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Decompositions of Continuity

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Abstract

In 2004, Al-Hawary and Al-Omari introduced and explored the class of ω^o-open sets which is strictly stronger than the class of $\omega-$open sets and weaker than that of open sets. In this paper, we introduce what we call ω^o- continuity and ω^o_X-continuity and we give several characterizations and two decompositions of ω^o-continuity. Finally, new decompositions of continuity are provided.

Key Words: ω^o-open, ω^o-continuity, Continuity.

1. Introduction

Let $(X, \mathfrak{T})$ be a topological space (or simply, a space). If $A \subseteq X$, then the closure of A and the interior of A will be denoted by $Cl_{\mathfrak{T}}(A)$ and $Int_{\mathfrak{T}}(A)$, respectively. If no ambiguity appears, we use $\overline{A}$ and A^o instead. By X, Y and Z we mean topological spaces with no separation axioms assumed. $\mathfrak{T}_{s\tan dard}$, $\mathfrak{T}_{indiscrete}$, $\mathfrak{T}_{leftray}$ and $\mathfrak{T}_{cocountable}$ stand for the standard, indiscrete, left ray and the cocountable topologies, respectively. A space $(X, \mathfrak{T})$ is *anti locally countable* if all non-empty open subsets are uncountable.

In [3], the concept of ω-closed subsets was explored where a subset A of a space $(X, \mathfrak{T})$ is ω-closed if it contains all of its condensation points. In [4], several characterizations of ω-continuity were provided where a map $f : X \to Y$ is *ω-continuous at* $x \in X$ if for every open subset V in Y containing $f(x)$, there exists an ω-open subset U in X containing x such that $f(U) \subseteq V$. f is *ω-continuous* if it is ω-continuous at every $x \in X$. Several properties of ω-continuous mappings were also explored. Analogous to [4, 5, 8, 9], in

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Section 2 we introduce the relatively new notion of ω^o-continuity, which is closely related to continuity and ω-continuity. In fact, properly placed between them. Moreover, we show that ω^o-continuity preserves Lindelof property and a space $(X, \mathfrak{T})$ is Lindelof if and only if $(X, \mathfrak{T}_{\omega^o})$ is Lindelof, where $\mathfrak{T}_{\omega^o}$ is the collection of all ω^o-open subsets of X. Sections 3 is devoted for studying four weaker notions of ω^o-continuity by which we provide two decompositions of ω^o-continuity. Finally, in Section 4 we give several decompositions of continuity which seem to be new.

Next, we recall several necessary definitions and results from [1].

Definition 1 *A subset A of a space $(X, \mathfrak{T})$ is called ω^o-*open *if for every $x \in A$, there exists an open subset $U_x \subseteq X$ containing x such that $U_x \setminus \overset{o}{A}$ is countable. The complement of an ω^o-open subset is called ω^o-*closed.

Clearly every open set is ω^o-open and every ω^o-open is ω-open.

Theorem 1 *If $(X, \mathfrak{T})$ is a space, then $(X, \mathfrak{T}_{\omega^o})$ is a space such that $\mathfrak{T} \subseteq \mathfrak{T}_{\omega^o} \subseteq \mathfrak{T}_\omega$, where $\mathfrak{T}_\omega$ is the collection of all ω-open subsets of X.*

Corollary 1 *If $(X, \mathfrak{T})$ is anti locally countable and A is ω^o-closed, then $Int_{\mathfrak{T}}(A) = Int_{\mathfrak{T}_{\omega^o}}(A)$.*

2. ω^o-Continuous Mappings

We begin this section by introducing the notion of ω^o-continuous mappings. Several characterizations of this class of mappings are also provided.

Definition 2 *A map $f : X \to Y$ is ω^o-*continuous at *$x \in X$ if for every open subset V in Y containing $f(x)$, there exists an ω^o-open subset U in X containing x such that $f(U) \subseteq V$. f is ω^o-*continuous *if it is ω^o-continuous at every $x \in X$.*

As every open set is ω^o-open and every ω^o-open set is ω-open, every continuous map is ω^o-continuous and every ω^o-continuous map is ω-continuous. The converses need not be true.

Example 1 *Let $X = \{a, b\}$, $\mathfrak{T}_1 = \{\varnothing, X, \{a\}\}$ and $\mathfrak{T}_2 = \{\varnothing, X, \{b\}\}$. Then the identity map $id : (X, \mathfrak{T}_1) \to (X, \mathfrak{T}_2)$ is ω^o-continuous but not continuous.*

Example 2 *Let* $Y = \{0,1\}$ *and* $\mathfrak{T} = \{\varnothing, Y, \{0\}\}$. *Then the map* $f : (\mathbb{R}, \mathfrak{T}_{s\tan dard}) \to (Y, \mathfrak{T})$ *defined by* $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$ *is* ω*-continuous but not* ω^o*-continuous.*

The proofs of the following three results are similar to those for ω-continuous maps given in [4] and are thus omitted.

Lemma 1 *Let* X, Y *and* Z *be spaces. Then*

(1) If $f : X \to Y$ *is* ω^o*-continuous surjection and* $g : Y \to Z$ *is continuous surjection, then* $g \circ f$ *is* ω^o*-continuous.*

(2) If $f : X \to Y$ *is* ω^o*-continuous surjection and* $A \subseteq X$, *then* $f|_A$ *is* ω^o*-continuous.*

(3) If $f : X \to Y$ *is a map such that* $X = X_1 \cup X_2$ *where* X_1 *and* X_2 *are closed and both* $f|_{X_1}$ *and* $f|_{X_2}$ *are* ω^o*-continuous, then* f *is* ω^o*-continuous.*

(4) If $f_1 : X \to X_1$ *and* $f_2 : X \to X_2$ *are maps and* $g : X \to X_1 \times X_2$ *is the map defined by* $g(x) = (f_1(x), f_2(x))$ *for all* $x \in X$, *then* g *is* ω^o*-continuous if and only if* f_1 *and* f_2 *are* ω^o*-continuous.*

Lemma 2 *For a map* $f : X \to Y$, *the following are equivalent:*

(1) f *is* ω^o*-continuous.*

(2) The inverse image of every open subset of Y *is* ω^o*-open in* X.

(3) The inverse image of every closed subset of Y *is* ω^o*-closed in* X.

(4) The inverse image of every basic open subset of Y *is* ω^o*-open in* X.

(5) The inverse image of every subbasic open subset of Y *is* ω^o*-open in* X.

Lemma 3 *A space* $(X, \mathfrak{T}_X)$ *is Lindelof if and only if* $(X, \mathfrak{T}_{\omega^o})$ *is Lindelof.*

Next we show that being Lindelof is preserved under ω^o-continuity.

Theorem 2 *If* $f : (X, \mathfrak{T}_X) \to (Y, \mathfrak{T}_Y)$ *is* ω^o*-continuous and* X *is Lindelof, then* Y *is Lindelof.*

Proof. Let $\mathfrak{B} = \{V_\alpha : \alpha \in \nabla\}$ be an open cover of Y. Since f is ω^o-continuous, $\mathfrak{A} = \{f^{-1}(V_\alpha) : \alpha \in \nabla\}$ is a cover of X by ω^o-open subsets and as X is Lindelof, by Lemma 3, $\mathfrak{A}$ has a countable subcover $\{f^{-1}(V_{\alpha_n}) : n \in \mathbb{N}\}$. Now $Y = f(X) = f(\cup\{f^{-1}(V_{\alpha_n}) : n \in \mathbb{N})\} \subseteq \cup\{V_{\alpha_n} : n \in \mathbb{N}\}$. Therefore Y is Lindelof. □

If X is a countable space, then every subset of X is ω^o-open and hence every map $f : X \to Y$ is ω^o-continuous. Next, we show that if X is uncountable such that every ω^o-continuous map $f : X \to Y$ is a constant map, then X has to be connected.

Theorem 3 *If X is uncountable space such that every ω^o-continuous map $f : X \to Y$ is a constant map, then X is connected.*

Proof. If X is disconnected, then there exists a non-empty proper subset A of X which is both open and closed. Let $Y = \{a, b\}$ and $\mathfrak{T}_Y = \{\varnothing, Y, \{b\}\}$ and $f : X \to Y$ defined by $f(A) = \{a\}$ and $f(X\backslash A) = \{b\}$. Then f is a non-constant ω^o-continuous map. □

The converse of the preceding result need not be true even when X is uncountable.

Example 3 *The identity map $id : (\mathbb{R}, \mathfrak{T}_{leftray}) \to (\mathbb{R}, \mathfrak{T}_{indiscrete})$ is a non-constant ω^o-continuous.*

3. Decompositions of ω^o-Continuity

We begin by recalling the following well-known two definitions.

Definition 3 *A map $f : X \to Y$ is* weakly continuous at $x \in X$ *if for every open subset V in Y containing $f(x)$, there exists an open subset U in X containing x such that $f(U) \subseteq \overline{V}$. f is* weakly continuous *if it is weakly continuous at every $x \in X$.*

Definition 4 *A map $f : X \to Y$ is* W^*-continuous *if for every open subset V in Y, $f^{-1}(Fr(V))$ is closed in X, where $Fr(V) = \overline{V} \backslash \overset{o}{V}$.*

Weakly continuity and W*-continuity are independent notions that are weaker than continuity and the two together characterize continuity (see for example [7]). Next we give two relatively new such definitions.

Definition 5 *A map $f : X \to Y$ is* weakly ω^o-continuous at $x \in X$ *if for every open subset V in Y containing $f(x)$, there exists an ω^o-open subset U in X containing x such that $f(U) \subseteq \overline{V}$. f is* weakly ω^o-continuous *if it is weakly ω^o-continuous at every $x \in X$.*

Clearly, every ω^o-continuous and every weakly continuous map is weakly ω^o-continuous. Non of the converses need be true as shown next.

Example 4 *Let* $Y = \{a, b, c\}$ *and* $\mathfrak{T} = \{\varnothing, Y, \{a\}, \{c\}, \{a, c\}\}$*. Then the map* $f : (\mathbb{R}, \mathfrak{T}_{cocountable}) \to (Y, \mathfrak{T})$ *defined by* $f(x) = a$ *for all* $x \in \mathbb{R}$*. Then* f *is weakly* ω^o*-continuous but not* ω^o*-continuous.*

Example 5 *Let* $Y = \{a, b, c\}$ *and* $\mathfrak{T} = \{\varnothing, Y, \{a\}, \{c\}, \{a, c\}\}$*. Then the map* $f : (\mathbb{R}, \mathfrak{T}_{cocountable}) \to (Y, \mathfrak{T})$ *defined by* $f(x) = \begin{cases} a & x \in \mathbb{Q} \\ b & x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$ *for all* $x \in \mathbb{R}$*. Then* f *is weakly continuous and hence weakly* ω^o*-continuous but not* ω^o*-continuous.*

Definition 6 *A map* $f : X \to Y$ *is* coweakly ω^o-continuous *if for every open subset* V *in* Y*,* $f^{-1}(Fr(V))$ *is* ω^o*-closed in* X*, where* $Fr(V) = \overline{V} \backslash \overset{o}{V}$ *.*

Clearly, every ω^o-continuous is coweakly ω^o-continuous. The converse need not be true.

Example 6 *Let* $X = Y = \{a, b\}$*,* $\mathfrak{T}_X = \{\varnothing, X\}$ *and* $\mathfrak{T}_Y = \{\varnothing, Y, \{a\}, \{b\}\}$ *Then the identity map* $id : X \to Y$ *is coweakly* ω^o*-continuous but not* ω^o*-continuous.*

Our first characterization of ω^o-continuity in terms of the preceding two notions of continuity is given next.

Theorem 4 *The following are equivalent for a map* $f : (X, \mathfrak{T}_X) \to (Y, \mathfrak{T}_Y)$ *:*

(1) f *is* ω^o*-continuous.*

(2) $f : (X, \mathfrak{T}_{\omega^o}) \to (Y, \mathfrak{T}_Y)$ *is continuous.*

(3) $f : (X, \mathfrak{T}_{\omega^o}) \to (Y, \mathfrak{T}_Y)$ *is weakly continuous and W*-continuous.*

Proof. (1) $\Rightarrow$ (2) : Obvious.

(2) $\Rightarrow$ (3) : Follows from Theorem 1.

(3) $\Rightarrow$ (1) : Since $f : (X, \mathfrak{T}_{\omega^o}) \to (Y, \mathfrak{T}_Y)$ is W*-continuous, it is coweakly ω^o-continuous and as it is weakly-continuous, it is weakly ω^o-continuous. Thus by Theorem 1, $f : (X, \mathfrak{T}_X) \to (Y, \mathfrak{T}_Y)$ is ω^o-continuous. □

We show that weakly ω^o-continuity and coweakly ω^o-continuity are independent notions, but together they characterize ω^o-continuity. This will be our first decomposition of ω^o-continuity which is analogous to the result that can be found in [2] for ω-continuity.

Example 7 *The map id in Example 6 is coweakly ω^o-continuous but not weakly ω^o-continuous.*

Example 8 *Let $Y = \{a, b\}$ and $\mathfrak{T} = \{\varnothing, Y, \{a\}\}$. Then the map $f : (\mathbb{R}, \mathfrak{T}_{cocountable}) \to (Y, \mathfrak{T})$ defined by $f(x) = \begin{cases} a & x \in \mathbb{Q} \\ b & x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$ for all $x \in \mathbb{R}$. Then f is weakly ω^o-continuous but not coweakly ω^o-continuous.*

Theorem 5 *A map $f : X \to Y$ is ω^o-continuous if and only if f is both weakly and coweakly ω^o-continuous.*

Proof. ω^o-continuity implies weakly and coweakly ω^o-continuity is obvious. Conversely, suppose $f : X \to Y$ is both weakly and coweakly ω^o-continuous and let $x \in X$ and V be an open subset of Y such that $f(x) \in V$. Then as f is weakly ω^o-continuous, there exists an ω^o-open subset U of X containing x such that $f(U) \subseteq \overline{V}$. Now $Fr(V) = \overline{V} \backslash V$ and hence $f(x) \notin Fr(V)$. So $x \in U \backslash f^{-1}(Fr(V))$ which is ω^o-open in X since f is coweakly ω^o-continuous. For every $y \in f(U \backslash f^{-1}(Fr(V)))$, $y = f(a)$ for some $a \in U \backslash f^{-1}(Fr(V))$ and hence $f(a) = y \in f(U) \subseteq \overline{V}$ and $y \notin Fr(V)$. Thus $f(a) = y \notin Fr(V)$ and thus $f(a) \in V$. Therefore, $f(U \backslash f^{-1}(Fr(V))) \subseteq V$ and hence f is ω^o-continuous. □

Next, we define a new class of open sets that is independent of ω-open class, but together they characterize ω^o-open.

Definition 7 *For a space $(X, \mathfrak{T})$, let $\omega_\omega^o =: \{A \subseteq X : Int_{\mathfrak{T}_{\omega^o}}(A) = Int_{\mathfrak{T}_\omega}(A)\}$. A is ω_ω^o-set if $A \in \omega_\omega^o$.*

Clearly every ω^o-open set is ω_ω^o-set, but the converse need not be true.

Example 9 *Consider $\mathbb{R}$ with the standard topology $\mathfrak{T}_{s\tan dard}$. Then $\mathbb{Q}$ is an ω_ω^o-set which is neither ω^o-open nor ω-open.*

Even an ω-open subset need not be an ω_ω^o-set.

Example 10 *Consider $\mathbb{R}$ with the standard topology $\mathfrak{T}_{s\tan dard}$. Then $\mathbb{R} \backslash \mathbb{Q}$ is an ω-open which is not an ω_ω^o-set.*

Theorem 6 *A subset A of a space X is ω^o-open if and only if A is ω-open and an ω_ω^o-set.*

Proof. Trivially every ω^o-open is ω-open and an ω^o_ω-set. Conversely, let A be an ω-open set that is ω^o_ω-set. Then $A = Int_{\mathfrak{T}_\omega}(A) = Int_{\mathfrak{T}_{\omega^o}}(A)$ and therefore A is ω^o-open. □

Definition 8 *A map $f : X \to Y$ is ω^o_ω-continuous if the inverse image of every open subset of Y is an ω^o_ω-set.*

Clearly every ω^o-continuous map is ω^o_ω-continuous, but the converse need not be true as not every ω^o_ω-set is ω^o-open. An immediate consequence of Theorem 6 is the following decomposition of ω^o-continuity.

Theorem 7 *A map $f : X \to Y$ is ω^o-continuous if and only if f is ω-continuous and ω^o_ω-continuous.*

4. Decompositions of Continuity

We begin this section by introducing the notion of an ω^o_X-set. We then introduce the notion of ω^o_X-continuity which gives an immediate decomposition of continuity.

Definition 9 *For a space $(X, \mathfrak{T})$, let $\omega^o_X =: \{A \subseteq X : Int_{\mathfrak{T}_{\omega^o}}(A) = Int_{\mathfrak{T}}(A)\}$. A is an ω^o_X-set if $A \in \omega^o_X$.*

The proof of the following result follows immediately from Corollary 1.

Corollary 2 *If $(X, \mathfrak{T})$ is anti locally countable, then ω^o_X contains all ω^o-closed subsets of X.*

We remark that, in general, an ω^o-closed set need not be an ω^o_X-set as shown in the next example.

Example 11 *Let $X = \{a, b\}$ and $\mathfrak{T} = \{\varnothing, X, \{a\}\}$. Set $A = \{b\}$. Then A is ω^o-closed but not an ω^o_X-set.*

As every open set is ω^o-open, every open set is an ω^o_X-set but the converse need not be true.

Example 12 *Consider $\mathbb{R}$ with the standard topology $\mathfrak{T}_{s\tan dard}$. Then $\mathbb{Q}$ is an ω^o_X-set which is not open.*

Next, we show that the notions of ω_X^o-set and ω^o-open are independent, but together they characterize open sets.

Example 13 *In Example 11, A is ω^o-open but not an ω_X^o-set.*

Example 14 *In Example 12, $\mathbb{Q}$ is an ω_X^o-set which is not ω^o-open.*

Theorem 8 *A subset A of a space X is open if and only if A is ω^o-open and an ω_X^o-set.*

Proof. Trivially every open set is ω^o-open and an ω_X^o-set.. Conversely, let A be an ω^o-open set that is ω_X^o-set. Then $A = Int_{\mathfrak{T}_{\omega^o}}(A) = Int_{\mathfrak{T}}(A)$ and therefore A is open. □

In a similar manner, for a space $(X, \mathfrak{T})$ let $\omega_X =: \{A \subseteq X : Int_{\mathfrak{T}_\omega}(A) = Int_{\mathfrak{T}}(A)\}$ and call a subset A is ω_X-set if $A \in \omega_X$. Then we have the following result.

Theorem 9 *A subset A of a space X is open if and only if A is ω-open and an ω_X-set.*

Definition 10 *A map $f : X \to Y$ is ω_X^o-continuous (respectively, ω_X-continuous) if the inverse image of every open subset of Y is an ω_X^o-set (respectively, ω_X-set).*

Clearly every continuous map is ω_X^o-continuous, but the converse need not be true as not every ω_X^o-set is open. An immediate consequence of Theorems 5, 7, 8 and 9 are the following decompositions of continuity, which seem to be new.

Theorem 10 *For a map $f : X \to Y$, the following are equivalent:*

(1) f is continuous.

(2) f is ω^o-continuous and ω_X^o-continuous.

(3) f is ω-continuous and ω_X^o-continuous.

(4) f is both weakly ω^o-continuous, coweakly ω^o-continuous and ω_X^o-continuous.

(5) f is ω-continuous, ω_ω^o-continuous and ω_X^o-continuous.

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