Turk J Math
30 (2006) , 385 – 401.

Remarks About Some Weierstrass Type Results

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Abstract

The Weierstrass type results of Gajek and Zagrodny [7] are not in general retainable in the precise context. Our first aim in this exposition is to show that a completion of the imposed conditions may be offered so that these results be true. As a second aim, alternate proofs of the statements in question are performed, via ordering principles comparable with the one in Brezis and Browder [3].

Key words and phrases: Relation, maximal element, extremal value, monotonically semicontinuous function, compact metric space, Brezis-Browder ordering principle, countably ordered structure.

1. Introduction

Let M be some nonempty set. By a *relation* over it we mean any (nonempty) part $\mathcal{S} \subseteq M \times M$; usually, we declare that $(x, y) \in \mathcal{S}$ is identical with $x\mathcal{S}y$. For each $n \geq 2$ denote $\mathcal{S}^n$= the n-th relational power of S:

$$x(\mathcal{S}^n)y \text{ iff } x = u_1 \perp ... \perp u_n = y \text{ (in the sense:}$$
$$u_i \perp u_{i+1}, \forall i \in \{1, ..., n-1\}), \text{ for some } u_1, ..., u_n \in M.$$

Further, put $\mathcal{I} := \{(x, x); x \in M\}$ (the *diagonal* of M); and $\mathcal{S}^{-1}$:=the (relational) inverse of $\mathcal{S}$ (introduced as: $x(\mathcal{S}^{-1})y$ iff $y\mathcal{S}x$). The relation $\mathcal{S}$ will be termed (a) *reflexive*, if $\mathcal{I} \subseteq \mathcal{S}$; (b) *transitive*, provided $\mathcal{S}^2 \subseteq \mathcal{S}$; (c) *irreflexive* if $\mathcal{I} \cap \mathcal{S} = \emptyset$; (d) *antisymmetric* when $\mathcal{S} \cap \mathcal{S}^{-1} \subseteq \mathcal{I}$; (e) *quasi-order*, provided it is reflexive and transitive; and (f) *order*, when it is

2000 *AMS Mathematics Subject Classification:* Primary 49J27; Secondary 49J53.

antisymmetric and quasi-order. Returning to the general case, denote $\widehat{\mathcal{S}} = \cup\{\mathcal{S}^n; n \geq 1\}$ (where $S^1 = \mathcal{S}$), $\widetilde{\mathcal{S}} = \mathcal{I} \cup \widehat{\mathcal{S}}$. These are, respectively, a transitive relation a quasi-order including $\mathcal{S}$ and minimal with such properties; we shall refer to them as the *transitive relation* (respectively, *quasi-order*) induced by $\mathcal{S}$. Finally, let $\mathcal{R}$ be another relation over M; and V some nonempty part of M. We say that $z \in V$ is $(\mathcal{S}, \mathcal{R})$-*maximal* on V if

$$w \in V \text{ and } z\mathcal{S}w \quad \text{imply } w\mathcal{R}z. \tag{1.1}$$

Note that, if $\mathcal{S} = \mathcal{R}$=quasi-order on M, then (1.1) is just the standard concept of $\mathcal{S}$-maximal element (over V); cf. Ward [14].

We may now describe the basic facts to be discussed. Let X, Y be nonempty sets; and $\mathcal{S} \subseteq Y \times Y$ be some relation. Take a function $f : X \to Y$, and some nonempty part $U \subseteq X$. We say that f achieves at $a \in U$ its $(\mathcal{S}, \widehat{\mathcal{S}})$-maximal value over U, provided

$$f(a) \text{ is } (\mathcal{S}, \widehat{\mathcal{S}})\text{-maximal in } V = f(U). \tag{1.2}$$

Equivalently, this will be referred to as $a \in U$ being a $(\mathcal{S}, \widehat{\mathcal{S}})$-*maximal point* for f over U. If (1.2) holds for at least one $a \in U$, we say that f achieves its $(\mathcal{S}, \widehat{\mathcal{S}})$-maximal value over U; or, equivalently, that f has a *Weierstrass type property* (modulo $(\mathcal{S}, \widehat{\mathcal{S}})$) over U. Note that, if

$$Y = R \cup \{\infty\} \text{ and } \mathcal{S}\text{=the usual dual order } (\geq) \text{ (hence } \mathcal{S} = \widehat{\mathcal{S}}),$$

the $(\mathcal{S}, \widehat{\mathcal{S}})$-maximal value property (1.2) means

$$f(a) \leq f(x), \text{ for all } x \in U; \quad \text{i.e., } f(a) = \min[f(U)].$$

Therefore, solving such questions is not without interest for optimization theory. To do this, some additional convergence structures over X must be considered. For example, the 1994 results in this area due to Gajek and Zagrodny [7] are based on the underlying structure introduced by a *metric* $d : X \times X \to R_+$. To state them, we need some conventions. Denote

$$X(a, \rho) = \{x \in X; d(a, x) < \rho\}, \quad X[a, \rho] = \{x \in X; d(a, x) \leq \rho\}$$

as the *open/closed sphere* centered at $a \in X$ with radius $\rho > 0$. The *convergence* of $(x_n) \subseteq X$ towards $a \in X$ (written as $x_n \to a$) means

$$\forall \varepsilon > 0, \exists n(\varepsilon) \text{ such that: } n \geq n(\varepsilon) \Longrightarrow x_n \in X(a, \varepsilon).$$

Given a sequence (y_n) in Y, call it *ascending* (modulo $\mathcal{S}$) if

$$y_n \mathcal{S} y_{n+1}, \text{ for all } n \text{ (also referred to as: } (y_n) \text{ is } \mathcal{S}\text{-}\textit{ascending}). \tag{1.3}$$

(Note that, if $\mathcal{S}$ is transitive, this reads: $y_n \mathcal{S} y_m$, whenever $n < m$. But, in general, these are distinct concepts). The point $v \in Y$ is called an *upper bound* (modulo $\mathcal{S}$) of (y_n) provided

$$y_n \mathcal{S} v, \text{ for all } n \quad \text{(written as: } (y_n)\mathcal{S}v); \tag{1.4}$$

and if v is generic, we say that (y_n) is *bounded above* (modulo $\mathcal{S}$). Further, call the function f (introduced as before) *monotonically semicontinuous* (in short: *msc*) at $a \in X$ provided

$$x_n \to a \text{ and } (f(x_n)) \text{ is } \widehat{\mathcal{S}}\text{-ascending imply } (f(x_n))\widehat{\mathcal{S}}f(a). \tag{1.5}$$

If this happens at each $a \in X$, then f will be termed *msc* over X. An intermediate property of this type is obtainable by restricting the latter one to open/closed neighborhoods of $a \in X$. Denote for simplicity

$$d_z(x) = d(z, x), \quad x \in X, \quad \text{ for each } z \in X.$$

We say that f is *monotonically semicontinuous* (in short: *msc*) towards a, if there exists $\gamma = \gamma(a) > 0$ such that

$$\begin{aligned} &(x_n) \subseteq X(a,\gamma), x_n \to u, \ (f(x_n)) \text{ is } \widehat{\mathcal{S}}\text{-ascending and} \\ &(d_a(x_n)) \text{ is strictly descending imply } (f(x_n))\widehat{\mathcal{S}}f(u). \end{aligned} \tag{1.6}$$

Finally, call the metric space X, *d-compact* when

$$\text{each sequence in } X \text{ has a convergent subsequence (in } X). \tag{1.7}$$

The following answers obtained in the quoted paper are available:

Theorem 1. *Assume that* $(X, Y; f, \mathcal{S}; a)$ *satisfy*

$$f \textit{ is msc towards } a \textit{ and } X \textit{ is d-compact.} \tag{1.8}$$

Then, f *achieves its* $(\mathcal{S}, \widehat{\mathcal{S}})$*-maximal value on* $X(a,\gamma)$*, where* $\gamma > 0$ *is subject to the same conditions as in (1.6).*

Theorem 2. *Assume that* $(X, Y; f, \mathcal{S})$ *fulfills in addition*

$$f \text{ is msc over } X \text{ and } X \text{ is } d\text{-compact.} \tag{1.9}$$

Then, f *achieves its* $(\mathcal{S}, \widehat{\mathcal{S}})$*-maximal value on* X.

The argument used there—based essentially on maximality principles involving countably ordered structures (cf. Gajek and Zagrodny [6])—is rather technical; moreover, as we shall see by concrete examples, the imposed conditions are not, in general, sufficient for the written conclusions be retainable. It is our aim in the following to show that *i)* a simplification of this argument is possible, so as to avoid the recursion to such structures; and *ii)* an appropriate completion of these conditions may be found in order that the precised conclusions be true. All necessary details will be given in Section 3 (for Theorem 1) and Section 4 (for Theorem 2). The basic tool of our investigations is a collection of ordering principles (exposed in Section 2) related to the Brezis-Browder's [3]. Further aspects will be discussed elsewhere.

2. Brezis-Browder ordering principles

Let M be some nonempty set. Take a *quasi-order* $(\leq)$ (i.e.: reflexive and transitive relation) over M; as well as a function $x \vdash \varphi(x)$ from M to $R_+ = [0, \infty[$. Call the point $z \in M$, $(\leq, \varphi)$*-maximal* when

$$w \in M \text{ and } z \leq w \text{ imply } \varphi(z) = \varphi(w). \tag{2.1}$$

A basic result about the existence of such points is the 1976 Brezis-Browder ordering principle [3]:

Proposition 1. *Suppose that;*

$$\textit{each ascending sequence in } M \textit{ has an upper bound} \tag{2.2}$$

$$\varphi \text{ is } (\leq)\text{-decreasing } (x \leq y \Longrightarrow \varphi(x) \geq \varphi(y)). \tag{2.3}$$

Then, for each $u \in M$ *there exists a* $(\leq, \varphi)$*-maximal* $v \in M$ *with* $u \leq v$.

This statement, including the well known Ekeland's variational principle [5], found useful applications to convex and nonconvex analysis (cf. the above references). So, it cannot be surprising that many extensions of Proposition 1 have been proposed. A basic one is the 1982 contribution in Altman [1]; various extensions of it were proposed by Kang [9] and Anisiu [2]. The obtained results are interesting from a technical viewpoint. However, we must emphasize that, in all concrete situations when a maximality principle of this type is to be applied, a substitution by the Brezis-Browder's is always possible. This raises the question of to what extent are these enlargements of Proposition 1 effective. It is our aim in the following to show that the answer is *essentially* negative.

To begin with, let $(M, \leq)$ be taken as above; and $x \vdash \varphi(x)$ stand for a function between M and $R_+ \cup \{\infty\} = [0, \infty]$. Denote, for each $x \in M$

$$M(x, \leq) = \{y \in M; x \leq y\} \quad \text{(the } x\text{-section of } (\leq)).$$

Proposition 2. *Assume (2.2)–(2.3) are true, as well as*

$$\textit{for each } x \in X, \varepsilon > 0, \textit{ there exists } y = y(x, \varepsilon) \geq x \textit{ with } \varphi(y) \leq \varepsilon. \tag{2.4}$$

Then, for each $u \in M$ there exists $v \in M$ with $u \leq v$ and $\varphi(v) = 0$ (hence v is $(\leq, \varphi)$-maximal).

Proof. Let $u \in M$ be arbitrary fixed. By (2.4), there must be some $z \geq u$ with $\varphi(z) < \infty$. Clearly, (2.2)–(2.3) apply to $M(z, \leq)$ and $(\leq, \varphi)$. So, for the starting point $z \in M(z, \leq)$ there exists $v \in M(z, \leq)$ with

$$z \leq v \text{ (hence } u \leq v) \text{ and } v \text{ is } (\leq, \varphi)\text{-maximal in } M(z, \leq). \tag{2.5}$$

Suppose, by contradiction, that $\eta := \varphi(v) > 0$. By (2.4) there must be $y = y(v, \eta/2) \geq v$ (hence $y \in M(z, \leq)$) with $\varphi(y) \leq \eta/2 < \varphi(v)$. This cannot be in agreement with (2.5). Hence, $\varphi(v) = 0$; and the conclusion follows. □

Clearly, Proposition 2 is a logical consequence of Proposition 1. But, the converse inclusion is also true. To verify this, we need a convention. By a generalized *pseudometric* over M we shall mean any map $d : M \times M \to R_+ \cup \{\infty\}$ fulfilling $d(x, x) = 0, \forall x \in M$. Suppose that we introduced such an object. Call the point $z \in M$, $(\leq, d)$-*maximal*, in case

$$u, v \in M \quad \text{and } z \leq u \leq v \text{ imply } d(u, v) = 0. \tag{2.6}$$

Note that, if the pseudometric d is *sufficient* ($d(x,y)=0$ implies $x=y$), this property may be written as

$$w \in M, z \le w \Longrightarrow z = w \quad \text{(referred to as: } z \text{ is } (\le)\text{-maximal).} \tag{2.7}$$

So, existence results involving such points may be viewed as "metrical" versions of the Zorn maximality principle (cf. Moore [11, Ch 4, Sect 4]). To get sufficient conditions for these, one may proceed as below. Let (x_n) be an ascending sequence in M. The d-Cauchy property for it is introduced as:

$$\forall \varepsilon > 0, \exists n(\varepsilon) \text{ such that } n(\varepsilon) \le p \le q \Longrightarrow d(x_p, x_q) \le \varepsilon.$$

Also, call this sequence d-*asymptotic*, when

$$d(x_n, x_{n+1}) \to 0, \quad \text{as } n \to \infty.$$

Clearly, each (ascending) d-Cauchy sequence is d-asymptotic too. The reverse implication is also true when all such sequences are involved; i.e., the global conditions below are equivalent each other:

$$\text{each ascending sequence is } d\text{-Cauchy} \tag{2.8}$$

$$\text{each ascending sequence is } d\text{-asymptotic.} \tag{2.9}$$

Moreover, either of these implies the structural property

$$\begin{aligned} &d \text{ is } (\le)\text{-admissible: } \forall x \in M, \forall \varepsilon > 0,\\ &\exists y = y(x,\varepsilon) \ge x \text{ such that } y \le u \le v \Longrightarrow d(u,v) \le \varepsilon. \end{aligned} \tag{2.10}$$

The following ordering principle is available (cf. Kang and Park [10]):

Proposition 3. *Assume that (2.2) and (2.10) are true. Then, for each $u \in M$ there exists a $(\le, d)$-maximal $v \in M$ with $u \le v$.*

Proof. Without loss of generality, one may assume that

$$\sup\{d(x,y); x, y \in M\} < \infty \quad (d \text{ is bounded on } M).$$

For, otherwise, the map (from $M \times M$ to R_+)

$$e(x,y) = \inf\{\mu, d(x,y)\}, \quad x, y \in M \quad (\text{for some } \mu > 0)$$

is a pseudometric over M fulfilling this property, as well as (2.10). And, if the conclusions of the statement are true modulo e, this will remain as such modulo d. Put $\varphi(x) = \sup\{d(u,v); x \leq u \leq v\}$, $x \in M$. Clearly, (2.3) holds; as well as (2.4) (if one takes (2.10) into account). Hence, Proposition 2 is applicable to M and $(\leq, \varphi)$. This, added to the remark

$$\varphi(z) = 0 \text{ if and only if } z \text{ is } (\leq, d)\text{-maximal}$$

gives the desired conclusion; and completes the argument. □

As a direct consequence of this, we have (cf. Turinici [12]):

Proposition 4. *Assume that (2.2) is true, as well as either of the (mutually equivalent) conditions (2.8)/(2.9). Then, the conclusion of Proposition 3 is retainable.*

So far, Proposition 4 is a logical consequence of Proposition 1. The reciprocal of this is also true, by simply taking

$$d(x,y) = |\varphi(x) - \varphi(y)|, \quad x, y \in M \qquad \text{(where } \varphi \text{ is the above one).}$$

We therefore deduce that

$$\text{Proposition 1} \Longleftrightarrow \text{Proposition 3 (from a logical viewpoint).} \tag{2.11}$$

Summing up, all these variants of the Brezis-Browder ordering principle are nothing but logical equivalents of it.

The following completion of these developments is to be made. Remember that, when the pseudometric $d : M \times M \to R_+$ is sufficient, the point $v \in M$ assured by Proposition 3 is $(\leq)$-maximal (according to (2.7)); hence, in particular, is $(\leq, \leq)$-maximal (cf. (1.1)). So, it is natural to ask whether genuine statements of this type are available in the absence of sufficiency. An appropriate answer is to be given below. Let dist(.,.) stand for the associated (to d) point to set distance function

$$\text{dist}(x, Z) = \inf\{d(x,z); z \in Z\}; \quad x \in M, \quad Z \subseteq M.$$

Call the ambient pseudometric d *eventually* $(\leq)$-*asymptotic,* provided

$$\begin{aligned} &\forall x \in M, \forall \varepsilon > 0, \text{ there exists } y = y(x,\varepsilon) \geq x \\ &\text{with the property: } y \leq u \leq v \Longrightarrow \text{dist}(u, M(v \leq)) \leq \varepsilon. \end{aligned} \tag{2.12}$$

Further, let $\to$ stand for the *(dual) convergence structure* over M

$$x_n \to x \text{ if and only if } [d(x, x_n) \to 0 \text{ as } n \to \infty].$$

This will be referred to as (x_n) *converges* to x; or, equivalently, that x is a *limit* of (x_n). Call the subset $Z \subseteq M$, $(\leq)$-*closed*, when

the limit of each ascending sequence in Z belongs to Z.

In particular, we say that $(\leq)$ is *self-closed*, if

$$\begin{aligned}&M(x, \leq) \text{ is } (\leq)\text{-closed, for each } x \in M; \text{ or, equivalently:}\\&\text{the limit of each ascending sequence is an upper bound of it.}\end{aligned} \tag{2.13}$$

We are now in position to state the following proposition.

Proposition 5. *Suppose that (2.2) and (2.12)–(2.13) are true. Then, for each* $u \in M$ *there exists a* $(\leq, \leq)$*-maximal* $v \in M$ *with* $u \leq v$*; i.e.,* $(\leq)$ *appears as a Zorn quasi-order.*

Proof. Define the function (from M to $R_+ \cup \{\infty\}$)

$$\varphi(x) = \sup\{\text{dist}(p, M(q, \leq)); x \leq p \leq q\}, \quad x \in M.$$

Clearly, φ satisfies (2.3), and (2.4) (if we take (2.12) into account); i.e., Proposition 2 is applicable here. So, for the starting point $u \in M$, there exists $v \in M$ with

$$u \leq v \text{ and } \varphi(v) = 0 \text{ (hence } v \text{ is } (\leq, \varphi)\text{-maximal).}$$

We now claim that the generic implication is valid:

$$(\forall z \in M)\ \varphi(z) = 0 \Longrightarrow z \text{ is } (\leq, \leq)\text{-maximal (cf. (1.1)).}$$

(And from this, the conclusion is clear). Take some $w \geq z$. Since

$$\text{dist}(z, M(y, \leq)) = 0, \quad \text{for each } y \geq z,$$

it is not hard to construct an ascending sequence (x_n) in M with

$$x_n \geq w, \text{ for all } n; \quad x_n \to z, \quad \text{as } n \to \infty.$$

But then, (2.13) gives at once $x_n \le z$, for all n; hence $w \le z$. □

The following particular aspect is to be noted. Call the (ascending) sequence (x_n) *eventually d-asymptotic* when

$$\forall n, \forall \varepsilon > 0, \exists (p,q) \text{ such that } [n \le p < q \text{ and } d(x_p, x_q) < \varepsilon].$$

The generic implication below is clear:

$$d\text{-asymptotic} \Longrightarrow \text{eventually } d\text{-asymptotic};$$

but, the converse is not in general valid. We claim that the global condition

$$\text{each ascending sequence is eventually } d\text{-asymptotic} \tag{2.14}$$

yields (2.12). In fact, if this were not true, then

$$\exists x \in M, \exists \varepsilon > 0 \text{ with the property:}$$
$$\forall y \ge x, \exists (u,v) \text{ such that } [y \le u \le v, \operatorname{dist}(u, M(v, \le)) \ge \varepsilon].$$

Put $x_0 = x$; with $y = x_0$ there must be (x_1, x_2) with $x_0 \le x_1 \le x_2$, $\operatorname{dist}(x_1, M(x_2, \le)) \ge \varepsilon$. Further, with $y = x_2$ there exist (x_3, x_4) with $x_2 \le x_3 \le x_4$, $\operatorname{dist}(x_3, M(x_4, \le)) \ge \varepsilon$; and so on. This finally yields an ascending sequence (x_n) with

$$d(x_{2p+1}, x_k) \ge \varepsilon, \quad \text{for all } k > 2p+1 \text{ and all } p \ge 0.$$

So, for the ascending sequence $(y_n = x_{2n+1})$ we must have

$$d(y_p, y_q) \ge \varepsilon, \quad \text{for all } p, q \ge 1 \text{ with } p < q;$$

in contradiction with (2.14). We thus deduced (cf. Turinici [13]): □

Proposition 6. *Suppose that (2.2) is true, as well as (2.13)–(2.14). Then, the conclusion of Proposition 5 is retainable.*

In particular, assume that $d : M \times M \to R_+ \cup \{\infty\}$ is a *semi-metric* over M. Then, a sufficient condition for (2.14) is

$$\begin{array}{l} M \text{ is } (\le, d)\text{-compact:} \\ \text{each ascending sequence has a convergent subsequence.} \end{array} \tag{2.15}$$

In addition, if (2.13) is admitted, the regularity condition (2.2) holds. In fact, let (x_n) be an ascending sequence of M. By (2.15), there exists a subsequence (y_n) of (x_n) which converges (in M). By (2.13), each of these limits is an upper bound of (y_n); hence of (x_n) (by the definition of a subsequence); and this proves our claim. We have therefore deduced

Proposition 7. *Let the quasi-ordered semi-metric space* $(M, d; \leq)$ *be such that (2.13) and (2.15) hold. Then, for each* $u \in M$ *there exists a* $(\leq, \leq)$*-maximal* $v \in M$ *with* $u \leq v$.

As before, these maximal principles are obtainable from Proposition 1. The question of the reciprocal inclusion being also true remains open; we conjecture that the answer is positive. Some related aspects may be found in Brunner [4]; see also Hyers, Isac and Rassias [8, Ch 5].

3. Technical discussions on Theorem 1

Let us now return to the framework of Section 1. We make some remarks on the proof of Theorem 1; which, as said above, is due to Gajek and Zagrodny [7]. Then (by the developments in the preceding section) an alternate proof is proposed, in a quasi-order context.

For a better understanding, we must begin with a description of the argument in question (under the precised notational conventions).

Original proof of Theorem 1. Define a relation $\mathcal{R}$ over X by

$$(x, u \in X) \quad x\mathcal{R}u \text{ iff } f(x)\mathcal{S}f(u) \text{ and } d_a(x) > d_a(u). \tag{3.1}$$

It is not hard to see that X is *countably ordered* (modulo $\mathcal{R}$):

$$\begin{array}{c}\text{each (nonempty) part } U \subseteq X \text{ endowed with} \\ \text{a well order } \mathcal{T} \subseteq \widetilde{\mathcal{R}} \text{ is at most countable.}\end{array} \tag{3.2}$$

Let $\gamma > 0$ be the number given by the condition (1.6). If

$$\text{for all } x \in X(a, \gamma) \quad \text{one has } f(x)\widehat{\mathcal{S}}f(a),$$

then f achieves its $(\mathcal{S}, \widehat{\mathcal{S}})$-maximal value on $X(a, \gamma)$ at the point a. So, without loss of generality, one may assume that

$$\text{there exists } x^* \in X(a, \gamma) \text{ such that } f(x^*)\widehat{\mathcal{S}}f(a) \text{ is false.} \tag{3.3}$$

Denote for simplicity $A = \{x \in X(a, \gamma); f(x^*)\widehat{\mathcal{S}}f(x)\}$ (hence $a \notin A$). The subset $C = \{x^*\} \cup A$ is nonempty; and (cf. (3.2)), countably ordered with respect to $\mathcal{R}$. In addition, by (1.8) it follows that

$$\text{each } \mathcal{R}\text{-ascending sequence in } C \text{ is bounded above (modulo } \mathcal{R}) \text{ in } C.$$

Summing up, the maximality principle in Gajek and Zagrodny [6, Section 3] applies to the structure $(C, \mathcal{R})$. Hence, there exists at least one $(\widehat{\mathcal{R}}, \widehat{\mathcal{R}})$-maximal element $z \in C$. Note that $z \neq a$; because

$$f(x^*)\widehat{\mathcal{S}}f(z) \quad \text{and } x^* \text{ is taken as in (3.3).}$$

This element has the desired properties, because $\mathcal{R}$ is irreflexive (cf. Section 1); hence the conclusion. □

For a technical analysis of this argument, it will be useful considering a particular case.

Example 1. Fix a certain $m \geq 1$; and let $E = R^m$ stand for the usual m-dimensional space. Denote by $||\cdot||$ the standard Euclidean norm over E; and let d stand for the associated metric. Fix some $a \in E, \alpha > 0$; and put $X = E[a, \alpha]$ (clearly, (1.7) is valid here). Further, take $Y = R$; and let $\mathcal{S}$ be some relation over it. Take a function $f : X \to R$ in such a way that

$$x, u \in X,\ f(x)\widehat{\mathcal{S}}f(u) \quad \text{imply} \quad d_a(x) \leq d_a(u). \tag{3.4}$$

Clearly, the regularity condition (1.6) is vacuously satisfied, for each γ in $]0, \alpha[$. So, by Theorem 1, it follows that, for any such γ, the function f achieves its $(\mathcal{S}, \widehat{\mathcal{S}})$-maximal value on $X(a, \gamma)$. This, however, may be not in general true, for certain pairs $(f, \mathcal{S})$ fulfilling (3.4); such as, e.g.,

$$f = d_a, \quad \mathcal{S}\text{=the usual order } (\leq) \text{ of } R \text{ (hence } \mathcal{S} = \widehat{\mathcal{S}} = \widetilde{\mathcal{S}}).$$

In fact, assume by contradiction that f achieves its $(\mathcal{S},\widehat{\mathcal{S}})$-maximal value on $X(a,\gamma)$ at some point z of this set. The property in question may be written as

$$f(z) \geq f(x) \text{ (i.e.: } d(z,a) \geq d(x,a)), \quad \text{for all } x \in X(a,\gamma).$$

This, along with d_a having no maximal value on $X(a,\gamma)$ proves our claim. So, the conclusion of Theorem 1 cannot be retainable for such data.

It follows that, in the quoted result, the imposed conditions are not in general sufficient for the written conclusions to hold. The motivation of such a "bad" property comes from $\mathcal{R} = \emptyset$ being not excluded; so, it is natural to ask whether or not is this removable. It is our aim in the following to show that a positive answer is available. This will necessitate a slight reformulation of (1.6). Precisely, call the function f, *Monotonically semicontinuous* (in short; *Msc*) towards a, if there exists $\gamma = \gamma(a) > 0$ such that

$$\begin{aligned} &(x_n) \subseteq X(a,\gamma),\ x_n \to u,\ (f(x_n)) \text{ is } \widetilde{\mathcal{S}}\text{-ascending} \\ &\text{and } (d_a(x_n)) \text{ is descending imply } (f(x_n))\widetilde{\mathcal{S}}f(u). \end{aligned} \tag{3.5}$$

Note that such a condition is not in general comparable with (1.6). In fact, if the premises of the last requirement are true, then so are the premises of (3.5). But, from this, one deduces $(f(x_n))\widetilde{\mathcal{S}}f(u)$; which, in general, may be strictly larger than $(f(x_n))\widehat{\mathcal{S}}f(u)$; hence the claim. The following counterpart of Theorem 1 is then available.

Theorem 3. *Assume that $(X,Y;f,\mathcal{S};a)$ satisfy*

$$f \text{ is Msc towards } a \text{ and } X \text{ is } d\text{-compact.} \tag{3.6}$$

Then, f achieves its $(\widetilde{\mathcal{S}},\widetilde{\mathcal{S}})$-maximal value on $X[a,\delta]$, for some δ in $[0,\gamma[$; where $\gamma > 0$ is that given by (3.5).

Proof. Define a relation $\mathcal{L}$ over X by the convention

$$(x,u \in X)\text{: } x\mathcal{L}u \text{ iff } f(x)\widetilde{\mathcal{S}}f(u) \text{ and } d_a(x) \geq d_a(u). \tag{3.7}$$

Clearly, $\mathcal{L}$ is a quasi-order on X; i.e., $\mathcal{L} = \widetilde{\mathcal{L}}$. If one happens that $f(x)\widetilde{\mathcal{S}}f(a)$, for all $x \in X(a,\gamma)$, then f achieves its $(\widetilde{\mathcal{S}},\widetilde{\mathcal{S}})$-maximal value on $X(a,\gamma)$ at the point a. So, without loss, one may assume that

$$f(x^0)\widetilde{\mathcal{S}}f(a) \text{ does not hold, for some } x^0 \in X(a,\gamma). \tag{3.8}$$

Denote for simplicity $B = \{x \in X(a, \gamma); f(x^0)\widetilde{\mathcal{S}}f(x)\}$ (clearly, $x^0 \in B$, $a \notin B$). By (3.5), $\mathcal{L}$ is self-closed on B:

the limit of each ascending (modulo $\mathcal{L}$) sequence in B
is an upper bound (modulo $\mathcal{L}$) of it (in B).

On the other hand, by (3.6) (the second half), B is $(\mathcal{L}, d)$-compact:

each $\mathcal{L}$-ascending sequence in B has a convergent (in B) subsequence.

Summing up, Proposition 7 is applicable to $(B, d; \mathcal{L})$; hence, there exists at least one $(\mathcal{L}, \mathcal{L})$-maximal element $z \in B$. Note that $z \neq a$ (by the remark above) so, $\delta := d(a, z)$ fulfills $0 < \delta < \gamma$. It is now clear that f achieves its $(\widetilde{\mathcal{S}}, \widetilde{\mathcal{S}})$-maximal value on $X[a, \delta]$ at z; and the proof is complete. □

Some remarks are in order. The very formulation of (3.5) and the proof of Theorem 3 show that one may replace *ab initio* $\mathcal{S}$ by $\widetilde{\mathcal{S}}$ in all these developments; i.e., to work with quasi-orders $(\widetilde{\mathcal{S}})$ over Y. Further, (3.5) may be expressed in terms of $\mathcal{L}$ above; and this is also valid for the regularity condition (1.7). Finally, a direct use of the Brezis-Browder principle is possible, without passing through Proposition 7; we do not give details.

4. A simplified proof of Theorem 2

Let us again return to the Section 1; precisely, to the second main result. As in Section 3, we intend to make some technical remarks about the original proof of it; due, as above said, to Gajek and Zagrodny [7]. Then (by the developments of Section 2) an alternate proof is proposed.

To begin with, it would be useful having a description of the argument in question (under the precised notational conventions).

Original proof of Theorem 2. Define a relation $\mathcal{R}$ over X by

$$\begin{aligned}&(x, u \in X)\text{: } x\mathcal{R}u \text{ iff } x \neq u,\ f(x)\widehat{\mathcal{S}}f(u) \text{ and}\\ &\text{there exists } \gamma > 0 \text{ such that } \{z \in X(x, \gamma); f(u)\widehat{\mathcal{S}}f(z)\} = \emptyset.\end{aligned} \tag{4.1}$$

Firstly, X is countably ordered (modulo $\mathcal{R}$) [cf. (3.2)]. Secondly, by the admitted hypothesis about f, one derives

each $\mathcal{R}$-ascending sequence in X is bounded above (modulo $\mathcal{R}$) in X.

Summing up, the maximality principle in Gajek and Zagrodny [6, Section 3] applies to $(X, \mathcal{R})$; and from here, it follows that there exists at least one $(\widehat{\mathcal{R}}, \widehat{\mathcal{R}})$-maximal element $z \in X$. This is the desired one; and the conclusion is clear. □

For a technical analysis of this reasoning it would be useful starting with a particular case.

Example 2. Let the metric space (X, d) be introduced as in Example 1 (hence (1.7) holds). Further, choose $(Y, \mathcal{S})$ according to

$$Y = R, \quad \mathcal{S} = \{(y, y); y \in R\} \text{ (hence } \mathcal{S} = \widehat{\mathcal{S}} = \widetilde{\mathcal{S}}).$$

Finally, take some continuous function $f : X \to R$. Clearly, (1.5) is true for each $a \in X$; i.e., f is msc (modulo $\mathcal{S}$) over X. On the other hand, f achieves its $(\mathcal{S}, \widehat{\mathcal{S}})$-maximal value over X; in fact, this holds at the level of each point of X. However, such a conclusion is not obtainable from the proof of Theorem 2. For, the definition (4.1) of $\mathcal{R}$ becomes

$$\begin{aligned} &(x, u \in X)\ x\mathcal{R}u \text{ iff } x \neq u,\ f(x) = f(u) \text{ and} \\ &\text{there exists } \gamma > 0 \text{ such that } \{z \in X(x, \gamma); f(z) = f(x)\} = \emptyset. \end{aligned} \tag{4.2}$$

The first half is false whenever f is injective. But, even if this cannot happen, the second half is false; because

$$f(x) = f(x) \text{ implies } [\{z \in X(x, \gamma); f(z) = f(x)\} \neq \emptyset\}, \text{ for all } \gamma > 0].$$

In other words, $\mathcal{R} = \emptyset$; and then, all subsequent constructions related to it remain without object.

Now, it is natural to ask whether this can be removed. A positive answer is available; but it requires a slight reformulation of (1.5). Precisely, call the function f, *Monotonically semicontinuous* (in short: *Msc*) at $a \in X$ provided

$$x_n \to a \text{ and } (f(x_n)) \text{ is } \widetilde{\mathcal{S}}\text{-ascending imply } (f(x_n))\widetilde{\mathcal{S}}f(a). \tag{4.3}$$

If this happens at each $a \in X$, then we say that f is Msc over X. Note that (4.3) is not comparable in general with (1.5). In fact, if the premises of this last requirement are true, then so are the ones of (4.3). But, from this, one has $(f(x_n))\widetilde{\mathcal{S}}f(a)$; which, in general, may be strictly larger than $(f(x_n))\widehat{\mathcal{S}}f(a)$. The following counterpart of Theorem 2 is available.

Theorem 4. *Assume that*

$$f \text{ is Msc over } X \text{ and } X \text{ is } d\text{-compact.} \tag{4.4}$$

Then, f achieves its $(\widetilde{\mathcal{S}}, \widetilde{\mathcal{S}})$-maximal value on X.

Proof. Define a relation $\mathcal{L}$ over X by

$$(x, u \in X) \quad x\mathcal{L}u \text{ if and only if } f(x)\widetilde{\mathcal{S}}f(u). \tag{4.5}$$

Clearly, $\mathcal{L}$ is a quasi-order on X; i.e., $\mathcal{L} = \widetilde{\mathcal{L}}$. By the admitted condition upon f, it is clear that $\mathcal{L}$ is self-closed over X:

the limit of each ascending (modulo $\mathcal{L}$) sequence
in X is an upper bound (modulo $\mathcal{L}$) of it (in X).

On the other hand, by (4.4) (the second half), X is $(\mathcal{L}, d)$-compact; i.e.,

each $\mathcal{L}$-ascending sequence in X has a convergent (in X) subsequence.

Summing up, Proposition 7 is applicable to $(X, d; \mathcal{L})$; hence, there exists at least one $(\mathcal{L}, \mathcal{L})$-maximal element $z \in X$. It is now clear that f achieves its $(\widetilde{\mathcal{S}}, \widetilde{\mathcal{S}})$-maximal value at z; and the proof is complete. □

Some remarks are in order. The very formulation of (4.3) and the proof of Theorem 4 show that one may replace *ab initio* $\mathcal{S}$ by $\widetilde{\mathcal{S}}$ in all these developments; i.e., to work with quasi-orders $(\widetilde{\mathcal{S}})$ over Y. Further, (4.3) may be expressed in terms of $\mathcal{L}$ above; and this is also valid for the regularity condition (1.7). Finally, a direct use of the Brezis-Browder ordering principle is possible, without passing through Proposition 7. Further aspects will be discussed elsewhere.

5. Conclusions

An interesting extension of the classical Weierstrass theorem was obtained in 1994 by Gajek and Zagrodny [7]. But, the argument used there—based essentially on maximality principles involving countably ordered structures (cf. Gajek and Zagrodny [6])—is rather technical. Moreover, the imposed conditions are not in general sufficient for the written conclusions be retainable. It was the first aim in the present exposition to verify this last claim, by concrete examples. Further, as a second aim, an appropriate completion of the imposed conditions has been proposed in order that the precised statements be true. Finally, a simplification of the initial proof is being obtained; the basic tool of our investigations is a lot of ordering principles related to the Brezis-Browder's [3].

Acknowledgement

The author is very indebted to the referee, for a number of useful suggestions.

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