as indeed all Sylow p -subgroups of J_1 or $SL(2, q)$, with q odd, have to be cyclic, p does not divide the Schur multiplier of any of these groups. Hence for this prime p , $p \nmid |J_1|$ and $p \nmid |SL(2, q)|$. Therefore ML splits centrally over M . Henceforth, by a similar argument as before, we conclude that the Theorem holds. Now assume that G contains a unique solvable minimal normal subgroup M , this time of order 2. If $L \trianglelefteq G$ would be simple, then $M \cap L = \{1\}$. Hence ML , whence also the B^* -group G , should possess a non-abelian Sylow 2-subgroup that is not generalized quaternion nor elementary abelian group; this is not the case! Assume now

that L is *not* simple, i.e. $\{1\} \neq Z(L) \leq Z(G)$ yields $Z(L) = M$ is of order 2. Consider $C_G(L)$ and assume that $C_G(L) \neq Z(L)$. Let u be an odd prime dividing $|C_G(L)/Z(L)|$. Then $u \nmid |L/Z(L)|$, because otherwise, on one hand a Sylow u -subgroup of $LC_G(L)$ is not cyclic, whereas on the other hand such a Sylow u -subgroup being of Exponent u now, should be completely contained in the normal subgroup L of the B^* -group G ; indeed this is an impossibility. Suppose $u = 2$. Then a Sylow 2-subgroup of $LC_G(L)$ is not abelian, and it turns out not to be a generalized quaternion 2-group, either! This contradicts the fact that G has only generalized quaternion groups as Sylow 2-subgroups. Therefore $Z(L) = C_G(L)$. Thus we have the existence of the chain $G \supseteq L \supseteq Z(L) = C_G(L) \supseteq \{1\}$ with $|Z(L)| = 2$. Since $G \in B^*$ and as $Z(L)$ is the unique normal solvable minimal subgroup of G , it follows from the (independent)(!) Theorem 4.5, that $G = L$. Then again Theorem 4.5 provides the result.

b) *Now assume* that $Z(L) \not\leq Z(G)$. Then $|Z(L)| = 2$. Again we look at a minimal normal subgroup M of G . If $|M| = 2^t$ with $t \geq 2$, then G admits M as its unique Sylow 2-subgroup, yielding the solvability of G which is not the case. Suppose M is an elementary abelian p -group with p odd and of order p^t with $t \geq 2$. Then $G/M \in B^*$. Since now G is non-solvable and $G/C_G(M) \in F(a)$ by the Theorems 2.2 and 2.3, we obtain by mathematical induction that $G/M = LM/M.K/M$ with LM/M quasi-simple and $LM/M \trianglelefteq G/M$ with the required structure, K/M being a solvable B -group with $(|LM/M|, |K/M|) = 1$ and $LM \leq C_G(M)$. Argue then further as under a), and we obtain the wanted result. A similar trick as under a) for the case $|M| = p$, p odd prime, is also applicable here, thereby providing the desired result, too. Now assume $|M| = 2$. Remember that L is not normal in G , by $|Z(L)| = 2$ and $Z(L) \not\leq Z(G)$. Hence $ML \cong C_2 \times C_2$ as $Z(L) \not\leq Z(G)$. Since G is a B^* -group, the elements of order 2 in $MZ(L)$ have to be conjugate in G , an impossibility by $M \trianglelefteq G$ with $|M| = 2$.

c) *Therefore it remains to investigate the case* where the B^* -group G contains only minimal normal subgroups M that are non-solvable. Hence, by ([11], I.9.12 Satz), such an M is isomorphic to $S \times \dots \times S$ (r - times), where S is a non-abelian simple group. Fix such an M . We have seen in Theorem 4.1, that $r = 1$, i.e., that M is non-abelian simple. Since $G \in B^*$, it follows that $2 \nmid |G/M|$, hence that G/M is solvable. Therefore the Jordan-Hölder theorem ([11], I.11.5 Satz) tells us, that $M \cong PSL(2, q)$ with q odd, $q \neq 5$, or that $M \cong J_1$. Consider $C_G(M)$. We have $M \cap C_G(M) = \{1\}$, as $C_G(M) \trianglelefteq G$. All Sylow 2-subgroups of G are *de facto* contained in our M , so $2 \nmid |C_G(M)|$. Thus $C_G(M) \trianglelefteq G$ is solvable. By the condition made in c), it therefore follows immediately

that $C_G(M) = \{1\}$. From Theorem 4.5 we observe that for $G \in B^*$, it must hold that $G = M$ and that for $PSL(2, q)$ the prime power q is of the form $q = p^f$ with p prime and f dividing 3. So we have shown the truth of the statements of the Theorem. The statements about the converse situation in the Theorem will be shown in Theorem 4.5 too. \square

Theorem 4.3 *Let G be a non-solvable B^* -group. Then either G is a (non-solvable) B -group or else G admits essentially one non-abelian chief factor, where that factor happens to be either isomorphic to the Janko simple group J_1 or to some simple group $PSL(2, q)$ with $q = p^f$, p odd prime and $f \mid 3, q \nmid 15$.*

Remark In Theorem 4.5 we will see that the simple groups $PSL(2, q)$ mentioned in Theorem 4.3 are all B^* -groups themselves if and only if $q \equiv 3$ or $5 \pmod{8}$; the group J_1 is a B^* -group too.

Proof of Theorem 4.3 Suppose G is a counterexample of minimal order. Let M be a minimal normal subgroup of G . Then either M is a direct product of isomorphic non-abelian simple groups or else M is elementary abelian. We split up:

a) Let $M \cong S \times \dots \times S$, S non-abelian simple. Then we have seen in Theorem 4.1, that $M \cong S$ is simple. It then follows from the (independent!) Theorem 4.6 that

$$M \in \{SL(2, 4); SL(2, 8); SL(2, 32); PSL(2, q),$$

$$\text{where } q > 3, q = p^f, p \text{ odd prime, } q \equiv 3 \text{ or } 5 \pmod{8}; J_1\}.$$

In the case $M \cong PSL(2, q)$, we refer the reader to the remarks at the end of the proof of this Theorem 4.3 in order to deduce that $f \mid 3$. Note that in all these possibilities for M any Sylow 2-subgroup of M is elementary abelian of order at least 4, implying (due to Propositions 1.1 and 1.2) that any Sylow 2-subgroup of G is contained in M . Hence G/M is solvable of odd order.

a.1) Assume $M \cong SL(2, 4) \cong A_5$. Then $G/C_G(M) \leq \text{Aut}(M) \cong S_5$. Now $MC_G(M)/C_G(M) \cong M \cong A_5$. Observe that $C_G(M) (\cong MC_G(M)/M)$ is of odd order, as $C_G(M)$ embeds in G/M . If $|C_G(M)| > 1$, then Lemma 1.3 yields that $G/C_G(M)$ is a B^* -group. Hence any Sylow 2-subgroup of $G/C_G(M)$ must be elementary abelian, as A_5 has already elementary abelian Sylow 2-subgroups of order 4; use Proposition 1.1. Even better now,

all involutions have to be contained in $MC_G(M)/M$, by Proposition 1.2. Hence $G \not\cong S_5$, so that $G = MC_G(M)$ with $M \cap C_G(M) = \{1\}$ yielding $[M, C_G(M)] = \{1\}$. Since $G \in B^*$, we see from Lemma 1.4, that also $M \in B^*$, $C_G(M) \in B^*$, $(|M|, |C_G(M)|) = 1$. Now A_5 is a B -group, (see [2], Lemma 6). Hence, Theorem 3.5 yields $C_G(M) \in B$ now, as $C_G(M)$ is solvable of odd order. Hence indeed $G \in B$ by ([2], Lemma 3).

a.2) Assume $M \cong SL(2, 8)$. Argue in the same spirit as in a.1). We know that $Aut(SL(2, 8))$ splits over $Inn(SL(2, 8))$, where

$$|Aut(SL(2, 8))/Inn(SL(2, 8))| = 3.$$

The Sylow 3-subgroups of $Aut(SL(2, 8))$ are non-cyclic of Exponent 9. Hence $Aut(SL(2, 8)) \notin B^*$. This means that $G = MC_G(M)$ with $M \cong SL(2, 8)$, $M \cap C_G(M) = \{1\}$. Hence $G \in B^*$, $[M, C_G(M)] = \{1\}$ and so $(|M|, |C_G(M)|) = 1$. Just by ([2], Lemma 6) we know that $SL(2, 8) \in B$. Again $C_G(M) \in B^*$ by Lemma 1.4. As $C_G(M)$ is solvable (of odd order), $C_G(M) \in B$ follows from Theorem 3.5. Therefore $G \in B$ by ([2], Lemma 3).

a.3) Assume $M \cong SL(2, 32)$. Consider $G/C_G(M) \hookrightarrow Aut(SL(2, 32)) \cong \langle SL(2, 32), \alpha \rangle$, where α is a field automorphism of F_{32} , of order 5, acting as such on the coefficients of the matrices in $SL(2, 32)$. We know from ([2], Lemma 6) that $SL(2, 32) \notin B^*$ but that $Aut(SL(2, 32)) \in B$. Now notice that here $(|M|, |C_G(M)|) = 1$ by Proposition 1.2. Also $5 \nmid |M|$. Hence $(|G/M|, |M|) = 1$. Thus the Schur-Zassenhaus theorem provides a subgroup T of G with $G = MT$, $T \cap M = \{1\}$. There exists a 5-element $\beta \in T$ such that $[\langle \beta \rangle, SL(2, 32)] \neq \{1\}$, just as $|G : MC_G(M)| = 5$. Since β acts “like a field automorphism on the coefficients of the matrices of $SL(2, 32)$ ”, it must be that $\beta^5 \in C_G(M)$. As $\langle C_G(M), \beta \rangle \leq T$ (all Sylow p -subgroups of $C_G(M)$ with $p \neq 5$ are Sylow p -subgroups of G and of T), it follows that $\langle C_G(M), \beta \rangle = T$. Therefore $G = (M \times C_G(M)) \langle \beta \rangle$ with $\beta^5 \in C_G(M)$. Let $H_i M/M$ with $i \in \{1, 2\}$, be abelian subgroups of equal order of G/M , where of course $H_i \leq G$ holds. Since $(|H_i/(H_i \cap M)|, |H_i \cap M|) = 1$, there exists by the Schur-Zassenhaus theorem, $L_i \leq G$ with $|L_i| = |H_i/(H_i \cap M)|$, satisfying $H_i = L_i(H_i \cap M)$. Therefore the L_1 and L_2 are abelian of equal order, whence conjugate in G . Therefore, as for $i = 1, 2$ $H_i M/M = L_i(H_i \cap M)M/M = L_i M/M$, we see that $H_1 M/M$ and $H_2 M/M$ are conjugate within G/M . Then $G/M \in B^*$, whence also $T \in B^*$. As $2 \nmid |T|$, T is solvable; whence $T \in B$ by Theorem 3.5. In ([2], Theorem 7'') it is indicated that our G is then also a B -group itself. The case a) is done.

b) Let $M \neq \{1\}$ be a solvable minimal normal subgroup of G . Then M is an

elementary abelian p -group for some prime p . We split up:

b.1) *Suppose $p \neq 2$.* Then G/M is a non-solvable B^* -group, by Lemma 1.3. Since G is a minimal counterexample to the Theorem, G/M is non-solvable B -group (such a strategem holds for any $T \trianglelefteq G$ with $T \neq \{1\}, |T| = \text{odd}$; we can assume $G/T \in B$). Look at the essentially unique non-abelian chief factor of G/M . Then, by ([2], Theorems 5 and 10), that factor is isomorphic to $SL(2, 4)$, to $SL(2, 8)$, to $SL(2, 32)$, or to $PSL(2, 5)$ (note that $SL(2, 4) \cong PSL(2, 5) \cong A_5$, the alternating group on five symbols). Suppose firstly that Sylow 2-subgroups of G are elementary abelian. Then ([2], Theorem 9) reveals the existence of $N \trianglelefteq G$ with $N/M \cong SL(2, 4), SL(2, 8)$ or $SL(2, 32)$, just by $G/M \in B$. By Theorem 2.4, $N = ML$ for some $L \leq G$ with $L' = L$ and $L/Z(L) \cong N/M, Z(L) = L \cap M$. By Theorem 2.4, it holds that $|Z(L)| \leq 2$. Assume $Z(L) \leq Z(G)$. Then $L \trianglelefteq G$ (by Theorem 2.6). If $Z(L) = \{1\}$, then we are in case a): we get $G \in B$. So, assume $|Z(L)| = 2$. This violates $Z(L) = L \cap M$ being of odd order. Now, as $G/M \in B$, we apparently have that Sylow 2-subgroups of G/M are not abelian, so that they are quaternion of order 8. This means, that the (essentially unique) non-abelian chief factor of G is isomorphic to $PSL(2, 5) (\cong SL(2, 4))$. Now look at the proof of ([2], Theorem 10). We work here in the situation where *any* $G/T \in B$, if $|T| = \text{odd} > 1$. A proper reading of the proof of ([2], Theorem 10) (of course, one has to be careful in handling B^* -situations and B -situations !) does instruct you, as it did to us, that G is *not* a counterexample to the Theorem, *unless* G does not contain *precisely one* minimal normal non-trivial subgroup whereas that subgroup happens to be of order 2! This brings us immediately to part b.2) of the proof.

b.2) *It remains to investigate the case* where G has only one minimal normal subgroup M , where moreover M happens to be an elementary abelian 2-group. Now, if $|M| \geq 4$, then M is the unique, whence normal, Sylow 2-subgroup of G ; so G/M is solvable of odd order. But then G is solvable, which is not the case. Hence we must have $|M| = 2$. It follows now from Propositions 1.1 and 1.2, in conjunction with ([11], IV.2.8 Satz) that the Sylow 2-subgroups of G are generalized quaternion. Thus by ([19], Ch.6, Theorems 8.6 and 8.7) a non-abelian chief factor $\overline{N}/\overline{M}$ of G exists satisfying the property that $\overline{M} = M$ with $\overline{N}/M \cong PSL(2, q)$ (q odd prime power with $q > 3$) or \overline{N}/M is isomorphic to the alternating group A_7 . Now if $\overline{N}/M \cong A_7$, it holds that $Aut(A_7) \cong S_7$, whence $G \notin B^*$ in that case [indeed, there are now (at least) two conjugacy classes of subgroups of order 6 inside \overline{N} which are *not* conjugate in G]. Therefore $\overline{N}/M \cong PSL(2, q)$ holds. Next observe, that $\overline{N} = \overline{N}'$ and that M , being of order 2, equals $\overline{N}' \cap Z(\overline{N})$. If $q \neq 9$, then

the Schur multiplier of our $PSL(2, q)$ has order 2, yielding the fact that $\overline{N} \cong SL(2, q)$; see ([11], V.25.7 Satz). Assume $q = 9$. Then $PSL(2, 9) \cong A_6$, the alternating group on 6 symbols. As $G \in B^*$, it holds that 2 does not divide $|G/\overline{N}|$. Note that here $|Aut(\overline{N}/M) : \overline{N}/M| = 4$; see ([5], Pages 4 and 5). Hence $G/M = (\overline{N}/M)C_{G/M}(\overline{N}/M)$ follows clearly. So as the group A_6 contains (at least) two conjugacy classes of subgroups of order 3, we conclude that G is *not* a B^* -group in this case too. Therefore, we land into the case where $\overline{N} = SL(2, q)$, with $q = p^f$, p odd prime, $q > 3$. Rewrite $\overline{N} = S$.

We will investigate now what it means to have $G \in B^*$, if $S \trianglelefteq G$ with $S \cong SL(2, q)$, $q = p^f$, p odd prime $q > 3$, *without assuming* per sè that $Z(S)$ is the *unique* minimal normal subgroup of G . We will prove that f equals at most 3 and that $f \neq 2$. Look at the group $K/Z(S) := C_{G/Z(S)}(S/Z(S))$ (thus $[K, S] \leq Z(S), |Z(S)| = 2, K \trianglelefteq G$). Suppose an odd prime t divides $|S/Z(S)|$ and $|K/Z(S)|$. Then $P_t \in Syl_t(G)$ is not cyclic; whence $Exp(P_t) = t$. Also, any element of P_t is now contained in S . Hence $t \nmid |K/Z(S)|$, a contradiction. Any Sylow 2-subgroup of G is generalized quaternion (hence generated by its elements of order 4); see Propositions 1.1 and 1.2. So $2 \nmid |K/Z(S)|$, as $G \in B^*$. Hence it is clear that $SK = SV$, whence $(|S|, |V|) = 1, S \cap V = \{1\}, [S, V] = \{1\}$. Thus V is characteristic in SK , whence normal in G . Since $2 \nmid |V|$, it follows that $G/V \in B^*$. Therefore we focus our attention on the case where $V = \{1\}$ and the corresponding subgroup $K/Z(S)$ is trivial: $K/Z(S) = Z(S)/Z(S) = \{\overline{1}\}$. In the same way we see that it follows that $C_G(S) = Z(S)$. By ([19], Ch.6, §8) it holds then that altogether $G/S \cong \overline{G}/\overline{S}$ induces field automorphisms on S , i.e. $|G/S|$ divides f . Now we will concentrate on the groups $\overline{G} = G/Z(S)$ and $\overline{S} = S/Z(S)$. All the Sylow p -subgroups of S are elementary abelian of order p^f ; they are all conjugate to each other in G . Assume $f \geq 2$. Therefore, since $G \in B^*$, p does not divide $|G/S|$. Look at $\mathcal{P} := \{R \mid R \leq G \text{ and } |R| = p^2\}$. By Burnside's theorem ([11], IV.2.5 Hilfssatz) it follows now that all elements of \mathcal{P} which are contained in a given Sylow p -subgroup P of G (and of S) are conjugate within $N_G(P)$. Hence, for $R \in \mathcal{P}$ with $R \leq P$ it holds that

$$\#\{R \mid P \in \mathcal{P} \text{ with } R \leq P\} = \frac{p^f - 1}{p^2 - 1} \frac{p^{f-1} - 1}{p - 1} = |N_G(P) : N_{N_G(P)}(R)|.$$

This number should also divide $|G/S| |N_G(P) : C_G(P)| = f(p^f - 1)/2$. Hence, $2(p^{f-1} - 1) \leq f(p^2 - 1)(p - 1) < fp^3$. If $p \geq 7$ and $f \geq 5$, we get that $fp^3 < p^{f-4} \cdot p^3 = p^{f-1}$, a contradiction to the inequality of integers just found. If $p = 5$ and $f \geq 5$, we find already that $5^{f-1} - 1 \leq f \cdot 24 \cdot 4 = 96 \cdot f$, which does not hold when $f \geq 5$. And if

$p = 3$, we see that $3^{f-1} - 1 \leq f \cdot 8 \cdot 3 = 24 \cdot f$ is false when $f \geq 6$. If $p = 3$ and $f = 5$, we get that $[(3^5 - 1)/(3^2 - 1)] \cdot [(3^4 - 1)/(3 - 1)]$ does not divide $5 \cdot (3^5 - 1)/2$. The case $f = 4$ can be disposed of easily for all odd primes p , by the same strategy. The case $f = 2$ gives rise to $G = S$. As G is a B^* -group by assumption, all subgroups U of order p of S are conjugate in S , that is $|S : N_S(U)| = p + 1$ (we know that there are precisely $(p + 1)$ subgroups of order p in S , anyway). The “actual” value of $|S : N_S(U)|$, where $U = \left\{ \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} : x \in \mathbb{F}_p \right\}$, however, equals $(p + 1)/2$, as a straightforward computation shows. So $G \cong SL(2, p^2)$ is not a B^* -group when p is an odd prime! Therefore, *inside* this proof of Theorem 4.3, we wish to stress the result just obtained, as a separate property.

Theorem 4.4 *Let G be a B^* -group and assume there exists $S \trianglelefteq G$ with $S \cong SL(2, q)$, $q = p^f$, $q > 3$, p odd prime. Then f divides 3.*

End of proof of Theorem 4.3 Now, we will elucidate here a structure that was left over from the beginning of the proof of Theorem 4.3. Namely, look at the case $M \cong PSL(2, q)$, $M \trianglelefteq G$, $G \in B^*$, with $q = p^f$, p odd prime, $q > 3$. By means of the independent Theorem 4.6, we also derive that $q \equiv 3$ or $5 \pmod{8}$. As it turns out f will divide 3 in that case, as well. In order not to annoy the reader, we omit the details of the proof of this last assertion. Namely halfway that “adapted” proof to the corresponding way of proving Theorem 4.4 as shown above, the reader will observe that the structure of $Aut(PSL(2, q))/Inn(PSL(2, q))$ can be retrieved from the structure of $Aut(SL(2, q))/Inn(SL(2, q))$. For a more precise statement of this remark, see for instance ([1], Theorem 1.3), which is a modern reference. With these remarks given, the proof of Theorem 4.3 is finished.

Now we have reached the point where the statements of the next Theorem 4.5 (closed on its own account!) make the proof of the Theorem 4.2 totally complete, and where it reflects on Theorem 4.3. In fact, it contributes so the full classification of the (non-solvable) B^* -groups; see the Main Theorem after the proof of Theorem 4.6. \square

Theorem 4.5 *Suppose S is a non-solvable normal subgroup of a group G , where either $S \cong SL(2, s)$ with s an odd prime or the third power of an odd prime, or $S \cong PSL(2, q)$ with $q = p^f$, $q \equiv 3$ or $5 \pmod{8}$, $q > 3$, p prime, f dividing 3, or $S \cong J_1$, the simple first Janko group.*

Assume $C_G(S) = Z(S)$ in any of the before mentioned cases. Then G is a B^* -group if and only if $G = S$.

Proof. a) Assume G is a B^* -group. Suppose $S \cong PSL(2, q)$ where q satisfies the given numerical conditions. Then $G \cong (G/C_G(S))$ is isomorphic to a subgroup of $Aut(S) = P\Gamma L(2, q)$ with G/S of odd order, due to ([19], Ch.6, Th. 8.10). So $|G/S| = 1$ or 3. If $G = S$ we are done. Thus assume $\{1\} \neq R \in Syl_3(G)$ with $|G/S| = 3$. As 3 divides $|PSL(2, q)|$, we get $R \cap S \neq \{1\}$. Thus R should be cyclic by Proposition 1.2. On the other hand, a Sylow 3-subgroup of $P\Gamma L(2, p^3)$ is not cyclic, since $P\Gamma L(2, p^3)$ is a semi-direct product of $PGL(2, p^3)$ with a group of order 3. Thus G is *not* a B^* -group if $G > S$ with $S \cong PSL(2, q)$ for the indicated numerical constraints on q .

b) Assume G is a B^* -group with $S \cong SL(2, q)$ where $q = p^f, p$ odd prime, $f = 1$ or $f = 3$. Put $K/Z(S) = C_{G/Z(S)}(S/Z(S))$. Then $K \trianglelefteq G$. The group $S/Z(S)$ is simple. By Proposition 1.2, we get $(|K/S|, 2|S/Z(S)|) = 1$. Hence by the Schur-Zassenhaus theorem, there exists $U \leq K$ with $K = UZ(S)$ and $U \cap Z(S) = \{1\}$. As $U \text{ char } K, U \trianglelefteq G$ follows with $[U, S] = \{1\}$. Since it is given that $C_G(S) = Z(S)$, we have $U \leq C_G(S)$, yielding $U = \{1\}$. Thus $K = Z(S)$. Altogether we thus have the embedding chain

$$\begin{aligned} G/Z(S) / C_{G/Z(S)}(S/Z(S)) &= G/Z(S) / Z(S) / Z(S) \\ &\cong G/Z(S) \hookrightarrow Aut(S/Z(S)), \end{aligned}$$

and so $|(G/Z(S)) / (S/Z(S)) C_{G/Z(S)}(S/Z(S))| = 1$ or 3. If that value equals 1, we get $G = S$. Also, that value cannot be equal to 3. Indeed, if so, $G/Z(S)$ (hence also G) contains non-cyclic Sylow 3-subgroups (see the corresponding argument under a)) so that, due to Proposition 1.1, 3 does not divide $|G/S|$, a contradiction. So $G = S$ holds anyway.

c) Assume G is a B^* -group with $S \cong J_1$. From ([5], Page 36) we borrow that $|Aut(J_1) / Inn(J_1)| \leq 2$. Now $G (\cong G/C_G(S))$ embeds in $Aut(J_1)$. So $|G/S| \leq 2$. Since J_1 contains elementary abelian Sylow 2-subgroups of order at least 4, see [14], it follows that $|G/S| = 1$, i.e. $G = S$.

So now our task is to show that the given $SL(2, q)$, $PSL(2, q)$ and J_1 with their respective constraints on the prime powers q , are indeed B^* -groups. *Let us look firstly at J_1 .* Suppose H_1 and H_2 are two abelian subgroups of J_1 of equal order. Assume the prime p divides $|H_1|$. Since $|J_1| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$ and since J_1 contains precisely one class of involutions (by [14]), we can apply Sylow's theorem in order to see that it is of no loss

of generality to assume that $\langle H_1, H_2 \rangle \leq C_{J_1}(a)$, where $a \in H_1$ has order p . It follows from ([14], § 1) that any of the groups $C_{J_1}(b)$, where $b \in J_1$ is of odd prime order, is a B -group and that $C_{J_1}(a_2) \in B^*$ for any involution a_2 of J_1 . That is, H_1 and H_2 are conjugate within J_1 after all. Hence indeed $J_1 \in B^*$. [Observe that $J_1 \notin B$; it contains two subgroups both of order 6 which are not isomorphic; see also ([5], Page 36)]. *Now let us look at the groups $PSL(2, q)$, where q is a power of an odd prime p . From ([11], Kapitel II. §8) we recall the following properties in respect to the group $PSL(2, q)$, to be mentioned as “the list”.*

Any Sylow p -subgroup of $PSL(2, q)$ is isomorphic to the additive group of the field \mathbb{F}_q ; $PSL(2, q)$ contains cyclic subgroup of order $(q - 1)/2$ (they are all conjugate to each other) and also cyclic subgroups of order $(q + 1)/2$ (they are all conjugate to each other; *all* these beformentioned proper subgroups of $PSL(2, q)$ intersect pairwise trivially; in fact *all* these subgroups of $PSL(2, q)$ constitute a *partition* of $PSL(2, q)$, i.e. every non-identity element of $PSL(2, q)$ is contained in precisely one of all these beformentioned proper subgroups of $PSL(2, q)$; when $D \leq PSL(2, q)$ with $|D| = (q \pm 1)/2$, then for any $t \in D$ with $t \neq 1$, the normalizer $N_{PSL(2, q)}(\langle t \rangle)$ is a dihedral group of order $2|D|$).

Thus, let A be an abelian subgroup of $PSL(2, q)$. Hence, by the above, A is contained in a Sylow p -subgroup of $PSL(2, q)$ as soon as $|A|$ is divisible by p . Now *let us assume firstly* that p does *not* divide $|A|$; assume furthermore that an odd prime s does divide $|A|$. Then it follows immediately from the above that all abelian subgroups of order $|A|$ of $PSL(2, q)$ are conjugate to A in $PSL(2, q)$! In that situation, also any two abelian subgroups of $SL(2, q)$ of the same order, are conjugate in $SL(2, q)$, as they are cyclic and as they correspond one-to-one to cyclic groups of $PSL(2, q)$ of half that order. Furthermore, let now A be a 2-subgroup of $PSL(2, q)$ (or $SL(2, q)$); assume in addition, that $q \equiv 3$ or $5 \pmod{8}$ when we focus our attention to $PSL(2, q)$. Those $PSL(2, q)$ contain Klein-Fourgroups. By one of the above principles we conclude here, that any two abelian 2-subgroups of equal order of these $PSL(2, q)$ are conjugate in $PSL(2, q)$. Another condition we will add to these groups $PSL(2, q)$ is that $q = p$ or that $q = p^3$. In the case $q = p$, *it remains to consider* $A \in Syl_p(PSL(2, p))$, thereby finishing the proof of $PSL(2, q) \in B^*$ when $p = q \equiv 3$ or $5 \pmod{8}$. In the case $q = p^3$, *the following remains*. Let A be a subgroup of a Sylow p -subgroup of $SL(2, q)$ with $|A| = p$. In particular, the

group

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \mid x \in \mathbb{F}_q \right\}$$

is an elementary abelian Sylow p -subgroup of $SL(2, q)$, of order $p^3 = q$.

Put

$$A := \left\{ \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} : a \in \mathbb{F}_p \right\}.$$

We have

$$\overline{N} := N_{SL(2, q)}(S) = \left\langle S, \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix} : b \in \mathbb{F}_q^* \right\rangle.$$

The group

$$C := \left\{ \begin{bmatrix} c & 0 \\ 0 & c^{-1} \end{bmatrix} : c \in \mathbb{F}_q, c^2 \in \mathbb{F}_p^* \right\}$$

satisfies $N_{\overline{N}}(A) = \langle S, C \rangle$; note that $c^2 \in \mathbb{F}_p^*$ implies $c \in \mathbb{F}_p^*$. Now consider

$$\begin{bmatrix} d^{-1} & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} d & 0 \\ 0 & d^{-1} \end{bmatrix}$$

with $d \in \mathbb{F}_q$; this expression equals $\begin{bmatrix} 1 & 0 \\ d^2 & 1 \end{bmatrix}$. Suppose that $\begin{bmatrix} 1 & 0 \\ e^2 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ g^2 & 1 \end{bmatrix}$ ($eg \neq 0$) are contained in the same cyclic subgroup of $SL(2, q)$. Then $te^2 = g^2$ for some $t \in \mathbb{F}_p^*$; in particular, t is a square of an element $w \in \mathbb{F}_p$, as $|w| \mid p-1$. So $g = \pm we$. This means that there are precisely $(q-1)/(p-1)$ conjugate subgroups to A in $SL(2, q)$, *de facto* they represent *all* of the subgroups of order p of S . So from “the list” of properties for any $PSL(2, q)$, with odd q , we conclude that all subgroups of order p are conjugate, either contained in $PSL(2, q)$ or in $SL(2, q)$. In the case where $q = p^3$, we see that $\#\{A \mid A \leq S, |A| = p\} = p^2 + p + 1 = \#\{B \mid B \leq S, |B| = p^2\}$. Again we observe that all subgroups of order p^2 are conjugate, either contained in $PSL(2, q)$, or in $SL(2, q)$; note that $N_{SL(2, q)}(S)$ acts transitively on the set of all the subgroups of order p^2 under

conjugation, just as it does on the set of all the subgroups of order p (remember Maschke's theorem ([11], I.17.7 Satz). Therefore, we see that $PSL(2, p^3) \in B^*$ when $p \equiv 3$ or $5 \pmod{8}$). Now return to $SL(2, q)$ with $q = p^f$, p any odd prime, $f = 1$ or 3 . Let A be an abelian 2-subgroup of $SL(2, q)$. Since Sylow 2-subgroups of $SL(2, q)$ are generalized quaternion, A will be cyclic. Let C be another cyclic group of order $|A|$. This can only be done if

$$A \neq \left\{ \left[\begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array} \right], \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] \right\} =: Z.$$

Due to a property of the above "list", $A/Z \leq S_2/Z$ and $C/Z \leq \overline{S_2}/Z$, where S_2/Z and $\overline{S_2}/Z$ are distinct cyclic Sylow 2-subgroups of $SL(2, q)/Z$. Hence indeed A and C are conjugate in $SL(2, q)$.

Thus, all contingencies are accounted for. The proof of Theorem 4.5 is complete. \square

The final touch of §4 is the following Theorem, needed in the proof of Theorem 4.3.

Theorem 4.6 *Let G be a B^* -group. Suppose there exists a non-abelian simple normal subgroup M of G . Then either $M \cong SL(2, 2^f)$, where f is a prime dividing 30, or $M \cong PSL(2, q)$ for prime powers q with $q \equiv 3$ or $5 \pmod{8}$ and $q \geq 5$, or $M \cong J_1$. Each type of group does indeed occur as a normal subgroup of a B^* -group.*

Proof. Each Sylow 2-subgroup of M has at least four elements by ([11], IV.2.8 Satz). As a Sylow 2-subgroup of M is isomorphic to a subgroup of a Sylow 2-subgroup S of G , where S is generalized quaternion, we see that each of the Sylow 2-subgroups of M must be elementary abelian (use [11], IV.2.8 Satz and Proposition 1.1). Therefore, due to Walter's theorem mentioned in [23], M is a group from one of the following four classes of groups:

- 1) $PSL(2, 2^n)$, $n \geq 2$;
- 2) $PSL(2, q)$, where q is a prime power, $q > 3$, $q \equiv 3$ or $5 \pmod{8}$;
- 3) Janko's simple group J_1 ;
- 4) the simple groups of Ree type in the sense of Thompson and Ree.

Let us first tackle class 4). It has been shown by Bombieri (Inventiones Math., Vol. 58 (1980), p.77–100) that indeed Walter's theorem provides precisely the Ree-type groups that were described earlier in time as the so-called Ree-groups by Thompson and Ree;

see ([20]) and Bombieri's paper for the precise story. Therefore, by Thompson, a Sylow 3-subgroup T of such a Ree group satisfies precisely all of the properties as described in Theorem 1 of [20]. In particular, it happens that $[T, T] = \langle a \in T \mid a^3 = 1 \rangle$ and that $[T, T]$ is not cyclic, whereas of course $[T, T] < T$, yielding now $Exp(T) > 3$. Hence, in this case, G cannot be a B^* -group by Proposition 1.1.

Let us now consider the groups of class 1). As to class 1) the reader is referred to an analogue of the proof of Theorem 5 of [2]. It has to do with the fact, that all Klein four subgroups of $SL(2, 2^n)$, where $n \geq 2$, are all conjugate to each other within the B^* -group G , whereas that property can only be accomplished when $n \in \{2, 3, 5\}$. The rest of the assertion in Theorem 4.6 has been shown above in this paper.

The proof of the Theorem 4.6 is complete. □

The proof of Theorem 4.6 concludes the classification of the B^* -groups. It is provided by compiling Theorems 3.5, 4.2, 4.3, 4.4, 4.5 and 4.6, and it has been formulated in the statement of the Main Theorem as it appeared in the Introduction.

Main Theorem

a) *The class consisting of all solvable B^* -groups coincides with the class consisting of all solvable B -groups.*

b) *Every non-solvable B^* -group either is a non-solvable B -group, or else it is isomorphic to a direct product of the groups M and H , where H is any solvable B -group whose order is relatively prime to the order of M , and where M is*

either isomorphic to the B^ -group J_1 (Janko's first simple group of order 175560),*

or to any of the simple B^ -groups $PSL(2, q)$ with $q = p^f$, p odd prime, $f = 1$ or 3 , $q \geq 11$, $q \equiv 3$ or $5 \pmod{8}$,*

or isomorphic to any of the quasi-simple B^ -groups $SL(2, u)$ with $u = p^f$, p odd prime, $f = 1$ or $f = 3$; $u \geq 7$. □*

Herewith the classification of all the B^* -groups is complete.

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