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Two-weight Norm Inequalities for Some Anisotropic Sublinear Operators

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Abstract

In this paper, we establish several general theorems for the boundedness of the anisotropic sublinear operators on a weighted Lebesgue space. Conditions of these theorems are satisfied by many important operators in analysis. We also give some applications the boundedness of the parabolic singular integral operators, and the maximal operators associated with them from one weighted Lebesgue space to another one. Using this results, we prove weighted embedding theorems for the anisotropic Sobolev spaces $W^{l_1,\dots,l_n}_{\omega_0,\omega_1,\dots,\omega_n}(\mathbb{R}^n)$.

Key Words: Weighted Lebesgue space, sublinear operators, anisotropic singular integral, two-weighted inequality.

1. Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space of points $x = (x_1, \dots, x_n)$ with norm $|x| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$, $\mathbb{R}^n_0 = \mathbb{R}^n \setminus \{0\}$ and $\mathbb{Z}$ be the set of integer numbers. Consider a real $n \times n$ matrix A with eigenvalues λ_j, $\mathrm{Re}\lambda_j > 0$, and let $Q = trA$ be its trace. The matrix A determines a one-parameter group $A_t = \exp(A \ln t), t > 0$ of nonsingular transformations of $\mathbb{R}^n$. Denote by diag $\{a_1, \dots, a_n\}$ the matrix with numbers $a_1, \dots, a_n$ on the main diagonal and zero off-diagonal elements. Associated with the group A_t is the

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A_t-homogeneous metric $\rho\ :\ \mathbb{R}^n_0 \to R_+$, $\rho(A_t x) = t\rho(x)$, which is smooth in $\mathbb{R}^n_0$.

For $x \in \mathbb{R}^n$ and $r > 0$ the ρ - ball (ρ - sphere) about x with radius r is defined to be the set

$$B(x,r) = \{y :\ \rho(x-y) < r\} \qquad (S(x,r) = \{y :\ \rho(x-y) = r\}).$$

Almost everywhere positive and locally integrable function $\omega : \mathbb{R}^n \to \mathbb{R}$ will be called a weight. We shall denote by $L_{p,\omega}(\mathbb{R}^n)$ the set of all measurable functions f on $\mathbb{R}^n$ such that the norm

$$\|f\|_{L_{p,\omega}(\mathbb{R}^n)} \equiv \|f\|_{p,\omega} = \left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx\right)^{1/p}, \qquad 1 \le p < \infty$$

is finite.

We say that the weighted pair (ω, ω_1) satisfies the Muckenhoupt's condition $A_p = A_p(\mathbb{R}^n, \rho)$ (briefly, $(\omega, \omega_1) \in A_p$), $1 < p < \infty$, if there is a constant $C = C(\omega, p)$ such that for any ρ - ball $B \subset \mathcal{B}$

$$\left(|B|^{-1}\int_B \omega(x)dx\right)\left(|B|^{-1}\int_B \omega_1^{1-p'}(x)dx\right) \le C, \quad \frac{1}{p} + \frac{1}{p'} = 1. \tag{1}$$

Here, $\mathcal{B} = \mathcal{B}(\rho)$ is the collection of all ρ-balls $B(x,r)$, $x \in \mathbb{R}^n$, $r > 0$.

For $p = 1$, we say $(\omega, \omega_1) \in A_1$, if

$$\left(|B|^{-1}\int_B \omega(x)dx\right)\left(\sup_B \omega_1^{-1}\right) \le C$$

for any ρ - ball $B \subset \mathcal{B}$.

We write also $\omega \in A_p$, if $\omega = \omega_1$. It is easy to verify that, $\rho(x)^\alpha \in A_p$ if and only if $-Q < \alpha < Q(p-1)$ for $1 < p < \infty$ and $\rho(x)^\alpha \in A_1$ if and only if $-Q < \alpha \le 0$.

A condition (1) was first introduced by Muckenhoupt [16] for weighted estimates of Hardy maximal functions.

Definition 1 *Function K defined on $\mathbb{R}^n_0$, is said to be an anisotropic Calderon-Zygmund (ACZ) kernel in the space $\mathbb{R}^n$ if*

i) $K \in C^\infty(\mathbb{R}^n \setminus \{0\})$ *;*

ii) $K(A_t x) = t^{-Q} K(x),\ \ t > 0,\ x \in \mathbb{R}^n_0$ *;*

iii) $\int_{S(0,1)} K(x)d\sigma(x) = 0$, *where $d\sigma$ is the element of area of the ρ-sphere $S(0,1)$.*

In this paper, we shall prove the boundedness of some anisotropic sublinear operators on a weighted L_p spaces. We point out that condition (2) (see below) in the isotropic case was first introduced by Soria and Weiss in [19]. Condition (2) is satisfied by many interesting operators in harmonic analysis, such as the anisotropic Calderon–Zygmund operators, anisotropic Hardy–Littlewood maximal operators, anisotropic R. Fefferman's singular integals, anisotropic Ricci–Stein's oscillatory singular integrals, the anisotropic Bochner–Riesz means and so on (see, for example [19]).

We also give some applications the boundedness of the parabolic singular integral operators and the maximal operators associated with them from one weighted Lebesgue space to another. Using these results, we prove two-weighted inequalities for linear means of Fourier integrals defined by a single function with support in a specially organized set. Weighted embedding theorems are obtained for the anisotropic Sobolev spaces $W^{l_1,\dots,l_n}_{\omega_0,\omega_1,\dots,\omega_n}(\mathbb{R}^n)$.

The structure of the paper is as follows. In Section 1 we present some definitions. In Section 2 we prove the boundedness of some anisotropic sublinear operators on a weighted L_p spaces. In Section 3 we give some applications. The main result of the paper is Theorem 1, established in Section 2.

For the sake of simplicity, the letter C always denotes a positive constant which may change from one step to the next.

2. Main theorems and their proofs

First, we shall establish the boundedness of a large class of sublinear operators in weighted L_p spaces.

In [18], in particular is proved the following theorem.

Theorem 1 *Let $p \in (1,\infty)$ and let T be a sublinear operator bounded from $L_p(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$ such that, for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin suppf$*

$$|Tf(x)| \le c_0 \int\limits_{\mathbb{R}^n} \frac{|f(y)|}{\rho(x-y)^Q} dy, \tag{2}$$

where c_0 is independent of f and x.

Let also ω be a positive function for which there exists a constant $\tilde{b} > 0$ such that

$$\sup_{2^{k-2} \le \rho(x) < 2^{k+1}} \omega(x) \le \tilde{b} \inf_{2^{k-2} \le \rho(x) < 2^{k+1}} \omega(x), \quad k \in \mathbb{Z}$$

and $\omega \in A_p$, then T is bounded in $L_{p,\omega}(\mathbb{R}^n)$.

Theorem 2 *Let $p \in (1, \infty)$, T be a sublinear operator bounded from $L_p(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$ and satisfying (2).*

Moreover, let $\omega(x)$, $\omega_1(x)$ be weight functions on $\mathbb{R}^n$ and the following three conditions are satisfied:

(a) *there exists $b > 0$ such that*

$$\sup_{\rho(x)/4 < \rho(y) \le 4\rho(x)} \omega_1(y) \le b\, \omega(x) \quad \text{for a.e. } x \in \mathbb{R}^n,$$

$$(b) \quad \mathcal{A} \equiv \sup_{r>0} \left(\int_{\rho(x)>2r} \omega_1(x) \rho(x)^{-Qp} dx \right) \left(\int_{\rho(x)<r} \omega^{1-p'}(x) dx \right)^{p-1} < \infty,$$

$$(c) \quad \mathcal{B} \equiv \sup_{r>0} \left(\int_{\rho(x)<r} \omega_1(x) dx \right) \left(\int_{\rho(x)>2r} \omega^{1-p'}(x) \rho(x)^{-Qp'} dx \right)^{p-1} < \infty.$$

Then there exists a constant C, independent of f, such that for all $f \in L_{p,\omega}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |Tf(x)|^p \omega_1(x) dx \le C \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx. \tag{3}$$

Moreover, condition (a) can be replaced by condition

(a') *and there exists $b > 0$ such that*

$$\omega_1(x) \left(\sup_{\rho(x)/4 \le \rho(y) \le 4\rho(x)} \frac{1}{\omega(y)} \right) \le b \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Proof. For $k \in Z$ we define $E_k = \{x \in \mathbb{R}^n : 2^k < \rho(x) \le 2^{k+1}\}$, $E_{k,1} = \{x \in \mathbb{R}^n : \rho(x) \le 2^{k-1}\}$, $E_{k,2} = \{x \in \mathbb{R}^n : 2^{k-1} < \rho(x) \le 2^{k+2}\}$, $E_{k,3} = \{x \in \mathbb{R}^n : \rho(x) > 2^{k+2}\}$. Then $E_{k,2} = E_{k-1} \cup E_k \cup E_{k+1}$ and the multiplicity of the covering $\{E_{k,2}\}_{k \in Z}$ is equal to 3.

Given $f \in L_{p,\omega}(\mathbb{R}^n)$, we write

$$|Tf(x)| = \sum_{k \in Z} |Tf(x)| \chi_{E_k}(x) \le \sum_{k \in Z} |Tf_{k,1}(x)|\, \chi_{E_k}(x)$$

$$+\sum_{k\in Z}|Tf_{k,2}(x)|\,\chi_{E_k}(x)+\sum_{k\in Z}|Tf_{k,3}(x)|\,\chi_{E_k}(x)$$

$$\equiv T_1f(x)+T_2f(x)+T_3f(x), \tag{4}$$

where χ_{E_k} is the characteristic function of the set E_k, $f_{k,i}=f\chi_{E_{k,i}}$, $i=1,2,3$.

First we shall estimate $\|T_1f\|_{L_{p,\omega_1}}$. Note that for $x\in E_k$, $y\in E_{k,1}$ we have $\rho(y)\le 2^{k-1}\le\rho(x)/2$. Moreover, $E_k\cap supp f_{k,1}=\emptyset$ and $\rho(x-y)\ge\rho(x)/2$. Hence by (2)

$$T_1f(x)\le c_0\sum_{k\in Z}\left(\int_{\mathbb{R}^n}\frac{|f_{k,1}(y)|}{\rho(x-y)^Q}dy\right)\chi_{E_k}$$

$$\le c_0\int_{\rho(y)\le\rho(x)/2}\rho(x-y)^{-Q}|f(y)|dy$$

$$\le 2^Qc_0\rho(x)^{-Q}\int_{\rho(y)\le\rho(x)/2}|f(y)|dy$$

for any $x\in E_k$. Hence we have

$$\int_{\mathbb{R}^n}|T_1f(x)|^p\omega_1(x)dx\le\left(2^Qc_0\right)^p\int_{\mathbb{R}^n}\left(\int_{\rho(y)\le\rho(x)/2}|f(y)|dy\right)^p\rho(x)^{-Qp}\omega_1(x)dx.$$

Since $\mathcal{A}<\infty$, the Hardy inequality

$$\int_{\mathbb{R}^n}\omega_1(x)\rho(x)^{-Qp}\left(\int_{\rho(y)\le\rho(x)/2}|f(y)|dy\right)^p dx\le C\int_{\mathbb{R}^n}|f(x)|^p\omega(x)dx$$

holds and $C\le c'\mathcal{A}$, where c' depends only on n, a and p. In fact, the condition $\mathcal{A}<\infty$ is necessary and sufficient for the validity of this inequality (see [2] and [12]). Hence, we obtain

$$\int_{\mathbb{R}^n}|T_1f(x)|^p\omega_1(x)dx\le c_1\int_{\mathbb{R}^n}|f(x)|^p\omega(x)dx, \tag{5}$$

where c_1 is independent of f.

Next we estimate $\|T_3f\|_{L_{p,\omega_1}}$. As it is easy to verify, for $x\in E_k$, $y\in E_{k,3}$ we have $\rho(y)>2\rho(x)$ and $\rho(x-y)\ge\rho(y)/2$. Since $E_k\cap supp f_{k,3}=\emptyset$, for $x\in E_k$ by (2), we obtain

$$T_3f(x)\le c_0\int_{\rho(y)>2\rho(x)}\frac{|f(y)|}{\rho(x-y)^Q}dy\le 2^Qc_0\int_{\rho(y)>2\rho(x)}|f(y)|\rho(y)^{-Q}dy.$$

Hence we have

$$\int_{\mathbb{R}^n} |T_3 f(x)|^p \omega_1(x)dx \le \left(2^Q c_0\right)^p \int_{\mathbb{R}^n} \left(\int_{\rho(y)>2\rho(x)} |f(y)|\rho(y)^{-Q}\, dy\right)^p \omega_1(x)dx.$$

Since $\mathcal{B} < \infty$, the Hardy inequality

$$\int_{\mathbb{R}^n} \omega_1(x) \left(\int_{\rho(y)>2\rho(x)} |f(y)|\rho(y)^{-Q} dy\right)^p dx \le C \int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx$$

holds and $C \le c'\mathcal{B}$, where c' depends only on n and p. In fact the condition $\mathcal{B} < \infty$ is necessary and sufficient for the validity of this inequality (see [2] and [12]). Hence, we obtain

$$\int_{\mathbb{R}^n} |T_3 f(x)|^p \omega_1(x)dx \le c_2 \int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx, \tag{6}$$

where c_2 is independent of f.

Finally, we estimate $\|T_2 f\|_{L_{p,\omega_1}}$. By the $L_p(\mathbb{R}^n)$ boundedness of T and condition (a) we have

$$\int_{\mathbb{R}^n} |T_2 f(x)|^p \omega_1(x)dx = \int_{\mathbb{R}^n} \left(\sum_{k\in Z} |Tf_{k,2}(x)|\, \chi_{E_k}(x)\right)^p \omega_1(x)dx$$

$$= \int_{\mathbb{R}^n} \left(\sum_{k\in Z} |Tf_{k,2}(x)|^p\, \chi_{E_k}(x)\right) \omega_1(x)dx = \sum_{k\in Z} \int_{E_k} |Tf_{k,2}(x)|^p\, \omega_1(x)dx$$

$$\le \sum_{k\in Z} \sup_{x\in E_k} \omega_1(x) \int_{\mathbb{R}^n} |Tf_{k,2}(x)|^p\, dx \le \|T\|^p \sum_{k\in Z} \sup_{x\in E_k} \omega_1(x) \int_{\mathbb{R}^n} |f_{k,2}(x)|^p\, dx$$

$$= \|T\|^p \sum_{k\in Z} \sup_{y\in E_k} \omega_1(y) \int_{E_{k,2}} |f(x)|^p dx,$$

where $\|T\| \equiv \|T\|_{L_p(\mathbb{R}^n)\to L_p(\mathbb{R}^n)}$. Since, for $x \in E_{k,2}$ $2^{k-1} < \rho(x) \le 2^{k+2}$, we have by condition (a)

$$\sup_{y\in E_k} \omega_1(y) = \sup_{2^{k-1}<\rho(y)\le 2^{k+2}} \omega_1(y) \le \sup_{\rho(x)/4<\rho(y)\le 4\rho(x)} \omega_1(y) \le b\,\omega(x)$$

for almost all $x \in E_{k,2}$. Therefore

$$\int_{\mathbb{R}^n} |T_2 f(x)|^p \omega_1(x) dx$$

$$\leq \|T\|^p b \sum_{k \in Z} \int_{E_{k,2}} |f(x)|^p \omega(x) dx \leq c_3 \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx, \tag{7}$$

where $c_3 = 3\|T\|^p b$, since the multiplicity of covering $\{E_{k,2}\}_{k \in Z}$ is equal to 3.

Inequalities (4), (5), (6) and (7) imply (3) which completes the proof. □

Similarly we prove the following weak variant of Theorem 2.

Theorem 3 *Let $p \in [1, \infty)$ and let T be a sublinear operator bounded from $L_p(\mathbb{R}^n)$ to $WL_p(\mathbb{R}^n)$, i.e.,*

$$\int_{\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}} dx \leq \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p dx$$

and satisfying (2). Moreover, let $\omega(x)$, $\omega_1(x)$ be weight functions on $\mathbb{R}^n$ and conditions (a), (b), (c) be satisfied.

Then there exists a constant C_1, independent of f, such that for all $f \in L_{p,\omega}(\mathbb{R}^n)$

$$\int_{\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}} \omega_1(x) dx \leq \frac{C_1}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx. \tag{8}$$

Let K be an ACZ kernel and T be the corresponding integral operator

$$Tf(x) = p.v. \int_{\mathbb{R}^n} K(x-y) f(y) dy \equiv \lim_{r \to 0} \int_{\mathbb{R}^n \setminus B(x,r)} K(x-y) f(y) dy.$$

Then T satisfies the condition (2). See [19] for details. Thus, we have the following corollary.

Corollary 1 *Let $p \in (1, \infty)$ ($p \in [1, \infty)$), K be an ACZ kernel and T be the corresponding integral operator. Moreover, let $\omega(x)$, $\omega_1(x)$ be weight functions on $\mathbb{R}^n$ and conditions (a), (b), (c) be satisfied. Then inequality (3) ((8)) holds.*

Remark 1 *A sufficient condition for the Calderon–Zygmund operator* $T : L_{p,\omega}(\mathbb{R}^n) \to L_{p,\omega_1}(\mathbb{R}^n)$ *was found by N. Fuji [11], however the condition he introduced is not easy to check for given weights. Recently, Guliyev [9] and Edmunds and Kokilashvili [13] found new sufficient conditions easily verifiable for Calderon–Zygmund operator* $T : L_{p,\omega}(\mathbb{R}^n) \to L_{p,\omega_1}(\mathbb{R}^n)$*, whenever* $\omega(\cdot)$ *and* $\omega_1(\cdot)$ *are radial monotone weights. In the paper by Y. Rakotondratsimba [23], Corollary 1 was proved for Calderon–Zygmund operator* T*. Note that, for singular integral operators, defined on homogeneous groups analog, Corollary 1 was proved in [15] and [8] (see also [10], [1], [3] and [9]).*

Theorem 4 *Let* $p \in (1,\infty)$*,* T *be a sublinear operator bounded from* $L_p(\mathbb{R}^n)$ *to* $L_p(\mathbb{R}^n)$ *and satisfying* (2).

1) Let $\omega(t)$ *be a weight function on* $(0,\infty)$*,* $\omega_1(t)$ *be a positive increasing function on* $(0,\infty)$ *and the weighted pair* $(\omega(\rho(x)),\, \omega_1(\rho(x)))$ *satisfies conditions* (a)*,* (b)*. Then there exists a constant* C*, independent of* f*, such that for all* $f \in L_{p,\omega(\rho(x))}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |Tf(x)|^p \omega_1(\rho(x))dx \le C \int_{\mathbb{R}^n} |f(x)|^p \omega(\rho(x))dx. \tag{9}$$

2) Let $\omega(t)$ *be a weight function on* $(0,\infty)$*,* $\omega_1(t)$ *be a positive decreasing function on* $(0,\infty)$ *and the weighted pair* $(\omega(\rho(x)),\, \omega_1(\rho(x)))$ *satisfies the conditions* (a)*,* (c)*. Then inequality* (9) *is valid.*

Proof. 1) Suppose that $f \in L_{p,\omega}(\mathbb{R}^n)$, $\omega(t)$ be a weight function on $(0,\infty)$ and ω_1 is a positive increasing function on $(0,\infty)$ and $(\omega(\rho(x)),\, \omega_1(\rho(x)))$ satisfies conditions (a), (b).

Without the loss of generality we can suppose that ω_1 may be represented by

$$\omega_1(t) = \omega_1(0+) + \int_0^t \psi(\lambda)d\lambda,$$

where $\omega_1(0+) = \lim_{t\to 0}\omega_1(t)$ and $\omega_1(t) \ge 0$ on $(0,\infty)$. In fact there exists a sequence of increasing absolutely continuous functions ϖ_n such that $\varpi_n(t) \le \omega_1(t)$ and $\lim\limits_{n\to\infty} \varpi_n(t) = \omega_1(t)$ for any $t \in (0,\infty)$ (see [7] and [9] for details).

We have

$$\int_{\mathbb{R}^n} |Tf(x)|^p \omega_1(\rho(x))dx = \omega_1(0+) \int_{\mathbb{R}^n} |Tf(x)|^p dx +$$

$$+\int_{\mathbb{R}^n} |Tf(x)|^p \left(\int_0^{\rho(x)} \psi(\lambda)d\lambda\right) dx = J_1 + J_2.$$

If $\omega_1(0+) = 0$, then $J_1 = 0$. If $\omega_1(0+) \neq 0$, then by the boundedness of T in $L_p(\mathbb{R}^n)$, thanks to (a), we have

$$J_1 \leq \|T\|^p \omega_1(0+) \int_{\mathbb{R}^n} |f(x)|^p dx \leq$$

$$\leq \|T\|^p \int_{\mathbb{R}^n} |f(x)|^p \omega_1(\rho(x)) dx \leq b\, \|T\|^p \int_{\mathbb{R}^n} |f(x)|^p \omega(\rho(x)) dx.$$

After changing the order of integration in J_2 we have

$$J_2 = \int_0^\infty \psi(\lambda) \left(\int_{\rho(x)>\lambda} |Tf(x)|^p dx\right) d\lambda \leq$$

$$\leq 2^{p-1} \int_0^\infty \psi(\lambda) \left(\int_{\rho(x)>\lambda} |T(f\chi_{\{\rho(x)>\lambda/2\}})(x)|^p dx +\right.$$

$$\left.+\int_{\rho(x)>\lambda} |T(f\chi_{\{\rho(x)\leq\lambda/2\}})(x)|^p dx\right) d\lambda = 2^{p-1}\left(J_{21} + J_{22}\right).$$

Using the boundedness of T in $L_p(\mathbb{R}^n)$ and condition (a) we have

$$J_{21} \leq \|T\|^p \int_0^\infty \psi(t) \left(\int_{\rho(y)>\lambda/2} |f(y)|^p dy\right) dt$$

$$= \|T\|^p \int_{\mathbb{R}^n} |f(y)|^p \left(\int_0^{2\rho(y)} \psi(\lambda)d\lambda\right) dy$$

$$\leq \|T\|^p \int_{\mathbb{R}^n} |f(y)|^p \omega_1(2\rho(y)) dy$$

$$\leq b\, \|T\|^p \int_{\mathbb{R}^n} |f(y)|^p \omega(\rho(y)) dy.$$

Let us estimate J_{22}. For $\rho(x) > \lambda$ and $\rho(y) \leq \lambda/2$ we have $\rho(x)/2 \leq \rho(x-y) \leq 3\rho(x)/2$, and so

$$J_{22} \leq c_4 \int_0^\infty \psi(\lambda) \left(\int_{\rho(x)>\lambda} \left(\int_{\rho(y)\leq 2\lambda} \frac{|f(y)|}{\rho(x-y)^Q} dy\right)^p dx\right) d\lambda$$

$$\le c_5 \int_0^\infty \psi(\lambda) \left(\int_{\rho(x)>\lambda} \left(\int_{\rho(y)\le 2\lambda} |f(y)|dy \right)^p \rho(x)^{-Qp} dx \right) d\lambda$$

$$= c_6 \int_0^\infty \psi(\lambda) \lambda^{-Qp+Q} \left(\int_{\rho(y)\le \lambda/2} |f(y)|dy \right)^p d\lambda.$$

The Hardy inequality

$$\int_0^\infty \psi(\lambda) \lambda^{-Qp+Q} \left(\int_{\rho(y)\le \lambda/2} |f(y)|dy \right)^p d\lambda$$

$$\le C \int_{\mathbb{R}^n} |f(y)|^p \omega(\rho(y))dy$$

for $p \in (1, \infty)$ is characterized by the condition $C \le c'\mathcal{A}'$ (see [2], [12], see also [5], [14]), where

$$\mathcal{A}' \equiv \sup_{\tau>0} \left(\int_{2\tau}^\infty \psi(t) t^{-Qp+Q} d\tau \right) \left(\int_{\rho(x)<\tau} \omega^{1-p'}(x)dx \right)^{p-1} < \infty.$$

Note that

$$\int_{2t}^\infty \psi(\tau)\tau^{-Qp+Q} d\tau$$

$$= Q(p-1) \int_{2t}^\infty \psi(\tau)d\tau \int_\tau^\infty \lambda^{Q-1-Qp} d\lambda$$

$$= Q(p-1) \int_{2t}^\infty \lambda^{Q-1-Qp} d\lambda \int_{2t}^\lambda \psi(\tau)d\tau$$

$$\le Q(p-1) \int_{2t}^\infty \lambda^{Q-1-Qp} \omega_1(\lambda) d\lambda$$

$$= c_7 \int_{\rho(x)>2t} \omega_1(\rho(x))\rho(x)^{-Qp} dx.$$

Condition (b) of the theorem guarantees that $\mathcal{A}' \le c_7\mathcal{A} < \infty$. Hence, applying the Hardy inequality, we obtain

$$J_{22} \le c_8 \int_{\mathbb{R}^n} |f(x)|^p \omega(\rho(x))dx.$$

By the combining estimates of J_1 and J_2, we get (9) for $\omega_1(t) = \omega_1(0+) + \int_0^t \psi(\tau)d\tau$. By Fatou's theorem on passing to the limit under the Lebesgue integral sign, this implies (9). The first part of the Theorem 4 is proved.

2) Without loss of generality we can suppose that ω_1 may be represented by

$$\omega_1(t) = \omega_1(+\infty) + \int_t^\infty \psi(\tau)d\tau,$$

where $\omega_1(+\infty) = \lim\limits_{t\to\infty} \omega_1(t)$ and $\omega_1(t) \geq 0$ on $(0,\infty)$. In fact there exists a sequence of decreasing absolutely continuous fuctions ϖ_n such that $\varpi_n(t) \leq \omega_1(t)$ and $\lim_{n\to\infty} \varpi_n(t) = \omega_1(t)$ for any $t \in (0,\infty)$(see [7] and [9] for details).

We have

$$\int_{\mathbb{R}^n} |Tf(x)|^p \omega_1(\rho(x))dx = \omega_1(+\infty) \int_{\mathbb{R}^n} |Tf(x)|^p dx +$$

$$+ \int_{\mathbb{R}^n} |Tf(x)|^p \left(\int_{\rho(x)}^\infty \psi(\tau)d\tau \right) dx = I_1 + I_2.$$

If $\omega_1(+\infty) = 0$, then $I_1 = 0$. If $\omega_1(+\infty) \neq 0$, by the boundedness of T in $L_p(\mathbb{R}^n)$ and condition (a) we have

$$J_1 \leq \|T\| \omega_1(+\infty) \int_{\mathbb{R}^n} |f(x)|^p dx \leq$$

$$\leq \|T\| \int_{\mathbb{R}^n} |f(x)|^p \omega_1(\rho(x))dx$$

$$\leq b\,\|T\| \int_{\mathbb{R}^n} |f(x)|^p \omega(\rho(x))dx.$$

After changing the order of integration in J_2, we have

$$J_2 = \int_0^\infty \psi(\lambda) \left(\int_{\rho(x)<\lambda} |Tf(x)|^p dx \right) d\lambda$$

$$\leq 2^{p-1} \int_0^\infty \psi(\lambda) \left(\int_{\rho(x)<\lambda} |T(f\chi_{\{\rho(x)<2\lambda\}})(x)|^p dx \right.$$

$$\left. + \int_{\rho(x)<\lambda} |T(f\chi_{\{\rho(x)\geq 2\lambda\}})(x)|^p dx \right) d\lambda = J_{21} + J_{22}.$$

Using the boundedness of T in $L_p(\mathbb{R}^n)$ and condition (a'), we obtain

$$J_{21} \le \|T\| \int_0^\infty \psi(t) \left(\int_{\rho(y)<2\lambda} |f(y)|^p dy \right) dt$$

$$= \|T\| \int_{\mathbb{R}^n} |f(y)|^p \left(\int_{\rho(y)/2}^\infty \psi(\lambda) d\lambda \right) dy$$

$$\le \|T\| \int_{\mathbb{R}^n} |f(y)|^p \omega_1(\rho(y)/2) dy$$

$$\le b \, \|T\| \int_{\mathbb{R}^n} |f(y)|^p \omega(\rho(y)) dy.$$

Let us estimate J_{22}. For $\rho(x) < \lambda$ and $\rho(y) \ge 2\lambda$ we have $\rho(y)/2 \le \rho(x-y) \le 3\rho(y)/2$, and so

$$J_{22} \le c_8 \int_0^\infty \psi(\lambda) \left(\int_{\rho(x)<\lambda} \left(\int_{\rho(y)\ge 2\lambda} \frac{|f(y)|}{\rho(x-y)^Q} dy \right)^p dx \right) d\lambda \le$$

$$\le 2^Q c_8 \int_0^\infty \psi(\lambda) \left(\int_{\rho(x)<\lambda} \left(\int_{\rho(y)\ge 2\lambda} \rho(y)^{-Q} |f(y)| dy \right)^p dx \right) d\lambda =$$

$$= c_9 \int_0^\infty \psi(\lambda) \lambda^Q \left(\int_{\rho(y)\ge 2\lambda} \rho(y)^{-Q} |f(y)| dy \right)^p d\lambda.$$

The Hardy inequality

$$\int_0^\infty \psi(\lambda) \lambda^Q \left(\int_{\rho(y)\ge 2\lambda} \rho(y)^{-Q} |f(y)| dy \right)^p d\lambda \le C \int_{\mathbb{R}^n} |f(y)|^p \omega(\rho(y)) dy$$

for $p \in (1, \infty)$ is characterized by the condition $C \le c\mathcal{B}'$ ([2], [12], see also [5], [14]), where

$$\mathcal{B}' \equiv \sup_{\tau>0} \left(\int_0^\tau \psi(t) t^Q d\tau \right) \left(\int_{\rho(x)>2\tau} \omega^{1-p'}(x) \rho(x)^{-Qp'} dx \right)^{p-1} < \infty.$$

Note that

$$\int_0^\tau \psi(t) t^Q dt = Q \int_0^\tau \psi(t) dt \int_0^t \lambda^{Q-1} d\lambda =$$

$$= Q \int_0^\tau \lambda^{Q-1} d\lambda \int_\lambda^t \psi(\tau) d\tau \le Q \int_0^\tau \lambda^{Q-1} \omega_1(\lambda) d\lambda =$$

$$= c_9 \int_{\rho(x)<\tau} \omega_1(\rho(x)) dx.$$

Condition (c) of the theorem guarantees that $\mathcal{B}' \le c_9 \mathcal{B} < \infty$. Hence, applying the Hardy inequality, we obtain

$$J_{22} \le c_{10} \int_{\mathbb{R}^n} |f(x)|^p \omega(\rho(x)) dx.$$

Combining the estimates of J_1 and J_2, we get (9) for $\omega_1(t) = \omega_1(+\infty) + \int_t^\infty \psi(\tau) d\tau$. By Fatou's theorem on passing to the limit under the Lebesgue integral sign, this implies (9). The theorem is proved. □

Corollary 2 *Let $p \in (1,\infty)$, K be ACZ kernel and T be the corresponding operator.*

1) Let $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive increasing function on $(0,\infty)$ and the weighted pair $(\omega(\rho(x)), \omega_1(\rho(x)))$ satisfies conditions (a), (b). Then inequality (9) is valid.

2) Let $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive decreasing function on $(0,\infty)$ and the weighted pair $(\omega(\rho(x)), \omega_1(\rho(x)))$ satisfies conditions (a), (c). Then inequality (9) is valid.

Similarly we prove the following weak variant Theorem 4.

Theorem 5 *Let $p \in [1,\infty)$, T be a sublinear operator bounded from $L_p(\mathbb{R}^n)$ to $WL_p(\mathbb{R}^n)$ and satisfying (2).*

1) Let $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive increasing function on $(0,\infty)$ and the weighted pair $(\omega(\rho(x)), \omega_1(\rho(x)))$ satisfies conditions (a), (b). Then there exists a constant c, independent of f, such that for all $f \in L_{p,\omega(\rho(x))}(\mathbb{R}^n)$

$$\int_{\{x\in\mathbb{R}^n : |Tf(x)|>\lambda\}} \omega_1(\rho(x)) dx \le \frac{c}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p \omega(\rho(x)) dx. \qquad (10)$$

2) Let $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive decreasing function on $(0,\infty)$ and the weighted pair $(\omega(\rho(x)), \omega_1(\rho(x)))$ satisfies conditions (a), (c). Then inequality (10) is valid.

Corollary 3 *Let $p \in [1,\infty)$, K be ACZ kernel and T be the corresponding operator.*

1) Let $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive increasing function on $(0,\infty)$ and the weighted pair $(\omega(\rho(x)), \omega_1(\rho(x)))$ satisfies conditions (a), (b). Then inequality (10) *is valid.*

2) Let $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive decreasing function on $(0,\infty)$ and the weighted pair $(\omega(\rho(x)), \omega_1(\rho(x)))$ satisfies conditions (a), (c). Then inequality (10) *is valid.*

Example 1 *Let*

$$\omega(t) = \begin{cases} t^{Q(p-1)} \ln^p \frac{1}{t}, & for \quad t \in \left(0, \frac{1}{2}\right) \\ \left(2^{\beta-p+1} \ln^p 2\right) t^{\beta}, & for \quad t \in \left[\frac{1}{2}, \infty\right), \end{cases}$$

$$\omega_1(t) = \begin{cases} t^{Q(p-1)}, & for \quad t \in \left(0, \frac{1}{2}\right), \\ 2^{\alpha-p+1} t^{\alpha}, & for \quad t \in \left[\frac{1}{2}, \infty\right), \end{cases}$$

where $0 < \alpha \le \beta < Q(p-1)$. Then the pair $(\omega(\rho(x)), \omega_1(\rho(x)))$ satisfies the condition of Theorem 4.

Example 2 *Let*

$$\omega(t) = \begin{cases} \frac{1}{t^Q} \ln^{\nu} \frac{1}{t}, & for \quad t < d \\ \left(d^{-Q-\alpha} \ln^{\nu} \frac{1}{d}\right) t^{\alpha}, & for \quad t \ge d \end{cases}$$

$$\omega_1(t) = \begin{cases} \frac{1}{t^Q} \ln^{\beta} \frac{1}{t}, & for \quad t < d, \\ \left(d^{-Q-\lambda} \ln^{\beta} \frac{1}{d}\right) t^{\lambda}, & for \quad t \ge d, \end{cases}$$

where $\beta < \nu \le 0$, $-Q < \lambda < \alpha < 0$, $d = e^{\frac{\beta}{Q}}$. Then the pair $(\omega(\rho(x)), \omega_1(\rho(x)))$ satisfies the condition of Theorem 4.

3. Some applications

In this section, we will give some applications of Theorem 1 in section 2.

3.1. The Euclidean space

In [17] the weak and strong estimates in weighted L_p spaces are obtained for linear means of Fourier integrals defined by a single function with support in a specially organized set.

For a function f integrable on the n-dimensional Euclidean space $\mathbb{R}^n$, written f in $L_1(\mathbb{R}^n)$, its Fourier transform is well defined

$$\hat{f}(x) = \int_{\mathbb{R}^n} f(y) e^{-ixy} dy,$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, $xy = x_1 y_1 + \ldots + x_n y_n$. Let

$$\int_D \hat{f}(x) e^{xy} dx$$

be the partial Fourier integral defined by a set D. The behavior of partial Fourier integrals with respect to a specifically organized family of such sets characterizes approximation properties of f. It is natural to define such a family as a sequence of dilations of a fixed set D. This has been extensively studied when D is the cube (cubic case)

$$D = \{x \in \mathbb{R}^n \ : \ |x_j| \leq 1, \ j = 1, \ldots, n\}$$

or the ball (spherical case)

$$D = \{x \in \mathbb{R}^n \ : \ |x| \leq 1\}.$$

Their R-dilations are

$$RD = \{x \in \mathbb{R}^n \ : \ |x_j| \leq R, \ j = 1, \ldots, n\}$$

and

$$RD = \{x \in \mathbb{R}^n \ : \ |x| \leq R\},$$

respectively. The other example of a family of sets is the family of rectangles

$$\{x \in \mathbb{R}^n \ : \ |x_j| \leq R_j, \ R_j > 0, \ j = 1, \ldots, n\}$$

that cannot be expressed as a family of dilations of a fixed set. Numerous results on these (as well as references) may be found, e.g., in [24], Chapter 17 or [22], where similar problems are studied for multiple Fourier series as well.

Paper [17] considered linear means of multiple Fourier integrals rather than partial sums, and efined them by the family of dilations of a set D from some special class. The latter is closer to the spherical case rather than to the other ones. The estimates are obtained for the weighted L_p spaces.

Let λ be a function whose support is the closure of D and that is C^k-smooth inside D, of the form

$$\lambda(x) = \rho(x)^\alpha \varphi(x),$$

where $\varphi \in C^k(\mathbb{R}^n)$ and does not vanish on ∂D and ρ is a regularized distance to the boundary (see [20], Chapter 6, Theorem 2), that is $\rho \in C^\infty$ outside ∂D and

$$C_1 \mathrm{dist}(x, \partial D) \le \rho(x) \le C_2 \mathrm{dist}(x, \partial D)$$

for some positive constants C_1 and C_2. In addition, assume that $\rho(x) = 0$ when $x \notin D$.

Define the linear means of the Fourier integral

$$\Phi_R(f;x) = \Phi_R(f;x;\lambda) = (2\pi)^{-n} \int_{\mathbb{R}^n} f(x-y) R^n \hat{\lambda}(-Ry) dy.$$

Set

$$\Phi_*(f;x) = \sup_{R>0} |\Phi_R(f;x)|$$

In [17] was proved the following result.

Theorem 6 *Let $(\omega, \omega_1) \in A_p$, $p \ge 1$; then*

$$\|\Phi_*(f)\|_{L_{p,\omega_1}} \le C\|f\|_{L_{p,\omega}}, \qquad p > 1. \tag{11}$$

If we have in addition $\lambda(0) = 1$, then for every $f \in L_{p,\omega} \cap L_{p,\omega_1}$ the estimate (11) *is equivalent to*

$$\lim_{R\to\infty} \|\Phi_R(f) - f\|_{L_{p,\omega}} = 0. \tag{12}$$

Lemma 1 *[17] The following inequality*

$$\Phi_*(f;x) \le C f^*(x),$$

holds, where f^* is the Hardy-Littlewood maximal functions.

From Lemma 1 and the Theorem 2 we get the following theorem.

Theorem 7 *Let $\omega(x)$, $\omega_1(x)$ be weight functions on $\mathbb{R}^n$ and conditions (a), (b), (c) be satisfied for $Q = n$, $p \geq 1$; then the estimate* (11) *is valid.*

If we have in addition $\lambda(0) = 1$, then for every $f \in L_{p,\omega} \cap L_{p,\omega_1}$ the estimate (11) *is equivalent to* (12).

3.2. The parabolic Euclidean space

Let us now endow $\mathbb{R}^{n+1}$ with the following parabolic metric. For

$$x \equiv (x', t) = (x_1, \dots, x_n, t), \quad y \equiv (y', s) = (y_1, \dots, y_n, s) \in \mathbb{R}^{n+1},$$

we define

$$d(x, y) = \rho(x - y), \qquad \text{where} \quad \rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + 4t^2}}{2}}. \tag{13}$$

Obviously, $\mathbb{R}^{n+1}$ with its usual additive structure and the following parabolic dilation

$$x = (x', t) \longrightarrow \delta_r x = (rx', r^2 t)$$

becomes a homogeneous group. The above metric ρ is admissible with this homogeneous group structure of $\mathbb{R}^{n+1}$. This homogeneous group has a special meaning for the study on the solvability of parabolic equations. We will study the variable Calderon–Zygmund singular integrals on it. First, we present the following definition.

Definition 2 *A function K is said to be a parabolic Calderon–Zygmund kernel (for short, PCZ kernel) on $\mathbb{R}^{n+1}$ endowed with the above parabolic metric ρ, if*

i) $K \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$;

ii) $K(rx', r^2 t) = r^{-n-2} K(x', t)$, for any $r > 0$, $x \in \mathbb{R}^{n+1} \setminus \{0\}$;

iii) $\int_{S^n} K(x) d\sigma(x) = 0$, where $d\sigma$ is the element of area of the ellipsoid

$$S^n = \{x \in \mathbb{R}^{n+1} : \ \rho(x) = 1\}.$$

Let K be a PCZ kernel. We denote by $\mathrm{K}f$ the corresponding principle value singular integral operator:

$$\mathrm{K}f(x) = \text{p.v.} \int_{\mathbb{R}^{n+1}} \mathrm{K(x-y)\ f(y)\ dy}.$$

From our results for $\mathrm{K}f$ we get the following corollary.

Corollary 4 *Let $p \in (1,\infty)$, K be an PCZ kernel and Kf be the corresponding integral operator. Moreover, let $\omega(x)$, $\omega_1(x)$ be weight functions on $\mathbb{R}^{n+1}$ and conditions (a), (b), (c) be satisfied for $Q = n+2$. Then*

$$\int_{\mathbb{R}^{n+1}} |Kf(x)|^p \omega_1(x)dx \le c \int_{\mathbb{R}^{n+1}} |f(x)|^p \omega(x)dx.$$

3.3. The anisotropic Euclidean space

Corollary 5 *Let $p \in (1,\infty)$, K be a function defined on $\mathbb{R}^n_0$ the following properties: there exists $C > 0$ such that $|K(x)| \le C\rho(x)^{-Q}$, $x \in \mathbb{R}^n_0$ and for all x, $y \in \mathbb{R}^n$ satisfying $\rho(x) > 2\rho(y)$*

$$|K(x-y) - K(x)| \le C\phi\left(\frac{\rho(y)}{\rho(x)}\right) \rho(x)^{-Q},$$

where $\phi : [0,1] \to [0,\infty)$ is a non-decreasing function such that, $\phi(0) = 0$, $\phi(2s) \le C\phi(s)$ for any $s > 0$ and $\int_0^1 \phi(t)\frac{dt}{t} < \infty$.

Moreover, let $\omega(x)$, $\omega_1(x)$ be weight functions on $\mathbb{R}^n$ and conditions (a), (b), (c) be satisfied. Then the operator $T : f \to \text{p.v. } K * f$ is bounded from $L_{p,\omega}(\mathbb{R}^n)$ to $L_{p,\omega_1}(\mathbb{R}^n)$.

Remark 2 *Note that, the operator $T : f \to \text{p.v. } K * f$ is satisfies the condition (2). It is known, that this operator is bounded on $L_{p,\omega}(\mathbb{R}^n)$ for $1 < p < \infty$, if $\omega \in A_p$ (see [18] and [21]).*

Therefore, the Corollary 5 implies from Theorem 2.

Let $l = (l_1, \dots, l_n)$, $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}^n_0$ and $a = (a_1, \dots, a_n)$, $a_i > 0$, $i = 1, \dots, n$, $|a| = \sum_{i=1}^n a_i$. The weighted anisotropic Sobolev space $W^{l_1,\dots,l_n}_{p,\omega_0,\omega_1,\dots,\omega_n}(\mathbb{R}^n)$, is defined as the collection of all functions $f \in L_1^{loc}(\mathbb{R}^n)$, having the generalized derivatives $D_i^{l_i} f$ with the finite norm

$$\|f\|_{W^{l_1,\dots,l_n}_{p,\omega_0,\omega_1,\dots,\omega_n}(\mathbb{R}^n)} = \|f\|_{L_{p,\omega_0}(\mathbb{R}^n)} + \sum_{i=1}^n \left\|D_i^{l_i} f\right\|_{L_{p,\omega_i}(\mathbb{R}^n)},$$

where $1 \le p < \infty$.

We recall the Il'in-Besov integral representation of a function f in $\mathcal{R}(l)$ via its generalized derivatives (see [4]):

$$f(x) = f_{h^a}(x) + \sum_{i=1}^n \int_0^h v^{-|a|} dv \int_{\mathbb{R}^n} D_i^{l_i} f(x+y) \Phi_i(yh^{-a}) dy, \quad x \in \mathbb{R}^n,$$

where $a_i = 1/l_i$, $i = 1, \dots, n$, $f_{h^a}(x) = h^{-|a|} \int_{\mathbb{R}^n} \Phi_0(yh^{-a}) f(x+y)dy$ is the average of f and $\int_{\mathbb{R}^n} \Phi_0(x)dx = 1$. Here $\Phi_i \in C_0^\infty(\mathbb{R}^n)$ are concentrated in an arbitrary previously specified cube in the first coordinate angle and are such that

$$\int_{\mathbb{R}^n} \Phi_i(x)dx = 0, \quad i = 1, \dots, n.$$

By virtue of this integral representation we prove the following imbedding theorems.

Theorem 8 *Let $a = 1/l$, $1 < p \le q < \infty$, $æ = (\nu + 1/p - 1/q, 1/l) \le 1$, and $æ = (\nu, 1/l) = 1$, where $\nu = (\nu_1, \dots, \nu_n)$, and ν_i are nonnegative integer number. Suppose that the weight pairs (ω, ω_j) $j = 0, 1, \dots, n$, the conditions (a), (b), (c) are satisfied for $Q = |a|$.*

Then the continuous imbedding

$$D^\nu W^{l_1,\dots,l_n}_{p,\omega_0,\dots,\omega_n}(\mathbb{R}^n) \subset\!\!\succ L_{q,\omega}(\mathbb{R}^n)$$

is valid.

Further, the inequality

$$\|D^\nu f\|_{L_{q,\omega}(\mathbb{R}^n)} \le C \|f\|_{W^{l_1,\dots,l_n}_{p,\omega_0,\dots,\omega_n}(\mathbb{R}^n)},$$

holds, with a constant C is independent of f.

Proof. Applying the differentiation operation D^ν to equality

$$f_{\varepsilon^\lambda}(x) = f_{h^\lambda}(x) + \sum_{i=1}^n \lambda_i \int_\varepsilon^h \vartheta^{|\lambda|} d\vartheta \int_{\mathbb{R}^n} L_i\left(\vartheta^{-\lambda} y\right) D_i^{l_i} f(x+y)dy$$

and the Remark 2, we get

$$\left\| \int_\varepsilon^h \vartheta^{|\lambda| - (\nu,\lambda)} d\vartheta \int_{\mathbb{R}^n} L_i^{(k)}\left(\vartheta^{-\lambda} y\right) D_i^{l_i} f(x+y)dy \right\|_{L_{p,\omega}(\mathbb{R}^n)} \le C \|D_i^{l_i} f\|_{L_{p,\omega_i}(\mathbb{R}^n)}.$$

Besides,

$$\|D^\nu f_{h^\lambda}\|_{L_{p,\omega}(\mathbb{R}^n)} \le C \|f\|_{L_{p,\omega_0}(\mathbb{R}^n)}.$$

Thus, combining the estimates, we obtain

$$\|D^\nu f_{\varepsilon^\lambda}\|_{L_{p,\omega}(\mathbb{R}^n)} \le C\|f\|_{W^{l_1,\dots,l_n}_{p,\omega_0,\omega_1,\dots,\omega_n}(\mathbb{R}^n)}.$$

To conclude the proof of the theorem two facts are established: first, it is proved that $D^\nu f_{\varepsilon^\lambda}$ converges to some element of $L_{p,\omega}(\mathbb{R}^n)$ for $\varepsilon \to 0$, second, it is proved that this limit element is the generalized derivative $D^\nu f$ of the function f to which the f_{ε^λ} converge for $\varepsilon \to 0$.

For the proved of converges $D^\nu f_{\varepsilon^\lambda}$ to some element of $L_{p,\omega}(\mathbb{R}^n)$ for $\varepsilon \to 0$, it is proved that the sequence $\{D^\nu f_{\varepsilon^\lambda}\}$ is fundamental at norm $L_{p,\omega}(\mathbb{R}^n)$.

We have

$$\|D^\nu f_{\varepsilon^\lambda} - D^\nu f_{\eta^\lambda}\|_{L_{p,\omega}(\mathbb{R}^n)} \le C\sum_{i=1}^n \int_\varepsilon^\eta v^{-æ} dv \|M_i\|_{L_{1,\omega}(\mathbb{R}^n)} \|D_i^{l_i} f\|_{L_{p,\omega}(\mathbb{R}^n)} \le$$

$$\le C\eta^{1-æ}\|D_i^{l_i} f\|_{L_{p,\omega_i}(\mathbb{R}^n)},$$

where $0 < \varepsilon < \eta$.

Then by theorem Lebesgue we conclude that the sequence $\left\{D^\nu f_\varepsilon^\lambda\right\}$ is a Cauchy sequence.

Hence in view of the fact that the space $L_{p,\omega}(\mathbb{R}^n)$ is complete, then $D^\nu f_{\varepsilon^\lambda}$ converges to some element g of $L_{p,\omega}(\mathbb{R}^n)$ for $\varepsilon \to 0$. By the definition of generalized derivative in the sense of Sobolev at each a fixed ε for arbitrary function $\psi \in C_0^\infty(\mathbb{R}^n)$ the equality

$$\int_{\mathbb{R}^n} D^\nu \psi(x) f_{\varepsilon^\lambda}(x)dx = (-1)^{|\nu|}\int_{\mathbb{R}^n} \psi(x) D^\nu f_{\varepsilon^\lambda}(x)dx$$

holds.

Taking into account that $f \in L_1^{loc}(\mathbb{R}^n)$ and mean $f_{\varepsilon^\lambda} \to f$ in $L_1^{loc}(\mathbb{R}^n)$, and passing to the limit for $\varepsilon \to 0$, we give

$$\int_{\mathbb{R}^n} D^\nu \psi(x) f(x)dx = (-1)^{|\nu|}\int_{\mathbb{R}^n} \psi(x) g(x)dx,$$

and from that imply the limit element g of the sequence $\{D^\nu f_{\varepsilon^\lambda}\}$ is generalized derivative $D^\nu f$ function f.

Theorem 8 is proved. □

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