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Existence of Linear-Quadratic Regulator for Degenerate Diffusions

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Abstract

This paper studies a linear regulatory quadratic control problem for degenerate Hamilton-Jacobi-Bellman (HJB) equation. We establish the existence of a unique viscosity and a classical solution of the degenerate HJB equation associated with this problem by the technique of viscosity solutions, and, hence, derive an optimal control from the optimality conditions in the HJB equation.

Key words and phrases: Stochastic differential equation, Hamilton-Jacobi-Bellman equation, Linear-Quadratic problem, Viscosity solutions, Applications to control theory.

1. Introduction

We are concerned with the quadratic control problem to minimize the expected cost with discount factor $\beta > 0$:

$$J(c) = E\left[\int_0^\infty e^{-\beta t}\{h(x_t) + |c_t|^2\}dt\right] \tag{1}$$

over $c \in \mathcal{A}$ and subject to the degenerate stochastic differential equation

$$dx_t = [Ax_t + c_t]dt + \sigma x_t dw_t, \quad x_0 = x \in \mathbf{R}, \quad t \geq 0. \tag{2}$$

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Here, A consists of non-zero constants, $\sigma \neq 0$, and a continuous function h on $\mathbf{R}$. x_t is the state variable of the system at time t, c_t is the control variable of the system at time t, w_t is a one-dimensinal standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$ endowed with the natural filtration $\mathcal{F}_t$ generated by $\sigma(w_s, s \leq t)$, $x_0 = x$ is the initial value of the state variable, and $\mathcal{A}$ denotes the class of all $\mathcal{F}_t-$progressively measurable processes $c = (c_t)$ with $J(c) < \infty$.

This kind of stochastic control problem has been studied by many authors [3, 6] for non-degenerate diffusions to (1) and (2). We also assume that h satisfies the properties that

$$h \text{ is convex;} \tag{3}$$

$$\text{there exists} \quad C > 0 \quad \text{such that} \quad h(x) \leq C(1 + |x|^n), x \epsilon \mathbf{R}, \tag{4}$$

for some constant $C > 0, n \geq 2$. We refer to [11] for the quadratic case of degenerate diffusions related to Ricatti equations in case of $h(x) = Cx^2$ and $n = 2$ with infinite horizon.

The purpose of this paper is to show the existence of a smooth solution u of the associated *Hamilton-Jacobi-Bellman (in short, HJB)* equation of the form:

$$-\beta u + \frac{1}{2}\sigma^2 x^2 u'' + Axu' + \min_{r \in \mathbf{R}}(r^2 + ru') + h(x) = 0 \quad \text{in } \mathbf{R}, \tag{5}$$

and to give a synthesis of optimal control. Our method consists in finding the viscosity solution u of (5) [5, 6], by the limit of the solution $v = v_L, L > 0$, to the *HJB* equation

$$-\beta v_L + \frac{1}{2}\sigma^2 x^2 v_L'' + Axv_L' + \min_{|r| \leq L}(r^2 + rv_L') + h(x) = 0 \quad \text{in } \mathbf{R}, \tag{6}$$

as $L \to \infty$, and then in considering the smoothness of u by its convexity. To show the existence of the *viscosity solution* v_L, we assume that h has the following property: there exists $C_\rho > 0$, for any $\rho > 0$, such that

$$|h(x) - h(y)| \leq C_\rho |x - y|^n + \rho(1 + |x|^n + |y|^n), \quad \forall x, y \in \mathbf{R}, \tag{7}$$

for a fixed integer $n \geq 2$.

This condition acts as the uniform continuity of h with order n, and plays an important role for the existence of viscosity solutions [7, 9]. We notice that (7) is fulfilled for $h(x) = |x|^{\bar{n}}, \bar{n} \in [\mathbf{2}, \mathbf{n}]$ closed interval.

In Section 2 we show that $u(x) := \lim_{L\to\infty} v_L(x)$ is a viscosity solution of (5), as $L \to \infty$. Section 3 is devoted to the study of smoothness of u. In Section 4 we present an optimal control to the optimization problem (1) and (2). Finally in Section 5, the major conclusions of this study is presented.

2. Viscosity solutions

In this subsection we show that $v_L(x)$ is a viscosity solution of the Bellman equation (5) for any fixed $L > 0$, and then converges to a viscosity solution $u(x)$ of the Bellman equation (5). In order to introduce solutions in the viscosity sense, given a continuous and degenerate elliptic map $H : \mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, we recall by [5] the definition of viscosity solutions of

$$H(x, w, w', w'') = 0 \quad \text{in} \quad \mathbf{R}. \tag{8}$$

Definition 2.1 *A function $w \in C(\mathbf{R})$ is called a viscosity subsolution (resp., supersolution) of (8) if, whenever for $\varphi \in C^2(\mathbf{R}), w - \varphi$ attains its local maximum (resp., minimum) at $x \in \mathbf{R}$, then*

$$H(x, w(x), \varphi'(x), \varphi''(x)) \leq 0, \tag{9}$$

$$H(x, w(x), \varphi'(x), \varphi''(x)) \geq 0, \tag{10}$$

respectively. We also call $w \in C(\mathbf{R})$ a viscosity solution of (8) if it is both a viscosity sub- and supersolution of (8).
According to Crandall, Ishii and Lions [5] and Fleming and Soner [6] this definition is equivalent to the following: for any $x \in \mathbf{R}$,

$$H(x, w(x), p, q) \leq 0 \quad \textit{for} \quad (p, q) \in J^{2,+}w(x)$$

$$H(x, w(x), p, q) \geq 0 \quad \textit{for} \quad (p, q) \in J^{2,-}w(x),$$

where $J^{2,+}$ and $J^{2,-}$ are the second-order superjets and subjets defined by

$$\begin{aligned} J^{2,+}w(x) &= \{(p, q) \in \mathbf{R}^2 : \limsup_{y\to x} \frac{w(y) - w(x) - p(y - x) - \frac{1}{2}q|y - x|^2}{|y - x|^2} \leq 0\}, \\ J^{2,-}w(x) &= \{(p, q) \in \mathbf{R}^2 : \liminf_{y\to x} \frac{w(y) - w(x) - p(y - x) - \frac{1}{2}q|y - x|^2}{|y - x|^2} \geq 0\}. \end{aligned}$$

Let us define the value function $v_L(x) := \inf_{c\in\mathcal{A}_L} J(c)$, where $\mathcal{A}_L = \{c = (c_t) \in \mathcal{A} : |c_t| \le L \text{ for all } t \ge 0\}$.

We assume that there exists $\beta_0 \in (0, \beta)$ satisfying

$$-\beta_0 + \sigma^2 n(2n-1) + 2n|A| < 0, \tag{11}$$

and we set $f_k(x) = \gamma + |x|^k$ for any $2 \le k \le 2n$ and a constant $\gamma \ge 1$ chosen later.

Lemma 2.2 *Assume (11). Then there exist $\gamma \ge 1$ and $\eta > 0$, depending on L, k, such that*

$$-\beta_0 f_k + \frac{1}{2}\sigma^2 x^2 f_k'' + Axf_k' + \max_{|r|\le L}(r^2 + rf_k') + \eta f_k \le 0. \tag{12}$$

Furthermore,

$$E[\int_0^\tau e^{-\beta_0 s}\eta f_k(x_s)ds + e^{-\beta_0\tau} f_k(x_\tau)] \le f_k(x) \quad \textit{for} \quad 2 \le k \le 2n, \tag{13}$$

$$E[\sup_t e^{-\beta_0 t} f_k(x_t)] < \infty \quad \textit{for} \quad 2 \le k \le n, \tag{14}$$

where τ is any stopping time and x_t is the response to $(c_t) \in \mathcal{A}_L$.

Proof. By (11), we choose $\eta \in (0, \beta_0)$ such that

$$-\beta_0 + \frac{1}{2}\sigma^2 k(k-1) + k|A| + \eta < 0, \tag{15}$$

and then $\gamma \ge 1$ such that

$$(-\beta_0 + \frac{1}{2}\sigma^2 k(k-1) + k|A| + \eta)|x|^k + Lk|x|^{k-1} + (L^2 + \eta\gamma - \beta_0\gamma) \le 0.$$

Then (12) follows immediately. By (12) and Itô formula we deduce (13). Moreover, by moment inequalities for martingales we get

$$\begin{aligned} E[\sup_t e^{-\beta_0 t} f_k(x_t)] &\le f_k(x) + E[\sup_t |\int_0^t e^{-\beta_0 s} f_k'(x_s)\sigma x_s dw_s|] \\ &\le f_k(x) + KE[(\int_0^\infty e^{-2\beta_0 s}\sigma^2 |x_s|^{2k} ds)^{1/2}], \end{aligned}$$

for some constant $K > 0$. Therefore, (14) follows from this relation together with (13).
□

Theorem 2.3 *We assume (3), (4), (7) and (11). Then,*

$$v_L \quad \textit{fulfills} \quad (3), (4), (7), \tag{16}$$

and the dynamic programming principle holds, i.e.,

$$v_L(x) = \inf_{c \in \mathcal{A}_L} E\left[\int_0^\tau e^{-\beta t}\{h(x_t) + |c_t|^2\}dt + e^{-\beta\tau} v_L(x_\tau)\right] \tag{17}$$

for any stopping time τ.

Proof. We suppress L of v_L for simplicity. The convexity of v follows by the same line as [5, Chap. 4, Lemma 10.6]. Let x_t^0 be the unique solution of

$$dx_t^0 = Ax_t^0 dt + \sigma x_t^0 dw_t, \qquad x_0^0 = x. \tag{18}$$

Then, by (13) and (4) it follows

$$v(x) \le E[\int_0^\infty e^{-\beta t} h(x_t^0)dt] \le CE[\int_0^\infty e^{-\beta_0 t} f_n(x_t^0)dt] \le Cf_n(x)/\eta. \tag{19}$$

For the solution y_t of (2) with $y_0 = y$, it is clear that $x_t - y_t$ fulfills (18) with initial condition $x - y$. We note by (15) with $k = n$ and Itô formula that

$$E[e^{-\beta_0 t}|x_t^0|^n] \le |x|^n.$$

Thus by (7) and (13)

$$\begin{aligned}
|v(x) - v(y)| &\le \sup_{c \in \mathcal{A}_L} E[\int_0^\infty e^{-\beta t}|h(x_t) - h(y_t)|dt] \\
&\le \sup_{c \in \mathcal{A}_L} E[\int_0^\infty e^{-\beta t}\left\{C_\rho|x_t - y_t|^n + \rho(1 + |x_t|^n + |y_t|^n)\right\}dt] \qquad (20) \\
&\le \sup_{c \in \mathcal{A}_L} \int_0^\infty e^{-\beta t}\left\{C_\rho|x - y|^n e^{\beta_0 t} + \rho(h_n(x) + h_n(y))e^{\beta_0 t}\right\}dt \\
&\le \frac{1}{\beta - \beta_0}[C_\rho|x - y|^n + 2\rho\gamma(1 + |x|^n + |y|^n)].
\end{aligned}$$

Therefore we get (16).

To prove (17), we denote by $v^r(x)$ the right hand side of (17). By the formal Markov property

$$\begin{aligned}
E[\int_\tau^\infty e^{-\beta t}\{h(x_t) + |c_t|^2\}dt|\mathcal{F}_\tau] &= E[\int_0^\infty e^{-\beta(t+\tau)}\{h(x_{\tau+t}) + |c_{\tau+t}|^2\}dt|\mathcal{F}_\tau] \\
&= e^{-\beta\tau} J_{\tilde{c}}(x_\tau),
\end{aligned}$$

with $\tilde{c}$ equal to c shifted by τ. Thus,

$$\begin{aligned} J_c(x) &= E\left[\int_0^\tau + \int_\tau^\infty e^{-\beta t}\{h(x_t)+|c_t|^2\}dt\right] \\ &= E\left[\int_0^\tau e^{-\beta t}\{h(x_t)+|c_t|^2\}dt\right] + E\left[\int_\tau^\infty e^{-\beta t}\{h(x_t)+|c_t|^2\}dt/\mathcal{F}_\tau\right] \\ &\geq E\left[\int_0^\tau e^{-\beta t}\{h(x_t)+|c_t|^2\}dt + e^{-\beta\tau}v_L(x_\tau)\right]. \end{aligned}$$

It is known in [6, 10] that this formal argument can be verified, and we deduce $v_L(x) \geq v^r(x)$.

To prove the reverse inequality, let $\rho > 0$ be arbitrary. We set

$$V_c(x) := E[\int_0^\infty e^{-\beta t}\{h(x_t)+|c_t|^2\}dt]. \tag{21}$$

By the same calculation as (20), there exists $C_\rho > 0$ such that

$$|V_c(x) - V_c(y)| \leq C_\rho|x-y|^n + \rho(1+|x|^n+|y|^n).$$

Take $0 < \delta < 1$ with $C_\rho\delta^n < \rho$. Then, we have for $|x-y| < \delta$,

$$\begin{aligned} |v(x)-v(y)| &\leq \sup_{c\in\mathcal{A}_L}|V_c(x)-V_c(y)| \\ &\leq \rho(2+|x|^n+|y|^n) \\ &\leq \rho[2+|x|^n+2^n(1+|x|^n)] \\ &= \rho[(2+2^n)+(1+2^n)|x|^n] \\ &\leq \Xi_\rho(x) := \rho(2^n+2)(1+|x|^n). \end{aligned}$$

Let $\{S_i\}$ be a sequence of disjoint subsets of $\mathbf{R}$ such that

$$diam(S_i) < \delta \quad and \quad \cup_i S_i = \mathbf{R}.$$

For any i, we take $x^{(i)} \in S_i$ and $c^{(i)} \in \mathcal{A}_L$ such that

$$V_{c^{(i)}}(x^{(i)}) \leq \inf_{c\in\mathcal{A}_L} V_c(x^{(i)}) + \rho.$$

Define $c^\tau \in \mathcal{A}_L$ by

$$c_t^\tau = c_t 1_{\{t<\tau\}} + c_{t-\tau}^{(i)} 1_{\{x_\tau\in S_i\}} 1_{\{t\geq\tau\}}, \quad \text{for} \quad x_\tau \in S_i.$$

Hence,

$$\begin{aligned} V_{c^{(i)}}(x_\tau) &= V_{c^{(i)}}(x_\tau) - V_{c^{(i)}}(x^{(i)}) + V_{c^{(i)}}(x^{(i)}) \\ &\le \Xi_\rho(x_\tau) + V_{c^{(i)}}(x^{(i)}) \\ &\le \Xi_\rho(x_\tau) + \inf_{c\in\mathcal{A}_L} V_c(x^{(i)}) + \rho \\ &= \Xi_\rho(x_\tau) + v(x^{(i)}) + \rho \\ &\le 2\Xi_\rho(x_\tau) + v(x_\tau) + \rho. \end{aligned}$$

Now, by the definition of $v^r(x)$, we can find $c \in \mathcal{A}_L$ such that

$$v^r(x) + \rho \ge E\left[\int_0^\tau e^{-\beta t}\{h(x_t) + |c_t|^2\}dt] + e^{-\beta\tau} v(x_\tau)\right].$$

Thus, using the formal Markov property [6], we have

$$\begin{aligned} v^r(x) + \rho &\ge \sum_i E\left[\int_0^\tau e^{-\beta t}\{h(x_t) + |c_t|^2\}dt + e^{-\beta\tau}(V_{c^{(i)}}(x_\tau) - 2\Xi_\rho(x_\tau) - \rho) :\ x_\tau \in S_i\right] \\ &= E\left[\int_0^\tau e^{-\beta t}\{h(x_t^\tau) + |c_t^\tau|^2\}dt + \int_\tau^\infty e^{-\beta t}\{h(x_t^\tau) + |c_t^\tau|^2\}dt|\mathcal{F}_\tau\right] \\ &\quad - 2E[e^{-\beta\tau}\Xi_\rho(x_\tau)] - \rho \\ &\ge v(x) - 2\Xi_\rho(x) - \rho, \end{aligned}$$

where x_t^τ is the response to c_t^τ with $x_0^\tau = x_\tau$. Letting $\rho \to 0$, we deduce $v^r(x) \ge v(x)$, which completes the proof. □

Theorem 2.4 *We assume (3), (4), (7) and (11). Then v_L is a viscosity solution of (5). Furthermore, v_L converges locally uniformly to a viscosity solution $u \in C(\mathbf{R})$ of (6) satisfying (4), (7) as $L \to \infty$.*

Proof. We note that (13) gives $E[\int_0^h |x_t|^2 dt] \le e^{\beta_0 h} h f_2(x)$ for $h > 0$, and

$$\begin{aligned} E\left[\sup_{0\le s\le h} |x_s - x|^2\right] &\le 3^2\left(E[(\int_0^h |Ax_t|dt)^2 + (\int_0^h |c_t|dt)^2 + (\sup_{0\le s\le h}|\int_0^s \sigma x_t dw_t|)^2]\right) \\ &\le 3^2\left(|A|^2 h E[(\int_0^h |x_t|^2 dt)] + h^2L^2 + CE[\int_0^h |x_t|^2 dt]\right) \end{aligned}$$

with some constant $C > 0$. Hence, we have

$$\lim_{h\to 0} \sup_{c\in\mathcal{A}_L} E[\sup_{0\le s\le h} |x_s - x|^2] = 0.$$

Thus we can apply a standard result of viscosity solutions ([5], Theorem 3.1, p. 220) to obtain the viscosity property of v_L, taking into account the uniform continuity of h on each compact interval. Since $v_L(x)$ is non-increasing, we can define $u(x)$ by $u(x) = \lim_{L\to\infty} v_L(x)$. By Theorem 2.3, it is clear that u fulfills (4), (7). Thus by Dini's theorem, we can observe the locally uniform convergence and the viscosity property of u [5]. The proof is complete. □

2.1. Uniqueness of HJB

In this subsection we give a proof of uniqueness result for the quadratic control problem that v is an unique viscosity solution of (6).

Theorem 2.5 *We assume (3), (4), (7) and (11). Let v_i $(i = 1, 2)$ be two viscosity solutions of (6) satisfying (16). Then we have $v_1 = v_2$.*

Proof. We first note that (11) and there exists $n < k < n + 1$ such that

$$-\beta\psi_k + \frac{1}{2}\sigma^2 x^2 \psi_k'' + Ax\psi_k' + \min_{r\in\mathbf{R}}(|r|^2 + r\psi_k') \le 0, \tag{22}$$

where $\psi_k(x) = (1 + |x|^k)$. Indeed, by (11) we choose $\vartheta \in (0, \beta)$ such that

$$-\beta + \frac{1}{2}k(k-1)\sigma^2 + k|A| + \vartheta < 0. \tag{23}$$

By (23), we have

$$(-\beta + \frac{1}{2}k(k-1)\sigma^2 + k|A| + \vartheta)|x|^k - \frac{k^2}{4}(|x|^{k-1})^2 - \beta - \vartheta|x|^k < 0.$$

Then (22) is immediate.

Suppose that $v_1(x_0) - v_2(x_0) > 0$ for some $x_0 \in \mathbf{R}$. Then we find $\eta > 0$ such that

$$\sup_{x\in\mathbf{R}} [v_1(x) - v_2(x) - 2\eta\psi_k(x)] > 0. \tag{24}$$

Since

$$v_1(x) - v_2(x) - 2\eta\psi_k(x) \le \bar{K}(1+|x|^n) - 2\eta(1+|x|^k) \longrightarrow -\infty \text{ as } x \to \infty,$$

there exists $\bar{x} \in \mathbf{R}$ such that

$$\sup_{x\in\mathbf{R}}[v_1(x) - v_2(x) - 2\eta\psi_k(x)] = v_1(\bar{x}) - v_2(\bar{x}) - 2\eta\psi_k(\bar{x}) > 0.$$

Define

$$\Phi(x,y) = v_1(x) - v_2(y) - \frac{m}{2}|x-y|^2 - \eta(\psi_k(x) + \psi_k(y)),$$

for any $m > 0$. It is clear that

$$\begin{aligned}\Phi(x,y) \quad &\le \quad C(1+|x|^n+|y|^n) - \eta(2+|x|^k+|y|^k) \\ &\quad \to \quad -\infty \quad as \quad x,y \to \infty,\end{aligned}$$

where $C > \max\{\bar{K}, \rho\}$. Hence we find $(x_m, y_m) \in \mathbf{R}^2$ such that

$$\begin{aligned}\Phi(x_m,y_m) \quad &= \quad \sup_{x,y}\Phi(x,y) \\ &= \quad v_1(x_m) - v_2(y_m) - \frac{m}{2}|x_m - y_m|^2 - \eta(\psi_k(x_m) + \psi_k(y_m)) \qquad (25) \\ &\ge \quad v_1(\bar{x}) - v_2(\bar{x}) - 2\eta\psi_k(\bar{x}) > 0,\end{aligned}$$

from which

$$\begin{aligned}\frac{m}{2}|x_m - y_m|^2 \quad &< \quad v_1(x_m) - v_2(y_m) - \eta(\psi_k(x_m) + \psi_k(y_m)) \\ &\le \quad C(2+|x_m|^n+|y_m|^n) - \eta(2+|x_m|^k+|y_m|^k) \\ &\qquad \to \quad -\infty \quad as \quad |x_m|, |y_m| \to \infty,\end{aligned}$$

where $C > \max\{\bar{K}, \rho\}$. Thus we deduce that the sequences $\{x_m\}, \{y_m\}$ are bounded and then $\{m|x_m - y_m|^2\}$ is bounded by some constant $C > 0$, and

$$|x_m - y_m| \le (C/m)^{\frac{1}{2}} \to 0 \quad as \quad m \to \infty. \qquad (26)$$

Now, we claim that

$$m|x_m - y_m|^2 \quad \to \quad 0 \quad as \quad m \to \infty \qquad (27)$$

$$x_m,\ y_m \quad \to \tilde{x} \quad as \quad m \to \infty. \qquad (28)$$

Indeed, by the definition of (x_m, y_m),

$$\Phi(x_m, y_m) \geq v_1(x_m) - v_2(x_m) - 2\eta\psi_k(x_m).$$

Hence, by (25) and (7)

$$\begin{aligned}
\frac{m}{2}|x_m - y_m|^2 &\leq v_2(x_m) - v_2(y_m) + \eta(\psi_k(x_m) - \psi_k(y_m)) \\
&= v_2(x_m) - v_2(y_m) + \eta(|x_m|^k - |y_m|^k) \\
&\leq v_2(x_m) - v_2(y_m) + \eta(n+1)C^{k-1}|x_m - y_m| \\
&\leq C_\rho|x_m - y_m|^n + \rho(1 + |x_m|^n + |y_m|^n) + \eta(n+1)C^{k-1}|x_m - y_m|.
\end{aligned}$$

Letting $m \to \infty$ and then $\rho \to 0$, we obtain (27). Moreover, by (26) we have, (28) taking a subsequence if necessary. (26) implies $\tilde{x} = \tilde{y}$. Passing to the limit in (25), we get

$$v_1(\tilde{x}) - v_2(\tilde{x}) - 2\eta(1 + |\tilde{x}|^k) > 0. \tag{29}$$

□

We apply Ishii's lemma below to

$$V_1(x) = v_1(x) - \eta\psi_k(x),$$

$$V_2(y) = v_2(y) + \eta\psi_k(y).$$

Lemma 2.6 *(Ishii) Let $V_1, -V_2$ be upper semi-continuous in an open domain, and set*

$$\Phi(x, y) = V_1(x) - V_2(y) - \frac{m}{2}|x - y|^2.$$

Let $(\hat{x}, \hat{y})$ be the local maximizer of $\Phi(x, y)$. Then there exist symmetric matrices X_1, X_2 such that

$$(m(\hat{x} - \hat{y}), X_1) \in \bar{J}^{2,+}V_1(\hat{x}),$$

$$(m(\hat{x} - \hat{y}), X_2) \in \bar{J}^{2,-}V_2(\hat{y}),$$

and

$$\begin{pmatrix} X_1 & 0 \\ 0 & -X_2 \end{pmatrix} \leq 3m \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \quad I = identity,$$

where

$$\begin{aligned}\bar{J}^{2,\pm}V_1(x) &= \{(p,X_1) : \exists x_r \to x, \exists (p_r, X_r) \in J^{2,\pm}V_1(x_r),\\ &\quad (V_1(x_r), p_r, X_r) \to (V_1(x), p, X_1)\}.\end{aligned}$$

Proof. For the proof, see ([5], Theorem 3.2), ([6], Lemma 6.1, p. 238) and ([8], Lemma 1, p. 149).

We remark that if $V_1, V_2 \in C^2$, then

$$\Phi_x(\hat{x}, \hat{y}) = \Phi_y(\hat{x}, \hat{y}) = 0,$$

from which

$$V_1'(\hat{x}) = m(\hat{x} - \hat{y}), \quad V_2'(\hat{y}) = -m(\hat{x} - \hat{y}).$$

Since

$$\Phi_{xx} = V_1''(x) - m, \quad \Phi_{xy} = m, \quad \Phi_{yy} = -V_2''(y) - m,$$

the maximum principle gives

$$0 \geq D^2\Phi(\hat{x}, \hat{y}) = \begin{pmatrix} V_1''(\hat{x}) & 0 \\ 0 & -V_2''(\hat{y}) \end{pmatrix} - m \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

We obtain $X_1, X_2 \in \mathbf{R}^1$ such that

$$\begin{aligned}&(m(x_m - y_m), X_1) \in \bar{J}^{2,+}V_1(x_m),\\ &(m(x_m - y_m), X_2) \in \bar{J}^{2,-}V_2(y_m),\\ &-3m \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X_1 & 0 \\ 0 & -X_2 \end{pmatrix} \leq 3m \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \quad I = \text{identity},\end{aligned}$$

where

$$\begin{aligned}\bar{J}^{2,\pm}V_i(x) &= \{(p,X) : \exists x_r \to x, \exists (p_r, X_r) \in J^{2,\pm}V_i(x_r),\\ &\quad (V_i(x_r), p_r, X_r) \to (V_i(x), p, X)\}, \ i = 1, 2.\end{aligned}$$

Recall that

$$\begin{aligned}&J^{2,+}v_1(x) = \{(p + \eta k|x|^{k-1}sgn(x), X + \eta k(k-1)|x|^{k-2}) : \ (p, X) \in J^{2,+}V_1(x)\},\\ &J^{2,-}v_2(y) = \{(p - \eta k|y|^{k-1}sgn(y), X - \eta k(k-1)|y|^{k-2}) : \ (p, X) \in J^{2,-}V_2(y)\}.\end{aligned}$$

Hence

$$\begin{aligned}(p_1,\bar{X}_1) &:= \left(m(x_m-y_m)+\eta k|x_m|^{k-1}sgn(x_m), X_1+\eta k(k-1)|x_m|^{k-2}\right)\\ &\qquad \in \bar{J}^{2,+}v_1(x_m),\\ (p_2,\bar{X}_2) &:= \left(m(x_m-y_m)-\eta k|x_m|^{k-1}sgn(y_m), X_2-\eta k(k-1)|y_m|^{k-2}\right)\\ &\qquad \in \bar{J}^{2,-}v_2(y_m),\\ x_m^2X_1 &\le y_m^2X_2.\end{aligned}$$

By virtue of (9), (10) and (6) gives

$$-\beta v_1(x)+\frac{1}{2}\sigma^2x^2\bar{X}_1+Axp_1-\frac{|p_1|^2}{4}+h(x)|_{x=x_m}\ge 0,$$

$$-\beta v_2(y)+\frac{1}{2}\sigma^2y^2\bar{X}_2+Ayp_2-\frac{|p_2|^2}{4}+h(y)|_{y=y_m}\le 0.$$

Putting these inequalities together, we get

$$\begin{aligned}\beta[v_1(x_m)-v_2(y_m)] &\le \frac{1}{2}\sigma^2({x_m}^2\bar{X}_1-{y_m}^2\bar{X}_2)+A(x_mp_1-y_mp_2)\\ &\quad -\frac{1}{4}\left((|p_1|)^2-(|p_2|)^2\right)+h(x_m)-h(y_m)\\ &\le \frac{1}{2}\sigma^2\eta k(k-1)[|x_m|^k+|y_m|^k]+Am(x_m-y_m)^2+A\eta k[|x_m|^k+|y_m|^k]\\ &\quad -\frac{1}{4}\Big[(|m(x_m-y_m)+\eta k|x_m|^{k-1}sgn(x_m)|)^2\\ &\quad -(|m(x_m-y_m)-\eta k|y_m|^{k-1}sgn(y_m)|)^2\Big]+h(x_m)-h(y_m).\end{aligned}$$

Letting $m\to\infty$, we have

$$\begin{aligned}\beta[v_1(\tilde{x})-v_2(\tilde{x})] &\le 2\eta\left[\frac{1}{2}\sigma^2\tilde{x}^2k(k-1)|\tilde{x}|^{k-2}+A\tilde{x}k|\tilde{x}|^{k-1}-\frac{k^2}{4}(|\tilde{x}|^{k-1})^2\right]\\ &\le 2\eta\beta(1+|\tilde{x}|^k),\end{aligned}$$

which follows from (22). This is contrary with (29), completing the proof of Theorem 2.5. □

3. Classical solutions

We study here the smoothness of the viscosity solution u of (5).

Proposition 3.1 *We assume (3), (4), (7) and (11); further, we assume that the solution is convex. Then, $v_L(x)$ and $u(x)$ are convex.*

Proof. For any $\epsilon > 0$, there exist $c, \hat{c} \in \mathcal{A}_L$ such that

$$\begin{aligned} E[\int_0^\infty e^{-\beta t}\{h(x_t) + |c_t|^2\}dt] &< v_L(x) + \epsilon, \\ E[\int_0^\infty e^{-\beta t}\{h(\hat{x}_t) + |\hat{c}_t|^2\}dt] &< v_L(\hat{x}) + \epsilon, \end{aligned}$$

where

$$\begin{aligned} dx_t &= [Ax_t + c_t]dt + \sigma x_t dw_t, \quad x_0 = x \in \mathbf{R}, \\ d\hat{x}_t &= [A\hat{x}_t + \hat{c}_t]dt + \sigma \hat{x}_t dw_t, \quad \hat{x}_0 = \hat{x} \in \mathbf{R}. \end{aligned}$$

We set

$$\begin{aligned} \tilde{c}_t &= \xi c_t + (1-\xi)\hat{c}_t, \\ \tilde{x}_t &= \xi x_t + (1-\xi)\hat{x}_t, \\ \tilde{x}_0 &= \xi x + (1-\xi)\hat{x} \equiv \tilde{x}, \end{aligned}$$

for $0 < \xi < 1$. Clearly,

$$d\tilde{x}_t = [A\tilde{x}_t + \tilde{c}_t]dt + \sigma \tilde{x}_t dw_t.$$

Hence, by convexity

$$\begin{aligned} v_L(\tilde{x}) &\leq E[\int_0^\infty e^{-\beta t}\{h(\tilde{x}_t) + |\tilde{c}_t|^2\}dt] \\ &\leq \xi E[\int_0^\infty e^{-\beta t}\{h(x_t) + |c_t|^2\}dt] + (1-\xi)E[\int_0^\infty e^{-\beta t}\{h(\hat{x}_t) + |\hat{c}_t|^2\}dt] \\ &\leq \xi(v_L(x) + \epsilon) + (1-\xi)(v_L(\hat{x}) + \epsilon). \end{aligned}$$

Letting $\epsilon \longrightarrow 0$, we get

$$v_L(\tilde{x}) = v_L(\xi x + (1-\xi)\hat{x}) \leq \xi v_L(x) + (1-\xi)v_L(\hat{x}),$$

which completes the convexity of $v_L(x)$. From the definition of $v_L(x)$, for each positive integer L, we have $0 \le v_{L+1}(x) \le v_L(x),\ x \in \mathbf{R}$. Since $v_L(x)$ is non-increasing, we can define $u(x)$ by $u(x) = \lim_{L\to\infty} v_L(x)$. Hence, we see that $u(x)$ is also convex. $\square$

Theorem 3.2 *We assume (3), (4), (7) and (11). Then we have*

$$u \in C^2(\mathbf{R} \setminus \{0\}). \tag{30}$$

Proof. **Step 1**: By the convexity of u we recall a classical result of Alexandrov [6] to see that Lebesgue measure of $\mathbf{R} \setminus \mathcal{D} \cup \{0\}$ is 0, where $\mathcal{D} = \Big\{x \in \mathbf{R} : u \text{ is twice differentiable at x}\Big\}$. By the definition of twice-differentiability, we have $(u'(x), u''(x)) \in \mathbf{J}^{+2}u(x) \cap \mathbf{J}^{-2}u(x)$ for all $x \in \mathcal{D}$, and hence

$$-\beta u + \frac{1}{2}\sigma^2 x^2 u'' + Axu' - \frac{(u')^2}{4} + h(x) = 0, \quad \forall x \in \mathcal{D}.$$

Let $d^+u(x)$ and $d^-u(x)$ denote the right- and left-hand derivatives respectively. Define $r^{\pm}(x)$ by

$$-\beta u(x) + \frac{1}{2}\sigma^2 x^2 r^{\pm}(x) + Axd^{\pm}u(x) - \frac{(d^{\pm}u(x))^2}{4} + h(x) = 0 \quad \forall x \in (\mathbf{R} \setminus \{\mathbf{0}\}). \tag{31}$$

Since $d^+u = d^-u = u'$ *on* $\mathcal{D}$, we have $r^+ = r^- = u''$ a.e. By definition, $d^+u(x)$ is right continuous, and so is $r^+(x)$. Hence it is easy to see that

$$u(y) - u(x) = \int_x^y d^+u(s)ds$$

$$d^+u(s) - d^+u(x) = \int_x^s r^+(t)dt, \quad s > x.$$

Thus we get

$$\begin{aligned} R(u; y) : &= \Big\{u(y) - u(x) - d^+u(x)(y-x) - \frac{1}{2}r^+(x)|y-x|^2\Big\}/|y-x|^2 \\ &= \int_x^y \Big(d^+u(s) - d^+u(x) - r^+(x)(s-x)\Big)ds/|y-x|^2 \qquad (32) \\ &= \int_x^y \Big\{\int_x^s \Big(r^+(t) - r^+(x)\Big)dt\Big\}ds/|y-x|^2 \longrightarrow 0 \text{ as } y \downarrow x. \end{aligned}$$

Step 2: We claim that $u(x)$ is differentiable at $x \in \mathbf{R} \setminus \mathcal{D} \cup \{0\} = 0$. It is well known in [2] and [4] that $\delta u(x) = \Big[d^+u(x), d^-u(x)\Big]$ $\forall\, x \in (\mathbf{R} \setminus \{0\})$, where $\delta u(x)$ is the generalized gradient of u at x. Suppose $d^+u(x) > d^-u(x)$. We set

$$\begin{aligned}\hat{p} &= \xi d^+u(x) + (1-\xi)d^-u(x)\\ \hat{r} &= \xi r^+(x) + (1-\xi)r^-(x), \quad 0 < \xi < 1.\end{aligned}$$

If $\liminf_{y \to x} R(u; y) < 0$, then we can find a sequence $y_m \longrightarrow x$ such that $\lim_{m\to\infty} R(u; y_m) < 0$. By (32), we may consider that $y_m \leq y_{m+1} < x$ for every m, taking a subsequence if necessary. Hence

$$\lim_{m\to\infty} \frac{u(y_m) - u(x) - d^+u(x)(y_m - x)}{|y_m - x|} \leq 0.$$

This leads to $d^+u(x) \leq d^-u(x)$, which is a contradiction. Thus we have $(d^+u(x), r^+(x)) \in J^{2,-}u(x)$ and similarly, $(d^-u(x), r^-(x)) \in J^{2,-}u(x)$. By the convexity of $J^{2,-}u(x)$, we get $(\hat{p}, \hat{r}) \in J^{2,-}u(x)$. Now we note that

$$(\hat{p})^2 < \xi(d^+u(x))^2 + (1-\xi)(d^-u(x))^2,$$

and hence by (31)

$$-\beta u(x) + \frac{1}{2}\sigma^2 x^2 \hat{r} + Ax\hat{p} - \frac{(\hat{p})^2}{4} + h(x) > 0.$$

On the other hand, by the definition of viscosity solution

$$-\beta u(x) + \frac{1}{2}\sigma^2 x^2 q + Axp - \frac{p^2}{4} + h(x) \leq 0 \quad \forall (p,q) \in J^{2,-}u(x),$$

which is a contradiction. Therefore we deduce that $\delta u(x)$ is a singleton, and so u is differentiable at x [2].

Step 3: We claim that u' is continuous on $(\mathbf{R} \setminus \{\mathbf{0}\})$. Let $x_m \longrightarrow x$ and $p_m = u'(x_m) \longrightarrow p$. Then we have by convexity $u(y) \geq u(x) + p(y - x)$, for all y. Hence we see that $p \in D^-u(x)$, where

$$D^-u(x) = \{p \in \mathbf{R} : \liminf_{y\to x}\{u(y) - u(x) - p(y-x)\}/|y-x| \geq 0\}.$$

Since $\delta u(x) = D^- u(x)$ and $\delta u(x)$ is a singleton, we deduce $p = u'(x)$ ([2], Proposition 4.7, p. 66). **Step 4**: We set $w = u'$. Since

$$-\beta w(x_m) + \frac{1}{2}\sigma^2 {x_m}^2 w'(x_m) + Ax_m w(x_m) - \frac{(w(x_m))^2}{4} + h(x_m) = 0 \quad x_m \in \mathbf{D},$$

the sequence $\{w'(x_m)\}$ converges uniquely as $x_m \longrightarrow x \in \mathbf{R} \setminus \mathcal{D} \cup \{0\}$, and w is Lipschitz near x by monotonicity. Hence, we have a well-known result [4] in nonsmooth analysis that $\delta w(x)$ coincides with the convex hull of the set

$$\mathbf{D}^* w(x) = \left\{ q \in \mathbf{R} : q = \lim_{m\to\infty} w'(x_m),\ x_m \to x \right\}.$$

Then

$$-\beta u(x) + \frac{1}{2}\sigma^2 x^2 q + Axw(x) - \frac{(w'(x))^2}{4} + h(x) = 0 \quad \forall q \in \delta w(x).$$

Hence we observe that $\delta w(x)$ is a singleton, and then $w(x)$ is differentiable at x. The continuity of $w'(x)$ follows immediately. Thus we conclude that $w \in C^1(\mathbf{R} \setminus \{0\})$ and $(\mathbf{R} \setminus \mathcal{D} \cup \{0\})$ is empty. The proof is complete. □

Theorem 3.3 *We assume (3), (4), (7) and (11). Further, we assume that*

$$h(x)/x^2 \to \hat{h} \in \mathbf{R}_+ \ \textit{as} \ \ x \to 0. \tag{33}$$

Then we have

$$u \in C^1(\mathbf{R}) \cap C^2(\mathbf{R} \setminus \{0\}). \tag{34}$$

In addition, if $\hat{h} = 0$, then

$$u \in C^2(\mathbf{R}). \tag{35}$$

Proof. We first observe that v_L is a viscosity solution of the boundary value problem:

$$V'' + G(x, V, V') = 0 \quad \text{in} \ \ (a, b) \tag{36}$$

$$V(a) = v_L(a), \quad V(b) = v_L(b),$$

for any interval $[a, b] \subset \mathbf{R} \setminus \{0\}$ where

$$G(x, V, V') = 2\{-\alpha V + AxV' + \min_{|r|\leq L}(|r|^2 + rV') + h(x)\}/\sigma^2 x^2 = 0.$$

Standard elliptic regularity theory Fleming and Soner ([6], Theorem 4.1) and the uniqueness of viscosity solutions by Crandall, Ishii and Lions [5] yield that v_L is smooth in (a, b). Thus

$$\begin{aligned} |\min_{|r|\leq L}(|r|^2 + rv_L')| &\leq |\min_{r\in\mathbf{R}}(|r|^2 + rv_L')| = (|v_L'|/2)^2 \\ &\leq \{(|v_L'|/2)^2 + 1\}. \end{aligned}$$

By Theorem 3.2, we have $u \in C^2(\mathbf{R} \setminus \{0\})$.

To prove (34), it suffices to show that u has the following property:

$$u'(x) = o(1) \text{ as } x \to 0. \tag{37}$$

By (33), there exists $\lambda > 0$, for any $\varepsilon > 0$ such that $h(x) \leq (\hat{h} + \varepsilon)x^2$ for $|x| < \lambda$, and hence, by (4)

$$h(x) \leq (\hat{h} + \varepsilon)x^2 + C(1/\lambda^n + 1)|x|^n, \quad \forall x \in \mathbf{R}. \tag{38}$$

Note that $u(x) \leq E[\int_0^\infty e^{-\beta t} h(x_t^0)dt]$. Then by (13) we have

$$u'(x) = 0(x^2) \text{ as } x \to 0. \tag{39}$$

Now, by convexity

$$u(y) \geq u(x) + u'(x)(y - x), x \neq 0.$$

Substituting $y = 2x$, and $y = 0$ we get $u(2x) \geq u(x) + u'(x)x$ and $u(x) - u'(x)x \leq u(0) = 0$ by (39). Hence

$$\frac{u(2x)}{x^2} \geq \frac{u'(x)}{x} \geq \frac{u(x)}{x^2}, \tag{40}$$

which implies (37).

Finally, suppose $\hat{h} = 0$. Then, by virtue of (38), we have $u(x) = o(x^2)$ as $x \to 0$. Moreover, by (40), $u'(x) = o(x)$ as $x \to 0$. Dividing (5) by x^2 and passing to the limit, we get $u''(0) = 0$, which implies (35). □

4. An application to quadratic control theory

We now study the quadratic control problem (1) over the class $\mathcal{A}_{ad}$ of admissible controls, subject to (2), where
$\mathcal{A}_{ad} = \{c = (c_t) \in \mathcal{A} : \lim_{T\to\infty} E[e^{-\beta T}|x_T|^n] = 0$ for the response x_t to $c_t\}$. We consider the stochastic differential equation

$$dx_t^* = [Ax_t^* - u'(x_t^*)/2]dt + \sigma x_t^* dw_t, \quad x_0^* = x. \tag{41}$$

Theorem 4.1 *We assume (3), (4), (7), (11) and (33). Then the optimal control c_t^* is given by*

$$c_t^* = -u'(x_t^*)/2. \tag{42}$$

Proof. Since u' is continuous, (41) admits a weak solution x_t^* up to explosion time $\sigma = \inf\{t : |x_t^*| = \infty\}$ [**?**]. Taking into account $xu'(x) \geq 0$, we can show $(x_t^*)^2 \leq (x_t^0)^2$ by the comparison theorem. Hence $\sigma = \infty$. By the monotonicity of $u'(x)$, the uniqueness of (41) holds. Thus we conclude that (41) has a unique strong solution (x_t^*).

It follows from (14) that

$$E[e^{-\beta T}(1 + |x_T^*|^n)] \leq e^{-(\beta-\beta_0)T} E[e^{-\beta_0 T} f_n(x_T^0)] \longrightarrow 0 \quad \text{as} \quad T \to \infty,$$

where x_t^0 is a unique solution of (18). So $(c_t^*) \in \mathcal{A}_{ad}$. Since u satisfies (4), we see by (40) and (13) that

$$\begin{aligned} E[\int_0^T e^{-2\beta t}(x_t^* u'(x_t^*))^2 dt] &\leq E[\int_0^T e^{-2\beta t} u(2x_t^*)^2 dt] \\ &\leq CE[\int_0^T e^{-2\beta t}(1 + |x_t^*|^{2n}) dt] \\ &\leq CE[\int_0^T e^{-2\beta t} f_{2n}(x_t^0) dt] < \infty, \end{aligned}$$

and hence $\int_0^t e^{-\beta s} \sigma x_s^* u'(x_s^*) dw_s$ is a martingale. Then we apply Ito's formula for convex functions [7, p. 219] to obtain

$$\begin{aligned} E[e^{-\beta T} u(x_T^*)] &= u(x) + E\left[\int_0^T e^{-\beta t}\Big(- \beta u + Axu' + c_t^* u' + \frac{1}{2}\sigma^2 x^2 u''\Big)|_{x=x_t^*} dt\right] \\ &= u(x) - E[\int_0^T e^{-\beta t}\{h(x_t^*) + |c_t^*|^2\} dt]. \end{aligned}$$

Passing to the limit, we have $J(c^*) = u(x)$. By the same calculation as above, we can see that

$$E[e^{-\beta T\wedge\tau_n} u(x_{T\wedge\tau_n})] \geq u(x) - E[\int_0^{T\wedge\tau_n} e^{-\beta t}\{h(x_t) + |c_t|^2\}dt],$$

where $\{\tau_n\}$ is a sequence of localizing stopping times for the local martingale. Letting $\tau_n \longrightarrow \infty$ and then $T \longrightarrow \infty$, we obtain $u(x) \leq J(c)$ for all $c \in \mathcal{A}_{ad}$. The proof is complete. □

5. Conclusion

We have studied the Linear quadratic regulatory control problem for degenerate diffusions. In this paper we have proved the existence of a viscosity and smooth solutions u of (5) by its convexity argument following that the value function $v_L(x)$ is a viscosity solution of (5), and have showed also this value function converges to a viscosity solution u, for large $L > 0$.

We can further study in general a *stochastic control problem* for linear degenerate systems to minimize the discounted expected cost:

$$J(c) = E\left[\int_0^\infty e^{-\beta t}\{h(x_t) + |c_t|^n\}dt\right]$$

over $c \in \mathcal{A}$ and subject to the degenerate stochastic differential equation (2) and a continuous function f on $\mathbf{R}$ such that (4) and (7); and, in addition,

$$k_0|x|^n - k_1 \leq h(x)$$

for some constants $k_0, k_1 > 0$ and for a fixed integer $n \geq 2$.

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