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Diagonal Lift in the Tangent Bundle of Order Two and its Applications*

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Abstract

In this paper we define a diagonal lift ^{D}g of Riemannian metric g of manifold M_n to the tangent bundle of order two denoted by T^2M_n of M_n, we associate to ^{D}g its Levi-civita connection of T^2 M and we investigate applications of the diagonal lifts in the killing vectors and geodesics.

Key Words: Tangent bundle of order two, Riemannian metric, Diagonal lift, Levi-civita connection, Killing vector field, Geodesic.

1. Introduction

Let M_n be an n-dimensional differentiable manifold endowed with a linear connection ∇. The tangent bundle of order two, T^2M_n of M_n is the 3n-dimensional manifold of 2-jets at $0 \in \mathbb{R}$ of differentiable curves $f : \mathbb{R} \rightarrow M_n$; T^2M_n has a natural bundle structure over M_n,

$$\pi_2 : T^2M_n \rightarrow M_n$$

denoting the canonical projection.

The tangent bundle TM_n is nothing by the manifold of 1-jets j^1f at $0 \in \mathbb{R}$ of the curves $f : \mathbb{R} \rightarrow M_n$.

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If we denote $\pi_{12} : T^2M_n \to TM_n$ be a canonical projection π_{12}, then T^2M has a bundle structure over TM_n, with projection π_{12}.

For any coordinate neighborhood (U, x^i) in M , $(\pi^{-1}(U), x^i, y^i)$ denotes the induced coordinate neighborhood in TM_n, that is, if $j^1f \in TU$ then

$$x^i = f^i(0), \quad y^i = \frac{df^i}{dt}(0)$$

and $((\overset{2}{\pi})^{-1}(U), x^i, y^i, z^i)$ denotes the induced coordinate neighborhood in T^2M_n, that is , if $j^2f \in T^2U$ then

$$x^i = f^i(0), \quad y^i = \frac{df^i}{dt}(0), \quad z^i = \frac{d^2f^i}{dt^2}(0)$$

where $x^i = f^i(t)$ are the local expression of the curve f in U.

Let $f : \mathbb{R} \to M_n$ be a curve in M_n, then the tangent vector $\dot{f}(0)$ to f at $f(0)$ will be called the velocity of f at $f(0)$ and the covariant derivative $(\nabla_{\dot{f}(0)}\dot{f})(0)$ of $\dot{f}$ at $f(0)$ with respect to $\dot{f}(0)$ will be called the covariant acceleration of f at $f(0)$.

If (U, x^i) is a coordinate neighborhood in M_n and $x^i = f^i(t)$ are the local expressions of f in U, we have

$$\begin{aligned} \dot{f}(0) &= \frac{df^i}{dt}\frac{\partial}{\partial x^i} \\ (\nabla_{\dot{f}(0)}\dot{f})(0) &= (\frac{d^2f^i}{dt^2} + \frac{df^j}{dt}\frac{df^k}{dt}\Gamma^i_{j\,k})\frac{\partial}{\partial x^i}, \end{aligned}$$

$\Gamma^i_{j\,k}$being the components of ∇ in U.

2. λ-lift from M_n to T^2M_n

For any $x \in M_n$, we define the map

$$\begin{aligned} S_x : T^2_xM &\to T_xM \oplus T_xM \\ j^2f &\to (\dot{f}(0), (\nabla_{\dot{f}(0)}\dot{f})(0)). \end{aligned}$$

Then, S_x is bijective and permits one to define a vector space structure on $T_x^2 M_n$ such that S_x is a vector space isomorphism. Therefore $T^2 M_n$ becomes a vector bundle over M_n with fibre $\mathbb{R}^{2n}$ and projection π_2.

Indeed, if (U, x^i) is a coordinate neighborhood in M_n, then U can be considered as a vector bundle chart by defining the diffeomrphism

$$\begin{array}{ccl} T^2 U & \rightarrow & U \times \mathbb{R}^{2n} \\ j^2 f & \rightarrow & (f(0), \dfrac{d\dot{f}^i}{dt}(0), (\nabla_{\dot{f}(0)} \dot{f})^i(0)), \end{array}$$

or in the induced coordinates

$$(x^i, y^i, z^i) \rightarrow (x^i, y^i, w^i),$$

where $w^i = z^i + y^j y^k \Gamma^i_{j\,k}$.

Morever, let $TM_n \oplus TM_n$ be the Whitney sum of TM_n with itself, then the map

$$S : T^2 M_n \rightarrow TM_n \oplus TM_n$$

defined on each fibre $T_x^2 M_n$ as S_x, becomes a vector bundle isomorphism.
Thus, we have the following theorem.

Theorem 1 *The linear connection ∇ on M_n determines a vector bundle structure on $\pi_2 : T^2 M_n \rightarrow M_n$ and a vector bundle isomorphism $S : T^2 M_n \rightarrow TM_n \oplus TM_n$.*

For any vector fields X on M_n, we shall denote by X^V (resp X^H) the vertical lift (resp the horizontal lift) with respect to ∇ of X to TM_n([3]).

If we have in TU

$$X^H = (\frac{\partial}{\partial x^i})^H = \frac{\partial}{\partial x^i} + y^i \Gamma^k_{ij} \frac{\partial}{\partial y^i}, \;\; X^V = (\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i},$$

consequently $\left\{X^H, X^V\right\}$ is a 2n-frame which will be called the **adapted frame** to ∇ in TU.

Now, for any vector field X on M_n we shall consider three vectors fields X^0, X^I and X^{II} on $T^2 M_n$ defined by

$$\begin{array}{rcl} X^0 & = & S_*^{-1}(X^H + X^H) \\ X^I & = & S_*^{-1}(X^V + 0) \\ X^{II} & = & S_*^{-1}(0 + X^V). \end{array} \tag{1}$$

If we put in T^2U, then

$$X^0 = (\tfrac{\partial}{\partial x_i})^0 \ ; \quad X^1 = (\tfrac{\partial}{\partial x_i})^I; \quad X^2 = (\tfrac{\partial}{\partial x_i})^{II} \tag{2}$$

and

$$y^h\Gamma^k_{ih} = \Gamma^k_i \quad ; \quad A^k_i = z^h\Gamma^k_{ih} + y^t y^r(\frac{\partial\Gamma^k_{tr}}{\partial x^i} + \Gamma^k_{ih}\,\Gamma^h_{tr} - \Gamma^k_{tl}\,\Gamma^l_{ir} - \Gamma^k_{lt}\,\Gamma^l_{ir}),$$

we thus obtain

$$\begin{array}{rcl} X^0 &=& \frac{\partial}{\partial x^i} - \Gamma^k_i \frac{\partial}{\partial y^k} - A^k_i \frac{\partial}{\partial z^k} \\ X^1 &=& \frac{\partial}{\partial y^i} - 2\Gamma^k_i \frac{\partial}{\partial z^k} \\ X^2 &=& \frac{\partial}{\partial z^i}, \end{array} \tag{3}$$

and therefore, $\{X^0, X^1, X^2\}$ is a 3n-frame which will be called the **adapted frame** to ∇ in $T^2U([7],[8])$.

From (1), (2) and theorem 1 we easily obtain

$$X^0 = S_*^{-1}(X^H, X^H),\ X^1 = S_*^{-1}(X^V, 0),\ X^2 = S_*^{-1}(0, X^V). \tag{4}$$

Now hawe the following definition.

Definition 2 *If X is a vector field on U, X^λ $(\lambda = 1,2,3)$ is called the **λ-lift** of X to T^2U.*

λ-lift were studied in [8] and applied to the tangent bundle of higher order $T\,^rU$; and in the case of $r = 1$, we have $X^1 = X^V$ and $X^0 = X^H$.

Proposition 3 *For any $\lambda = 0,1,2$ we have*

$$(fX)^\lambda = f(X^\lambda)$$

for all $f \in C^\infty(M)$.

For any 1-form w in M_n, there exists a unique 1-form w^λ $(\lambda = 0,1,2)$ in T^2M_n, which for any vectors field X on M_n we have

$$w^\lambda(X^i) = \delta_i^{2-\lambda}\ w(X) \circ \overset{2}{\pi} \tag{5}$$

Definition 4 *The 1-form w^λ in T^2M_n is called the $\lambda-$**lift** of w.*

If we put

$$\bar{A}_i^k = 2(y^h y^m \Gamma_{ih}^k \Gamma_{im}^k) + A_i^k,$$

and by taking account of (5), we have

$$\begin{aligned} dx_i^0 &= \bar{A}_i^k dx_k + 2\Gamma_i^k dy_k + dz_i \\ dx_i^1 &= \Gamma_i^k dx_k + dy_i \\ dx_i^2 &= dx_i \end{aligned} \tag{6}$$

Let now M_n be a Riemannian manifold with nondegenerate metric g whose components is a coordinate neighborhood U are g_{ij} and denote by Γ_{ij}^h the christoffel symbols formed with g_{ij}.

3. Lift ^{D}g of Riemannian g to T^2M_n

For any tensor field g of type $(0,2)$ in M_n, there exist a unique tensor field $^Dg \in \mathfrak{T}_2^0(T^2M_n)$ whitch for any vectors fields X, Y on M_n and any $i, j = 0, 1, 2$, we have ([9])

$$^Dg(X^i, Y^j) = \delta_j^i \; g(X,Y) \circ \overset{2}{\pi}, \tag{7}$$

and locally in T^2M_n we have

$$^Dg = g_{ij} dx_i^0 \otimes dx_j^0 + g_{ij} dx_i^1 \otimes dx_j^1 + g_{ij} dx_i^2 \otimes dx_j^2. \tag{8}$$

Thus from (7) and (8), ^{D}g has components of the form

$$(^Dg_{\beta\alpha}) = \begin{pmatrix} g_{ij} & 0 & 0 \\ 0 & g_{ij} & 0 \\ 0 & 0 & g_{ij} \end{pmatrix} \tag{9}$$

with respect to the **adapted frame** $\{X^0, X^1, X^2\}$ in T^2M_n, and components

$$
{}^{D}g = \begin{pmatrix} g_{ij} + g_{lm}\Gamma^i_l\,\Gamma^j_m \; + g_{lm}\bar{A}^i_l\bar{A}^j_m & g_{kj}\Gamma^i_k + 2g_{lm}\bar{A}^i_l\Gamma^j_m & g_{kj}\bar{A}^i_k \\ g_{ki}\Gamma^j_k + 2g_{lm}\bar{A}^j_l\Gamma^i_m & g_{ij} + 2g_{lm}(\Gamma^i_l \; + \Gamma^j_m \;) & 2g_{kj}\Gamma^i_k \\ g_{ki}\bar{A}^j_k & 2g_{ki}\Gamma^j_k & g_{ij} \end{pmatrix} \tag{10}
$$

with respect to the coordinates (x^i, y^i, z^i).

From (9) it follows that if g is a Riemannian metric in M_n, then ${}^{D}g$ is a Riemannian metric in T^2M_n.

Definition 5 *The metric ${}^{D}g$ is called the **diagonal lift** of the tensor field g to T^2M_n (see [9]).*

In the case of TM_n we find diagonal lift stadied by S.Sasaki ([13]).

4. Levi-Civita Connetion of ^{D}g

Let ∇ be a linear levi-Civita connection on M_n, and taking account that ∇ is torsion free we shall need the following identities:

$$
\begin{aligned}
\left[X^0, Y^0\right] &= [X,Y]^0 - \sum_{k=1}^{2}(R(X,Y)u)^k \\
\left[X^0, Y^j\right] &= (\nabla_X Y)^j \\
\left[X^i, Y^j\right] &= 0 \qquad \forall i,j = 1,2
\end{aligned} \tag{11}
$$

(for proof, see [7], [8], [12]).

And by koszule formula, the levi-Civita connection ${}^{D}\nabla$ of $(T^2M_n, {}^{D}g)$ is given as following

$$
\begin{aligned}
1/\ {}^{D}\nabla_{X^0}Y^0 &= (\nabla_X Y)^0 - \tfrac{1}{2}\sum_{k=1}^{2}(R(X,Y)u)^k \\
2/\ {}^{D}\nabla_{X^0}Y^1 &= (\nabla_X Y)^1 + \tfrac{1}{2}(R(u,Y)X)^0 \\
3/\ {}^{D}\nabla_{X^0}Y^2 &= (\nabla_X Y)^2 + \tfrac{1}{2}(R(u,Y)X)^0 \\
4/\ {}^{D}\nabla_{X^1}Y^0 &= {}^{D}\nabla_{X^2}Y^0 = \tfrac{1}{2}(R(u,X)Y)^0 \\
5/\ {}^{D}\nabla_{X^1}Y^1 &= {}^{D}\nabla_{X^1}Y^2 = {}^{D}\nabla_{X^2}Y^1 = {}^{D}\nabla_{X^2}Y^2 = 0
\end{aligned} \tag{12}
$$

for any vectors fields $X,\ Y \in C^{\infty}(M_n)$ and for all $(p,u) \in TM_n$.

Thus, according to (8), (9) and (12), the components ${}^{D}\Gamma^{\alpha}_{\beta\gamma}$ with respect to the adapted frame are given by

$$
\begin{aligned}
{}^{D}\Gamma^{h}_{ij} &= \Gamma^{h}_{ij} \ ;\ {}^{D}\Gamma^{h}_{\bar{i}j} = {}^{D}\Gamma^{h}_{\bar{\bar{i}}j} = \tfrac{1}{2}y^{k}R^{h}_{kij} \ ;\ {}^{D}\Gamma^{h}_{i\bar{j}} = {}^{D}\Gamma^{h}_{i\bar{\bar{j}}} = \tfrac{1}{2}y^{k}R^{h}_{kji} \\
{}^{D}\Gamma^{h}_{\bar{i}\bar{j}} &= {}^{D}\Gamma^{h}_{\bar{\bar{i}}\bar{j}} = {}^{D}\Gamma^{h}_{\bar{i}\bar{\bar{j}}} = {}^{D}\Gamma^{h}_{\bar{\bar{i}}\bar{\bar{j}}} = 0 \\
{}^{D}\Gamma^{\bar{h}}_{ij} &= -y^{k}\ \Gamma^{k}_{ij}\,\Gamma^{h}_{k} - y^{k}\tfrac{1}{2}R^{h}_{ijk} \ ;\ {}^{D}\Gamma^{\bar{h}}_{\bar{i}j} = {}^{D}\Gamma^{\bar{h}}_{\bar{\bar{i}}j} = -\tfrac{1}{2}y^{k}\ \Gamma^{h}_{s}\,R^{s}_{kij} \\
{}^{D}\Gamma^{\bar{h}}_{i\bar{j}} &= \Gamma^{h}_{ij} - \tfrac{1}{2}\ y^{k}\Gamma^{h}_{s}\,R^{s}_{kij} \ ;\ {}^{D}\Gamma^{\bar{h}}_{\bar{i}\bar{j}} = {}^{D}\Gamma^{\bar{h}}_{\bar{\bar{i}}\bar{j}} = 0 \qquad (13) \\
{}^{D}\Gamma^{\bar{h}}_{i\bar{\bar{j}}} &= -\tfrac{1}{2}y^{k}\ \Gamma^{h}_{s}\,R^{s}_{kji} \ ;\ \Gamma^{\bar{h}}_{\bar{i}\bar{\bar{j}}} = {}^{D}\Gamma^{\bar{h}}_{\bar{\bar{i}}\bar{\bar{j}}} = 0 \\
{}^{D}\Gamma^{\bar{\bar{h}}}_{ij} &= -\Gamma^{k}_{ij}\,A^{h}_{k} + \Gamma^{h}_{s}\,y^{k}R^{s}_{ijk} - \tfrac{1}{2}y^{k}R^{h}_{ijk} \ ;\ {}^{D}\Gamma^{\bar{\bar{h}}}_{\bar{i}j} = -\tfrac{1}{2}y^{k}R^{h}_{ijk}\,A^{h}_{s} \\
{}^{D}\Gamma^{\bar{\bar{h}}}_{\bar{\bar{i}}j} &= -\tfrac{1}{2}R^{s}_{kij}\,A^{h}_{s} \ ;\ {}^{D}\Gamma^{\bar{\bar{h}}}_{i\bar{j}} = -\tfrac{1}{2}y^{k}R^{s}_{kji}\,A^{h}_{s} - 2\Gamma^{s}_{ij}\,\Gamma^{h}_{s} \ ;\ {}^{D}\Gamma^{\bar{\bar{h}}}_{\bar{i}\bar{j}} = {}^{D}\Gamma^{\bar{\bar{h}}}_{\bar{\bar{i}}\bar{j}} = 0 \\
{}^{D}\Gamma^{\bar{\bar{h}}}_{i\bar{\bar{j}}} &= -\tfrac{1}{2}y^{k}R\,{}^{s}_{kji}\,A^{h}_{s} + \Gamma^{h}_{ij} \ ;\ {}^{D}\Gamma^{\bar{\bar{h}}}_{\bar{i}\bar{\bar{j}}} = {}^{D}\Gamma^{\bar{\bar{h}}}_{\bar{\bar{i}}\bar{\bar{j}}} = 0.
\end{aligned}
$$

5. Killing Vector Fields

A vector fields X is said to be **infinitesimal isometry** or a **Killing vector field** of a riemannian manifold with metric g, if

$$\mathfrak{L}_{X}\ g = Xg(Y,Z) - g([X,Y],Z) - g(Y,[X,Z]) = 0 \qquad (14)$$

for all $X,\ Y \in C^{\infty}(M_n)$. In the terms of components g_{ij} of g, X is an infinitesimal isometry if and only if

$$X^{h}\partial_{h}g_{ij} + g_{hj}\partial_{i}X^{h} + g_{ih}\partial_{j}X^{h} = 0$$

where X^{h} are components of X.(see [3])

We see by virtue of (8) that $\tilde{X}$ is a **Killing vector field** in $T^{2}M_n$ with metric ${}^{D}g$ if and only if

$$\mathfrak{L}_{\tilde{X}}\ {}^{D}g = \tilde{X}g(\tilde{Y},\tilde{Z}) - {}^{D}g([\tilde{X},\tilde{Y}],\tilde{Z}) - {}^{D}g(\tilde{Y},[\tilde{X},\tilde{Z}]) = 0 \qquad (15)$$

for all $\tilde{Z}, \tilde{Y} \in C^{\infty}(T^2M_n)$.

Then by (11) we have

$$\begin{cases} (\mathfrak{L}_{X^0}\ {}^Dg)(Y^i, Z^i) = Xg(Y,Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0 \\ (\mathfrak{L}_{X^0}\ {}^Dg)(Y^0, Z^{\,j}) = g(R(X,Y)u^{\,j}, Z^{\,j}) = g(R(X,Y)u, Z) \\ (\mathfrak{L}_{X^0}\ {}^Dg)(Y^{\,j}, Z^0) = g(R(X,Z)u^{\,j}, Y^{\,j}) = g(R(X,Y)u, Z) \\ (\mathfrak{L}_{X^0}\ {}^Dg)(Y^1, Z^2) = (\mathfrak{L}_{X^0}\ {}^Dg)(Y^2, Z^1) = 0 \end{cases} \tag{16}$$

$$\begin{cases} (\mathfrak{L}_{X^1}\ {}^Dg)(Y^i, Z^i) = 0 \\ (\mathfrak{L}_{X^1}\ {}^Dg)(Y^0, Z^1) = g(\nabla_Y X, Z) \\ (\mathfrak{L}_{X^1}\ {}^Dg)(Y^1, Z^0) = g(Y, \nabla_Z X) \\ (\mathfrak{L}_{X^1}\ {}^Dg)(Y^0, Z^2) = (\mathfrak{L}_{X^1}\ {}^Dg)(Y^2, Z^0) = 0 \\ (\mathfrak{L}_{X^1}\ {}^Dg)(Y^1, Z^2) = (\mathfrak{L}_{X^1}\ {}^Dg)(Y^2, Z^1) = 0 \end{cases} \tag{17}$$

$$\begin{cases} (\mathfrak{L}_{X^2}\ {}^Dg)(Y^i, Z^i) = 0 \\ (\mathfrak{L}_{X^2}\ {}^Dg)(Y^0, Z^1) = (\mathfrak{L}_{X^2}\ {}^Dg)(Y^1, Z^0) = 0 \\ (\mathfrak{L}_{X^2}\ {}^Dg)(Y^0, Z^2) = g(\nabla_Y X, Z) \\ (\mathfrak{L}_{X^2}\ {}^Dg)(Y^2, Z^0) = g(Y, \nabla_Z X) \\ (\mathfrak{L}_{X^2}\ {}^Dg)(Y^1, Z^2) = (\mathfrak{L}_{X^2}\ {}^Dg)(Y^2, Z^1) = 0 \end{cases} \tag{18}$$

for all $X \in C^{\infty}(M_n)$, $j = 1, 2$ and $i = 0, 1, 2$.

Since we have

$$\begin{aligned} Xg(Y,Z) - g([X,Y],Z) - g(Y,[X,Z]) \quad = \quad & Xg(Y,Z) - g(\nabla_X Y, Z) - \\ & g(Y, \nabla_X Z) + g(\nabla_Y X, Z) + g(Y, \nabla_Z X), \end{aligned} \tag{19}$$

we conclude by means of (16), (17) and (18) that if $\mathfrak{L}_{X^0}\ {}^Dg$ and $\mathfrak{L}_{X^1}\ {}^Dg$ or $\mathfrak{L}_{X^2}$, that ${}^Dg$ vanishes implies that $\mathfrak{L}_X\ g = 0$.

We next have

$$\begin{cases} R(X,Y)u = 0 \Leftrightarrow X^h R^k_{hij} = 0 \\ \nabla_Z X = \nabla X(Z) = 0 \end{cases} \tag{20}$$

and $\mathfrak{L}_X\ g = 0$ imply that $\mathfrak{L}_{X^i}\ {}^Dg = 0$ for $i = 0, 1, 2$. Thus, we have.

Theorem 6 *The vector field X in M_n is a killing vector field if its 0-lift and λ-lift ($\lambda = 1$ or 2) are killing vectors fields in T^2M_n. Conversely,*
If X is a killing vector field, parallel and $R(X,Y)u = 0$ vanishes for all $Y \in C^\infty(M_n)$ (i.e. $X^h R^k_{hij} = 0$), then λ-lift ($\lambda = 0, 1, 2$) of X is a killing vector field in T^2M_n.

6. Geodesics in T^2M_n with metric $^D\mathbf{g}$

Let C be a curve in M_n expressed locally by $x^i = x^i(t)$ and $y^i(t)$ be a vector field along C. Then, in the tangent bundle of order two T^2M_n over the Riemannian manifold M_n with metric ^Dg, we define a curve $\tilde{C}$ by

$$x^i = x^i(t), \quad x^{\bar{i}} = y^i(t), \quad x^{\bar{\bar{i}}} = z^i(t).$$

We consider now differentiel equations of the geodesics of the tangent bundle of order two T^2M_n with the metric ^Dg. If t is the arc length of the curve $x^A = x^A(t)$ in T^2M_n, equations of geodesic in T^2M_n have the usual form

$$\frac{\delta^2 x^A}{dt^2} = \frac{d^2 x^A}{dt^2} + {}^D\Gamma^A_{C\ B}\frac{dx^C}{dt}\frac{dx^B}{dt} \tag{21}$$

with respect to the induced coordinates $(x^i, x^{\bar{i}}, x^{\bar{\bar{i}}}) = (x^i, y^i, z^i)$ in T^2M_n.

We find it more convenient to refer equations (22) to the adapted frame $\left\{dx^0_i, dx^1_i, dx^2_i\right\}$.

Using (6), we write

$$\begin{aligned}
\theta^i &= dx_i \\
\theta^{\bar{i}} &= \delta y_i = \Gamma^k_i dx_k + dy_i \\
\theta^{\bar{\bar{i}}} &= \delta z_i = \bar{A}^k_i dx_k + 2\Gamma^k_i dy_k + dz_i
\end{aligned}$$

and put

$$\begin{aligned}
\frac{\theta^i}{dt} &= \frac{dx_i}{dt} \\
\frac{\theta^{\bar{i}}}{dt} &= \frac{\delta y_i}{dt}, \qquad \frac{\theta^{\bar{\bar{i}}}}{dt} = \frac{\delta z_i}{dt}
\end{aligned}$$

along a curve $x^A = x^A(t)$, i.e., $x^i = x^i(t)$, $\quad x^{\bar{i}} = y^i(t)$, $\quad x^{\bar{\bar{i}}} = z^i(t)$ in T^2M_n.

If we write, therefore, down the form equivalent to (22), namely,

$$\frac{d}{dt}(\frac{d\theta^\alpha}{dt}) +^D \Gamma^\alpha_{\beta\ \gamma} \frac{d\theta^\beta}{dt}\frac{d\theta^\gamma}{dt} = 0$$

with respect to the adapted frame and take account of (13), then the curve $x^A = x^A(t)$ in T^2M_n with the metric ${}^Dg$ is a geodesic in T^2M_n if and only if

$$\begin{cases} \frac{\delta^2 x^i}{dt^2} + y^h R^i_{hjk}\frac{\delta y^j}{dt}\frac{dx^k}{dt} + y^h R^i_{hjk}\frac{\delta z^j}{dt}\frac{dx^k}{dt} = 0 \\ \frac{\delta^2 y^i}{dt^2} - \Gamma^h_{jk}\Gamma^i_h\frac{dx^j}{dt}\frac{dx^k}{dt} - y^h R^s_{hjk}\Gamma^i_s\frac{\delta y^j}{dt}\frac{dx^k}{dt} - y^h R^s_{hjk}\Gamma^i_s\frac{\delta z^j}{dt}\frac{dx^k}{dt} = 0 \\ \frac{\delta^2 z^i}{dt^2} - A^i_h\Gamma^h_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} - 2\Gamma^h_{jk}\Gamma^i_h\frac{\delta z^j}{dt}\frac{\delta y^k}{dt} - y^h R^s_{hjk}A^s_i\frac{\delta y^j}{dt}\frac{dx^k}{dt} \\ -y^h R^s_{hjk}A^s_i\frac{\delta z^j}{dt}\frac{dx^k}{dt} = 0 \end{cases} \tag{22}$$

with

$$\begin{aligned} \frac{\delta^2 z^i}{dt^2} &= \frac{d}{dt}(\frac{\delta z^i}{dt}) + \Gamma^i_{\alpha\beta}\frac{\delta z^\alpha}{dt}\frac{dx^\beta}{dt} \\ \frac{\delta^2 y^i}{dt^2} &= \frac{d}{dt}(\frac{\delta y^i}{dt}) + \Gamma^i_{\alpha\beta}\frac{\delta y^\alpha}{dt}\frac{dx^\beta}{dt}. \end{aligned} \tag{23}$$

If a curve satisfying (22) lies on the a fiber given by $x^i = const,\ y^i = const$ in TM, then (22) reduce to

$$\frac{d^2 z^i}{dt^2} = 0 \tag{24}$$

so that

$$z^i = a^i t + b^i, \tag{25}$$

a^i and b^i being constant. Thus, we have the following theorem.

Theorem 7 *If the geodesic $x^i = x^i(t)$, $y^i = y^i(t)$ and $z = z^i(t)$ lies in fiber of T^2M_n with the metric ${}^Dg$, the geodesic is expressed by linear equations $x^i = c^i$, $y^i = d^i$ and $z^i = a^i t + b^i$ with induced coordinates (x^i, y^i, z^i), where a^i,b^i,c^i and d^i are constant.*

If a curve satisfying (22) lies on the a fiber given by $x^i = const$, then (22) reduce to

$$\begin{cases} \dfrac{\delta^2 y^k}{dt^2} = \dfrac{d^2 y^i}{dt^2} = 0 & (i) \\ \dfrac{\delta^2 z^i}{dt^2} - 2\Gamma^h_{jk}\Gamma^i_h \dfrac{\delta z^j}{dt}\dfrac{\delta y^k}{dt} = 0 & (ii) \end{cases} \tag{26}$$

so that from (26,i) $y^i = a^i t + b^i, a^i$ and b^i geing constant.

From (23) we have

$$\frac{\delta^2 z^i}{dt^2} = 2a^r a^l \Gamma^l_{rj} + \frac{d^2 z_j}{dt^2}$$

and (26,ii) become

$$\begin{aligned} \frac{\delta^2 z^i}{dt^2} - 2\Gamma^h_{jk}\Gamma^i_h \frac{\delta z^j}{dt}\frac{\delta y^k}{dt} &= 2a^r a^l \Gamma^l_{rj} + \frac{d^2 z_j}{dt^2} - 2\Gamma^h_{jk}\Gamma^i_h (2\Gamma^l_j a^l a^k + \frac{dz_j}{dt} a^k) \\ &= \frac{d^2 z_j}{dt^2} - 2\Gamma^h_{jk}\Gamma^i_h a^k \frac{dz_j}{dt} - 4\Gamma^h_{jk}\Gamma^i_h \Gamma^l_j a^l a^k; \end{aligned}$$

then (26,ii) is given by

$$\frac{d^2 z_j}{dt^2} - 2\Gamma^h_{jk}\Gamma^i_h a^k \frac{dz_j}{dt} = 4\Gamma^h_{jk}\Gamma^i_h \Gamma^l_j a^l a^k. \tag{27}$$

Thus, we have the following theorem.

Theorem 8 *If the geodesic $x^i = x^i(t)$, $y^i = y^i(t)$ and $z = z^i(t)$ lies in fiber of T^2M_n with the metric ${}^Dg$, the geodesic is expressed by linear equations $x^i = c^i$, $y^i = a^i t + b^i$ and z^i solution of differential system (27) with induced coordinates (x^i, y^i, z^i), where a^i, b^i, c^i and d^i are constant.*

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