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The Restriction and the Continuity Properties of Potentials Depending on λ-distance

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Abstract

In this study we establish theorems on the restriction and continuity of the generalized Riesz potentials with the non-isotropic kernels depending on λ-distance.

Key Words: Riesz Potential, Non-Isotropic distance.

1. Introduction

It is well known that the classical Riesz Potentials $I_\alpha\varphi = \varphi * |x|^{\alpha-n}$ are bounded operators from $L_p(R^n)$ to $L_q(R^n)$ for $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n},\ 0 < \alpha < n,\ 1 \le p < q < \infty$ [1]. For these potentials, Y. Mizuta showed continuity and restriction properties [2],[3]. In this article we define the non-isotropic generalized Riesz potential generated by λ-distance and study the restriction and continuity properties of these potentials. The generalized Riesz potential generated by λ-distance is the classical Riesz potential for $\lambda_i = \frac{1}{2},\ i = 1, 2, ..., n$. Here particular importance of the non-isotropic kernel is that it doesn't have the classical triangle inequality.

2. Preliminaries

The λ-distance between points $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ is defined by the following formula given in [4]:

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$$|x-y|_\lambda := (|x_1-y_1|^{\frac{1}{\lambda_1}} + |x_2-y_2|^{\frac{1}{\lambda_2}} + ... + |x_n-y_n|^{\frac{1}{\lambda_n}})^{\frac{|\lambda|}{n}},$$

where $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$, $\lambda_k > 0$, $k = 1, 2, ..., n$, $|\lambda| = \lambda_1 + \lambda_2 + ... + \lambda_n$. Note that this distance has the following properties of homogeneity for any positive t:

$$\left(\left|t^{\lambda_1}x_1\right|^{\frac{1}{\lambda_1}} + ... + \left|t^{\lambda_n}x_n\right|^{\frac{1}{\lambda_n}}\right)^{\frac{|\lambda|}{n}} = t^{\frac{|\lambda|}{n}}\,|x|_\lambda\,,\ \ t > 0.$$

This equality gives us that the non-isotropic λ-distance has order of homogeneous function $\frac{|\lambda|}{n}$. So the non-isotropic λ-distance has the following properties:

1 $\quad |x|_\lambda = 0 \Leftrightarrow x = \theta,\ \ \theta = (0, 0, ..., 0);$

2. $\quad \left|t^\lambda x\right|_\lambda = |t|^{\frac{|\lambda|}{n}}\,|x|_\lambda;$

3. $\quad |x+y|_\lambda \leq 2^{\left(1+\frac{1}{\lambda_{\min}}\right)\frac{|\lambda|}{n}}(|x|_\lambda + |y|_\lambda).$

Here, we consider λ-spherical coordinates by the following formulas

$$x_1 = (\rho\cos\varphi_1)^{2\lambda_1}, ..., x_n = (\rho\sin\varphi_1\sin\varphi_2...\sin\varphi_{n-1})^{2\lambda_n}.$$

We obtain that $|x|_\lambda = \rho^{\frac{2|\lambda|}{n}}$. It can be seen that the Jacobian $J_\lambda(\rho,\varphi)$ of this transformation is $J_\lambda(\rho,\varphi) = \rho^{2|\lambda|-1}\Omega_\lambda(\varphi)$, where $\Omega_\lambda(\varphi)$ is the bounded function, which only depends on angles $\varphi_1, \varphi_2, ..., \varphi_{n-1}$. It is clear that, if $\lambda_i = \frac{1}{2}$, $i = 1, ..., n$, the λ-distance is Euclidean distance.

Now for $0 < \alpha < n$, we shall consider the generalized Riesz potential with the non-isotropic kernel depending on λ-distance

$$I_{\alpha,\lambda}f(x) = \int\limits_{\mathbb{R}^n} |x-y|_\lambda^{\alpha-n}\, f(y)dy, \tag{2.1}$$

where $x \in \mathbb{R}^n$. Equality (2.1) is a well-known classical Riesz potential for $\lambda_i = \frac{1}{2}$, $i = 1, ..., n$. For a positive r and any $x \in \mathbb{R}^n$, we denote the open λ-ball $B_\lambda(x,r)$ with radius r and a center x as

$$B_\lambda(x,r) = \{y : |y-x|_\lambda < r\ \}.$$

In this article we need the following Theorems given in [3].

Theorem 2.1 ***(Young's inequality):*** *Let* $1 \le p, q \le \infty$ *and* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \ge 0$. *If* $f \in L_p(\mathbb{R}^n)$ *and* $g \in L_q(\mathbb{R}^n)$, *then*

$$\|f * g\|_r \le \|f\|_p \|g\|_q .$$

Theorem 2.2 ***(Hardy's inequalities):*** *If* f *is a nonnegative measurable function on* $\mathbb{R}^+$ *and* $r > 0$, *then*

$$\left\{\int_0^\infty \left(\int_0^x f(y)dy\right)^p x^{-r-1}dx\right\}^{\frac{1}{p}} \le \frac{p}{r}\left(\int_0^\infty [yf(y)]^p y^{-r-1}dy\right)^{\frac{1}{p}}$$

and

$$\left\{\int_0^\infty \left(\int_x^\infty f(y)dy\right)^p x^{r-1}dx\right\}^{\frac{1}{p}} \le \frac{p}{r}\left(\int_0^\infty [yf(y)]^p y^{r-1}dy\right)^{\frac{1}{p}}.$$

There are various ways of proving restriction and continuity of classical Riesz potentials [3]. In this paper we study the restriction and continuity properties of generalized Riesz potentials with the non-isotropic kernel depending on λ*-distance for functions in* L_p.

3. Restriction properties

Our main aim is to give a proof of restriction of $I_{\alpha,\lambda}$.

Theorem 3.1 *Let* $0 < \frac{|\lambda|}{n\lambda_1}(\alpha - 1) < \frac{1}{p}$. *Then*

$$\left(\int_{\mathbb{R}^{n-1}} \int_{|x'-y'|_\lambda<1} \frac{|I_{\alpha,\lambda}f(0,x') - I_{\alpha,\lambda}f(0,y')|^p}{|x'-y'|_\lambda^{n-2-(\frac{|\lambda|}{n\lambda_1}(\alpha-1)+1)p}} dx'dy'\right)^{\frac{1}{p}} \le M \|f\|_p$$

where $x \in \mathbb{R}^n$ *and* $x = (x_1, ..., x_n) = (x_1, x')$, $x' = (x_2, ..., x_n)$.

In order to prove the Theorem 3.1, we need the following Lemmas.

Lemma 3.1 *Let $0 < \alpha < n$. Then there is the following inequality.*

$$\left| |x-y|_\lambda^{\alpha-n} - |y-z|_\lambda^{\alpha-n} \right| \leq Mr\, |x-y|_\lambda^{\alpha-n-1}$$

where $y \in \mathbb{R}^n - B_\lambda(x, 2r)$ and M is a constant independent of x and y.

Proof. Let $r = |x-z|_\lambda$, $|x-y|_\lambda = a$, $|y-z|_\lambda = b$ and $a \neq 0$, $b \neq 0$. Thus $0 < a - r < b < a + r$. Now we consider $f(t) = \frac{1}{t^\beta}$, where $t \in [a, b]$ (or $t \in [b, a]$), $n - \alpha = \beta > 0$. Then function $f(t)$ is continuous and continuously differentiable in $[a, b]$ (or $[b, a]$). Therefore, there is the following equality from Lagrange Theorem, that

$$|f(b) - f(a)| = \left| f^{'}(\xi) \right| |b-a| \quad \xi \in [a, b] \text{ [or } \xi \in [b, a]].$$

Here, $|b - a| < r$ we have the inequality

$$\left| \frac{1}{b^\beta} - \frac{1}{a^\beta} \right| = \left| -\beta \frac{1}{\xi^{\beta+1}} \right| |b-a| \leq \beta \left| \frac{1}{\xi^{\beta+1}} \right| r.$$

If $a < \xi < b$, then we have

$$\left| \frac{1}{b^\beta} - \frac{1}{a^\beta} \right| \leq \beta \frac{1}{a^{\beta+1}} r \leq Mr\, |x-y|_\lambda^{\alpha-n-1}.$$

If $b < \xi < a$, $\xi \in (a - r, a)$, $\xi = a - \theta r$, $0 < \theta < 1$, then we have

$$\left| \frac{1}{b^\beta} - \frac{1}{a^\beta} \right| = \beta \frac{1}{(a-\theta r)^{\beta+1}} r \leq Mr\, |x-y|_\lambda^{\alpha-n-1}.$$

The proof is completed. □

Lemma 3.2 *If $\frac{|\lambda|}{n\lambda_1}\alpha < 1$, then*

$$\int |(z_1, z')|_\lambda^{\alpha-n}\, dz' \leq M\, |z_1|^{\frac{|\lambda|}{n\lambda_1}\alpha - 1}.$$

The proof of this Lemma can be easily seen with change of variable

$$z_2 = t_2 z_1^{\frac{\lambda_2}{\lambda_1}}, ..., z_n = t_n z_1^{\frac{\lambda_n}{\lambda_1}}$$

and using λ-spherical coordinates in the integral.

Lemma 3.3 *If* $\frac{|\lambda|}{n\lambda_1}(\alpha-1)<1$, *then*

$$\int\limits_{\{x' :\, |x'|_\lambda > 2|z_1|\}} \left| |(z_1, x'+h')|_\lambda^{\alpha-n} - |(z_1, x')|_\lambda^{\alpha-n} \right| dx' \leq M\, |h'|_\lambda\, |z_1|^{\frac{|\lambda|}{n\lambda_1}(\alpha-1)-1}. \tag{3.2}$$

Proof. From Lemma 3.1 we have the inequality

$$\int\limits_{\{x' :\, |x'|_\lambda > 2|z_1|\}} \left| |(z_1, x'+h')|_\lambda^{\alpha-n} - |(z_1, x')|_\lambda^{\alpha-n} \right| dx' \leq |h'|_\lambda \int\limits_{\{x' :\, |x'|_\lambda > 2|z_1|\}} |(z_1, x')|_\lambda^{\alpha-n-1}\, dx'.$$

Thus from Lemma 3.2 we obtain (3.2). □

Proof of Theorem 3.1 We will adapt to our paper the proof given by Mizuta [3] for the classical Riesz potential. Note that

$$I_{\alpha,\lambda} f(0, x') = \int\limits_{\mathbb{R}^1} \int\limits_{\mathbb{R}^{n-1}} |(-z_1, x'-z')|_\lambda^{\alpha-n} f(z_1, z')dz_1dz'$$

and

$$\begin{aligned} &|I_{\alpha,\lambda} f(0, x'+h') - I_{\alpha,\lambda} f(0, x')| \\ &\leq \int\limits_{\mathbb{R}^1} \left(\int\limits_{\mathbb{R}^{n-1}} \left| |(z_1, x'+h'-z')|_\lambda^{\alpha-n} - |(-z_1, x'-z')|_\lambda^{\alpha-n} \right| |f(z_1, z')|\, dz' \right) dz_1. \end{aligned}$$

Hence by Young's inequality we have the inequality

$$\begin{aligned} \|I_{\alpha,\lambda} f(0, .+h') - I_{\alpha,\lambda} f(0, .)\|_p \;\leq\; & \int\limits_{\mathbb{R}^1} \left(\int\limits_{\mathbb{R}^{n-1}} \left| |(-z_1, x'+h')|_\lambda^{\alpha-n} - |(-z_1, x')|_\lambda^{\alpha-n}\, dx' \right| \right) \\ & \times \left(\int\limits_{\mathbb{R}^{n-1}} |f(z_1, z')|^p\, dz' \right)^{\frac{1}{p}} dz_1. \end{aligned}$$

In case $\alpha < 1$, and in view of Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned}
\|I_{\alpha,\lambda}f(0,.+h') - I_{\alpha,\lambda}f(0,.)\|_p &\le M\int\limits_{\mathbb{R}^1} |h'|_\lambda |z_1|^{\frac{|\lambda|}{n\lambda_1}(\alpha-1)-1} \|f(z_1,z')\|_p\, dz_1 \\
&\le M|h'|_\lambda \int\limits_{|z_1|<|h'|_\lambda} |z_1|^{\frac{|\lambda|}{n\lambda_1}(\alpha-1)-1} \|f(z_1,z')\|_p\, dz_1 \\
&\quad + M\int\limits_{|z_1|\ge|h'|_\lambda} |z_1|^{\frac{|\lambda|}{n\lambda_1}(\alpha-1)} \|f(z_1,z')\|_p\, dz_1 \\
&= M[I_1(h') + I_2(h')].
\end{aligned}$$

Passing to the λ-spherical coordinates, we obtain

$$\begin{aligned}
\int\limits_{\mathbb{R}^{n-1}} \frac{[I_1(h')]^P}{|h'|_\lambda^{(n-2+(\frac{|\lambda|}{n\lambda_1}(\alpha-1)+1)p)}} dh' &= \int\limits_{\mathbb{R}^{n-1}} |h'|_\lambda^{(2-n-(\frac{|\lambda|}{n\lambda_1}(\alpha-1)+1)p)} \\
&\quad \times \left[M|h'|_\lambda \int\limits_{|z_1|<|h'|_\lambda} |z_1|^{\frac{|\lambda|}{n\lambda_1}(\alpha-1)-1} \|f(z_1,z')\|_p\, dz_1\right]^p dh' \\
&= M\int\limits_0^\infty r^{\frac{2|\lambda'|}{n-1}(2-n-\frac{|\lambda|}{n\lambda_1}(\alpha-1)p)-1} \\
&\quad \times \left[\int\limits_0^{r^{\frac{2|\lambda|}{n}}} |z_1|^{\frac{|\lambda|}{n\lambda_1}(\alpha-1)-1} \|f(z_1,z')\|_p\, dz_1\right]^p dr.
\end{aligned}$$

Here for $u = r^{\frac{2|\lambda'|}{n-1}}$, we have

$$= M\int\limits_0^\infty u^{2-\frac{|\lambda|}{n\lambda_1}(\alpha-1)p} \left[\int\limits_0^u |z_1|^{\frac{|\lambda|}{n\lambda_1}(\alpha-1)-1} \|f(z_1,z')\|_p\, dz_1\right]^p du.$$

By Hardy's inequality we get

$$\begin{aligned}
&\le M\int\limits_0^\infty |z_1|^{-\frac{|\lambda|}{n\lambda_1}(\alpha-1)p} \left[|z_1|^{\frac{|\lambda|}{n\lambda_1}(\alpha-1)} \|f(z_1,z')\|_p\right]^p dz_1 \\
&= M\int\limits_0^\infty \|f(z_1,z')\|_p^p\, dz_1 \\
&= M\|f\|_p^p .
\end{aligned}$$

In the same way, we find

$$\begin{aligned}
\int\limits_{\mathbb{R}^{n-1}} \frac{[I_2(h')]^P}{|h'|_\lambda^{(n-2+(\frac{|\lambda|}{n\lambda_1}(\alpha-1)+1)p)}}dh' &\leq M\int\limits_0^\infty r^{\frac{2|\lambda'|}{n-1}(1-(\frac{|\lambda|}{n\lambda_1}(\alpha-1)+1)p)-1} \\
&\times\left[\int\limits_{r^{\frac{2|\lambda|}{n}}}^\infty |z_1|^{\frac{|\lambda|}{n\lambda_1}(\alpha-1)}\left\|f(z_1,z')\right\|_p dz_1\right]^p dr \\
&\leq M\int\limits_0^\infty |z_1|^{-(\frac{|\lambda|}{n\lambda_1}(\alpha-1)+1)p}\left[|z_1|^{\frac{|\lambda|}{n\lambda_1}(\alpha-1)+1}\left\|f(z_1,z')\right\|_p\right]^p dz_1 \\
&= M\int\limits_0^\infty \left\|f(z_1,z')\right\|_p^p dz_1 \\
&= M\left\|f\right\|_p^p .
\end{aligned}$$

Thus the case $\alpha < 1$ is proved.

In case $\alpha = 1$, we must replace I_1 by

$$\begin{aligned}
J_1(h') &= \int\limits_{|z_1|<|h'|_\lambda} \log\left(\frac{2|h'|_\lambda}{|z_1|}\right)\left\|f(z_1,.)\right\|_p dz_1 \\
&\leq M\int\limits_{|z_1|<|h'|_\lambda}\left(\frac{2|h'|_\lambda}{|z_1|}\right)^\varepsilon \left\|f(z_1,.)\right\|_p dz_1
\end{aligned}$$

for $0 < \varepsilon < 1$ and apply Hardy' inequality.

In case $1 < \alpha < 2$,

$$I_1(h') \leq M\,|h'|_\lambda \int\limits_{|z_1|<|h'|_\lambda} [|h'|_\lambda + |z_1|]^{-\frac{|\lambda|}{n\lambda_1}(\alpha-1)-1}\left\|f(z_1,.)\right\|_p dz_1,$$

which can be treated similarly. □

Now we give the following theorem which is classical Sobolev's inequality for $\lambda_i = \frac{1}{2},\ i = 1,2,...,n$.

Theorem 3.2 *Let* $\frac{1}{p^*} = \frac{1}{p} - \frac{\alpha}{n}$. *If* $1 < p < \infty$ *and* $\frac{1}{p^*} > 0$, *then*

$$\left(\int |I_{\alpha,\lambda}f(x)|^{p^*} dx\right)^{\frac{1}{p^*}} \leq M\left\|f\right\|_p .$$

Proof. Let $\frac{1}{p^*} = \frac{1}{p} - \frac{\alpha}{n}$. We may assume that f is nonnegative. For $r > 0$, we write

$$\begin{aligned} I_{\alpha,\lambda}f(x) &= \int\limits_{B_\lambda(x,2r)} |x-y|_\lambda^{\alpha-n} f(y)dy + \int\limits_{\mathbb{R}^n - B_\lambda(x,2r)} |x-y|_\lambda^{\alpha-n} f(y)dy \\ &= I^1_{\alpha,\lambda}(x) + I^2_{\alpha,\lambda}(x). \end{aligned}$$

Since $(\alpha - n)p' + n > 0$, we have the following inequality by Hölder's inequality

$$\begin{aligned} I^1_{\alpha,\lambda}(x) &\leq \left(\int\limits_{B_\lambda(x,2r)} |x-y|_\lambda^{(\alpha-n)p'} dy \right)^{\frac{1}{p'}} \left(\int\limits_{B_\lambda(x,2r)} f^p(y)dy \right)^{\frac{1}{p}} \\ &\leq Mr^{\left[\frac{2|\lambda|}{n}(\alpha-n)p'+2|\lambda|\right]} \|f\|_p . \end{aligned}$$

By Hölder's inequality

$$\begin{aligned} I^2_{\alpha,\lambda}(x) &\leq \left(\int\limits_{\mathbb{R}^n - B_\lambda(x,2r)} |x-y|_\lambda^{(\alpha-n)p'} dy \right)^{\frac{1}{p'}} \left(\int\limits_{\mathbb{R}^n - B_\lambda(x,2r)} f^p(y)dy \right)^{\frac{1}{p}} \\ &\leq Mr^{\left[\frac{2|\lambda|}{n}(\alpha-n)p'+2|\lambda|\right]} \|f\|_p . \end{aligned}$$

For any $\rho > 0$, choose $r > 0$ so that

$$Mr^{\left[\frac{2|\lambda|}{n}(\alpha-n)p'+2|\lambda|\right]} \|f\|_p = \rho.$$

Then it follows that

$$\begin{aligned} |\{x : I_{\alpha,\lambda}f(x) > 2\rho\}| &\leq \left| \left\{ x : I^1_{\alpha,\lambda}(x) > \rho \right\} \right| \\ &\leq \int \left(\frac{I^1_{\alpha,\lambda}(x)}{\rho} \right)^p dx \\ &\leq M \left[r^{\frac{2|\lambda|}{n}\alpha} \rho^{-1} \|f\|_p \right]^p \\ &= M \left[\frac{\|f\|_p}{\rho} \right]^{p^*} . \end{aligned}$$

This implies that $f \to I_{\alpha,\lambda}f$ is of weak type (p, p^*). In view of the Marcinkiewicz interpolation theorem, the mapping is seen to be of strong type (p, p^*).

The proof is completed. □

Theorem 3.3 *Let $\alpha p > 1$ and $\frac{1}{p^*} = \frac{1}{p} - \frac{\alpha}{n-1} > 0$. Then there is the inequality*

$$\left(\int |I_{\alpha,\lambda} f(0,x')|^{p^*} dx'\right)^{\frac{1}{p^*}} \leq M \|f\|_p .$$

Proof. Note that

$$I_{\alpha,\lambda} f(0,x') = \int\limits_{\mathbb{R}^1} \int\limits_{\mathbb{R}^{n-1}} |(-z_1, x'-z')|_\lambda^{\alpha-n} f(z_1,z') dz_1 dz'.$$

Hence we have by Hölder's inequality ,

$$\begin{aligned} |I_{\alpha,\lambda} f(0,x')| &\leq \int\limits_{\mathbb{R}^{n-1}} \left(\int\limits_{\mathbb{R}^1} |(-z_1,x'-z')|_\lambda^{(\alpha-n)p'} dz_1\right)^{\frac{1}{p'}} \left(\int\limits_{\mathbb{R}^1} f^p(z_1,z') dz_1\right)^{\frac{1}{p}} dz' \\ &\leq M \int\limits_{\mathbb{R}^{n-1}} |x'-z'|_\lambda^{(\alpha-n)+\frac{n\lambda_1}{|\lambda| p'}} \left(\int\limits_{\mathbb{R}^1} f^p(z_1,z') dz_1\right)^{\frac{1}{p}} dz'. \end{aligned}$$

The required inequality can be established by applying Theorem 3.2 □

4. Continuity properties

In this chapter, assume that $\alpha p = n$. Let φ be a positive nondecreasing function on the interval $(0,\infty)$ satisfying

$$A^{-1}\varphi(r) \leq \varphi(r^2) \leq A\varphi(r). \tag{4.3}$$

By condition (4.3), we have the doubling condition

$$A^{-1}\varphi(r) \leq \varphi(2r) \leq A\varphi(r), \tag{4.4}$$

and for $v > 1$

$$A(v)^{-1}\varphi(r) \leq \varphi(r^v) \leq A(v)\varphi(r). \tag{4.5}$$

Our aim in this chapter is to discuss the continuity of $I_{\alpha,\lambda} f$ when

$$\int (1+|y|_\lambda)^{\alpha-n} f(y) dy < \infty \tag{4.6}$$

and

$$\int \Phi_p(|f(y)|)dy < \infty, \tag{4.7}$$

where $\Phi_p(r) = r^p \varphi(r)$.

Lemma 4.1 *If $v > 0$, then*

$$s^v \varphi(s^{-1}) \leq M t^v \varphi(t^{-1}) \quad \textit{whenever } 0 < s < t.$$

The proof of this Lemma is given in [5].

Theorem 4.1 *Let φ satisfy the following condition:*

$$\int_0^1 \varphi(r^{-1})^{-\frac{1}{p-1}} r^{-1} dr < \infty \tag{4.8}$$

and set

$$\varphi^*(r) = \left(\int_0^r \varphi(t^{-1})^{-\frac{1}{p-1}} t^{-1} dt \right)^{1-\frac{1}{p}}.$$

If f satisfies (4.6) and (4.7), then $I_{\alpha,\lambda} f$ is continuous on $\mathbb{R}^n$ and, moreover,

$$|I_{\alpha,\lambda} f(x) - I_{\alpha,\lambda} f(z)| = o(\varphi^*(|x-z|_\lambda)) \quad \textit{as } |x-z|_\lambda \to 0.$$

Proof. Let $r = |x-z|_\lambda < \frac{1}{2}$. We write

$$\begin{aligned} I_{\alpha,\lambda} f(z) &= \int_{B_\lambda(x,2r)} |z-y|_\lambda^{\alpha-n} f(y)dy + \int_{\mathbb{R}^n - B_\lambda(x,2r)} |z-y|_\lambda^{\alpha-n} f(y)dy \\ &= I_1(z) + I_2(z) . \end{aligned}$$

For $0 < \delta < \alpha$, we have by Hölder's inequality,

$$\begin{aligned}
|I_1(z)| &= \int\limits_{\{x:\, B_\lambda(z,3r);\, |f(y)|<|z-y|_\lambda^{-\delta}\}} |z-y|_\lambda^{\alpha-n} |f(y)|\, dy \\
&+ \int\limits_{\{x:\, B_\lambda(z,3r);\, |f(y)|>|z-y|_\lambda^{-\delta}\}} |z-y|_\lambda^{\alpha-n} |f(y)|\, dy \\
&\le \int\limits_{B_\lambda(z,3r)} |z-y|_\lambda^{\alpha-n-\delta}\, dy \\
&+ \int\limits_{\{x:\, B_\lambda(z,3r);\, |f(y)|>|z-y|_\lambda^{-\delta}\}} \left[|z-y|_\lambda^{\alpha-n} \varphi\left(|z-y|_\lambda^{-\delta}\right)^{-\frac{1}{p}}\right] \left[|f(y)|\, \varphi\left(|f(y)|\right)^{\frac{1}{p}}\right] dy \\
&\le Mr^{\frac{2|\lambda|}{n}\alpha-\delta} + \left(\int\limits_{B_\lambda(z,3r)} \left[|z-y|_\lambda^{\alpha-n} \varphi\left(|z-y|_\lambda^{-\delta}\right)^{-\frac{1}{p}}\right]^{p'} dy \right)^{\frac{1}{p'}} \\
&\times \left(\int\limits_{B_\lambda(z,3r)} \left[|f(y)|\, \varphi\left(|f(y)|\right)^{\frac{1}{p}}\right]^{p} dy \right)^{\frac{1}{p}} \\
&= Mr^{\frac{2|\lambda|}{n}\alpha-\delta} + \left(\int\limits_0^{3r} \varphi\left(t^{-\frac{2|\lambda|}{n}\delta}\right)^{-\frac{p'}{p}} t^{-1} dt \right)^{\frac{1}{p'}} \left(\int\limits_{B_\lambda(z,3r)} \Phi_p\left(|f(y)|\right) dy \right)^{\frac{1}{p}}.
\end{aligned}$$

Therefore, from (4.5) we have

$$|I_1(z)| \le Mr^{\frac{2|\lambda|}{n}\alpha-\delta} + M\varphi^*(r) \left(\int\limits_{B_\lambda(x,4r)} \Phi_p\left(|f(y)|\right) dy \right)^{\frac{1}{p}}.$$

On the other hand, from Lemma 3.1

$$\int\limits_{\mathbb{R}^n - B_\lambda(x,2r)} \left| |x-y|_\lambda^{\alpha-n} - |y-z|_\lambda^{\alpha-n} \right| |f(y)|\, dy \le Mr \int\limits_{\mathbb{R}^n - B_\lambda(x,2r)} |x-y|_\lambda^{\alpha-n-1} |f(y)|\, dy.$$

Hence for $\alpha - 1 < \delta < \alpha$, we have as above

$$\begin{aligned}
|I_2(x) - I_2(z)| &\leq Mr \int\limits_{\mathbb{R}^n - B_\lambda(x,2r)} |x-y|_\lambda^{\alpha-n-1} |f(y)|\, dy \\
&= Mr \int\limits_{\{x:\, \mathbb{R}^n - B_\lambda(x,2r);\ |f(y)|<r^{-\delta}\}} |x-y|_\lambda^{\alpha-n-1} |f(y)|\, dy \\
&+ Mr \int\limits_{\{x:\, \mathbb{R}^n - B_\lambda(x,2r);\ |f(y)|>r^{-\delta}\}} |x-y|_\lambda^{\alpha-n-1} |f(y)|\, dy \\
&\leq Mr \int\limits_{\mathbb{R}^n - B_\lambda(x,2r)} |x-y|_\lambda^{\alpha-n-\delta-1}\, dy + Mr\varphi^*(r^{-\delta})^{-\frac{1}{p}} \\
&\times \int\limits_{\{x:\, \mathbb{R}^n - B_\lambda(x,2r);\ |f(y)|>r^{-\delta}\}} |x-y|_\lambda^{\alpha-n-1} \left[|f(y)|\, \varphi(|f(y)|)^{\frac{1}{p}} \right] dy \\
&\leq Mr^{\frac{2|\lambda|}{n}(\alpha-\delta-1)+1} + Mr\left[\varphi(r^{-\delta})\right]^{-\frac{1}{p}} \\
&\times \left(\int\limits_{\mathbb{R}^n - B_\lambda(x,2r)} |x-y|_\lambda^{(\alpha-n-1)p'}\, dy \right)^{\frac{1}{p'}} \left(\int\limits_{\mathbb{R}^n - B_\lambda(x,2r)} \Phi_p(|f(y)|)\, dy \right)^{\frac{1}{p}} \\
&\leq Mr^{\frac{2|\lambda|}{n}(\alpha-\delta-1)+1} + Mr^{1-\frac{2|\lambda|}{n}} \left[\varphi(r^{-\delta})\right]^{-\frac{1}{p}} \left(\int \Phi_p(|f(y)|)\, dy \right)^{\frac{1}{p}}.
\end{aligned}$$

By (4.3), we see that

$$\varphi^*(r) \geq \left(\int\limits_{r^2}^{r} \left[\varphi(t^{-1})\right]^{-\frac{1}{p-1}} t^{-1} dt \right)^{\frac{1}{p'}} \geq M\left[\varphi(r^{-1})\right]^{-\frac{1}{p}} \left[\log(\frac{1}{r})\right]^{\frac{1}{p'}}.$$

Further, by an application of Lemma 4.1 with $\left[\varphi(r^{-1})\right]^{-1}$

$$Ms^{\alpha-n} \leq \left[\varphi(s^{-1})\right]^{-1} \qquad \text{whenever } 0 < s < 1. \tag{4.9}$$

Thus we establish the inequality

$$\begin{aligned}
|I_2(x) - I_2(z)| &\leq Mr^{\frac{2|\lambda|}{n}(\alpha-\delta-1)+1} + Mr^{1-\frac{2|\lambda|}{n}} \varphi^*(r) \\
&\quad \times \left[\log(\tfrac{1}{r})\right]^{-\frac{1}{p'}} \left(\int \Phi_p(|f(y)|)\, dy \right)^{\frac{1}{p}}.
\end{aligned}$$

Now it follows that

$$\begin{aligned}|I_{\alpha,\lambda}f(x)-I_{\alpha,\lambda}f(z)| \quad\leq\quad & Mr^{\frac{2|\lambda|}{n}\alpha-\delta}+M\varphi^{*}(r)\left(\int\limits_{B_{\lambda}(x,4r)}\Phi_{p}\left(|f(y)|\right)dy\right)^{\frac{1}{p}}\\ & +Mr^{\frac{2|\lambda|}{n}(\alpha-\delta-1)+1}+Mr^{1-\frac{2|\lambda|}{n}}\varphi^{*}(r)\\ & \times\left[\log(\tfrac{1}{r})\right]^{-\frac{1}{p'}}\left(\int\Phi_{p}\left(|f(y)|\right)dy\right)^{\frac{1}{p}},\end{aligned}$$

which together with (4.9) proves the required result. □

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