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On the Power Subgroups of the Extended Modular Group $\overline{\Gamma}$

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In [1], we proved that, if N is a non-trivial normal subgroup of $\overline{\Gamma}$ different from $\overline{\Gamma}$, Γ, Γ^2, Γ^3, then N is a free group. When we were doing this proof, we used the fact that an element of order 2 in $\overline{\Gamma}$ is conjugate to T or to R and an element of order 3 in $\overline{\Gamma}$ is conjugate to a power of S.

But while determining some low indexed normal subgroups of the extended modular group, we found two non-free normal subgroups of the extended modular group $\overline{\Gamma}$ having index 2 (except for the modular group Γ) and a non-free normal subgroup of the extended modular group having index 6 (except for the subgroup Γ^3). Also, when we were investigating conjugacy classes of finite order elements in $\overline{\Gamma}$ (see [2]), we determined a conjugacy class of reflection with representative TR, except the other conjugacy class of reflection with representative R. Thus we want to restate results related free normal subgroups of the extended modular group $\overline{\Gamma}$, specificially (the lemma 3.2, theorem 3.3 and theorem 3.4).

Before giving the main theorem we need the following lemmas.

Lemma 3.1 $\overline{\Gamma}$ *has no normal subgroups of index* 3.

Suppose $N \lhd \overline{\Gamma}$ with $\left|\overline{\Gamma} : N\right| = 3$. Let $A = \overline{\Gamma}/N$ and so $|A| = 3$ and thus A is abelian. Therefore $N \supset \overline{\Gamma}'$, which is impossible since $\left|\overline{\Gamma} : \overline{\Gamma}'\right| = 4$. □

Lemma 3.2 *There are exactly* 3 *normal subgroups of index* 2 *in* $\overline{\Gamma}$. *Explicitly these are:*

$\overline{\Gamma}_1 = \Gamma =< T, S \mid T^2 = S^3 = I >\cong C_2 * C_3$, $\overline{\Gamma}_2 =< R, S, TST \mid R^2 = S^3 = (TST)^3 = (RS)^2 = (RTST)^2 = I >\cong D_3 *_{\mathbb{Z}_2} D_3$, *and* $\overline{\Gamma}_3 =< TR, S \mid (TR)^2 = S^3 = I >\cong C_2 * C_3$.

Proof. Let $N \lhd \overline{\Gamma}$ with $\left|\overline{\Gamma} : N\right| = 2$. Since $\overline{\Gamma}/N$ is abelian we have $\overline{\Gamma} \supset N \supset \overline{\Gamma}'$.

Now $\overline{\Gamma}/\overline{\Gamma}' = C_2 \times C_2 = D_2$, a Klein 4−group. This has exactly 3 normal subgroups of index 2. Therefore these pull back to exactly 3 normal subgroups of index 2 in $\overline{\Gamma}$ containing $\overline{\Gamma}'$. Since N contains $\overline{\Gamma}'$, N must be one of $\overline{\Gamma}_1$, $\overline{\Gamma}_2$ and $\overline{\Gamma}_3$. □

Lemma 3.3 *There are exactly 2 normal subgroups of index 6 in $\overline{\Gamma}$. Explicity these are* $\Gamma^3 =< T, STS^2, S^2TS \mid T^2 = (STS^2)^2 = (S^2TS)^2 = I >\cong C_2 * C_2 * C_2$, *and* $\overline{\Gamma}_4 =< TR, RSTS, RS^2TS^2 \mid (TR)^2 = (RSTS)^2 = (RS^2TS^2)^2 = I >\cong C_2 * C_2 * C_2$.

Lemma 3.4 *Let N be a non-trivial normal subgroup of finite index in $\overline{\Gamma}$. Then N is free if and only if it contains no elements of finite order.*

Proof. Please see proof of the Lemma 3.1 in [1]. □

Lemma 3.5 *The only normal subgroups of finite index in $\overline{\Gamma}$ containing elements of finite order are*

$$\overline{\Gamma}, \overline{\Gamma}_1 = \Gamma, \overline{\Gamma}_2, \overline{\Gamma}_3, \Gamma^2,\ \Gamma^3 \text{ and } \overline{\Gamma}_4.$$

Proof. Let N be a normal subgroup of finite index in $\overline{\Gamma}$ containing an element of finite order. Then N contains an element of order 2 or an element of order 3, or two elements of order 2 or two elements of order 2 and 3, or three elements of with two elements are of order 2 and an element is of order 3. From [2], we know that an element of order 2 in $\overline{\Gamma}$ is conjugate to T or to R or to TR and an element of order 3 in $\overline{\Gamma}$ is conjugate to a power of S. Therefore if a normal subgroup N contains an element of finite order, then it contains T or R or TR or S. Therefore there are nine cases:

(i) N contains T, R and S (clearly TR). Then $N = \overline{\Gamma}$.

(ii) N contains T and S, but not R and TR. Then $N \neq \overline{\Gamma}$ and $\Gamma \subset N$, by (1) and the fact that N is normal. Since $\left|\overline{\Gamma} : \Gamma\right| = 2$, it follows that $N = \Gamma$.

(iii) N contains R and S, but not T and TR. Then $N \neq \overline{\Gamma}$ and $\overline{\Gamma}_2 \subset N$, and the fact that N is normal. Since $\left|\overline{\Gamma} : \overline{\Gamma}_2\right| = 2$, it follows that $N = \overline{\Gamma}_2$.

(iv) N contains TR and S, but not T and R. Then $N \neq \overline{\Gamma}$ and $\overline{\Gamma}_3 \subset N$, and the fact that N is normal. Since $\left|\overline{\Gamma} : \overline{\Gamma}_3\right| = 2$, it follows that $N = \overline{\Gamma}_3$.

(v) N contains T and R, but not S. This is impossible by (ii) and (iii).

(vi) N contains T but not R, TR and S. Then $N \neq \overline{\Gamma}$, $\Gamma, \overline{\Gamma}_2, \overline{\Gamma}_3$, and $\Gamma^3 \subset N$, as N is normal. Since$\left|\overline{\Gamma} : \Gamma^3\right| = 6$ and from lemma 3.3, we have $N = \Gamma^3$.

(vii) N contains S but not T and R. Then $N \neq \overline{\Gamma}$, $\Gamma, \overline{\Gamma}_2, \overline{\Gamma}_3$ and $\Gamma^2 \subset N$, by (2) and the fact that N is normal. Since $\left|\overline{\Gamma} : \Gamma^2\right| = 4$, it follows that $N = \Gamma^2$.

(viii) N contains TR but not T, R and S. Then $N \neq \overline{\Gamma}$, $\Gamma, \overline{\Gamma}_2, \overline{\Gamma}_3$, and $\overline{\Gamma}_4 \subset N$, as N is normal. Since$\left|\overline{\Gamma} : \overline{\Gamma}_4\right| = 6$ and from lemma 3.3, we have $N = \overline{\Gamma}_4$.

(ix) N contains R but not T, TR and S. This is impossible by (iii). □

Theorem 3.5 *Let N be a non-trivial normal subgroup of finite index in $\overline{\Gamma}$ different from $\overline{\Gamma}$, Γ, $\overline{\Gamma}_2$, $\overline{\Gamma}_3$, Γ^2, Γ^3 and $\overline{\Gamma}_4$. Then N is a free group.*

Proof. It can be easily seen as an immediate consequence of the lemmas. □

Theorem 3.6 *Let N be a normal subgroup of finite index in $\overline{\Gamma}$ different from $\overline{\Gamma}$, Γ, $\overline{\Gamma}_2$, $\overline{\Gamma}_3$, Γ^2, Γ^3 and $\overline{\Gamma}_4$ such that $\left|\overline{\Gamma} : N\right| = \mu < \infty$. Then μ is divisible by* 12.

Proof. Please see proof of Theorem 3.4 in [1]. □

References

[1] Sahin, R., İkikardes, S. and Koruoğlu, Ö.: On the power subgroups of the extended modular group $\overline{\Gamma}$, *Tr. J. of Math.*, 29, 143-151, (2004).

[2] Yılmaz Özgür, N. and Sahin, R.: On the extended Hecke groups $\overline{H}(\lambda_q)$, *Tr. J. of Math.*, 27, 473-480, (2003).

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