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A Note on Kaehlerian Manifolds

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Abstract

The main purpose of the present paper is to study nearly Kaehlerian manifolds. We give the condition for an almost Hermitian manifold to be nearly Kaehlerian.

Key Words: Hybrid tensor, Hermitian manifold, Kaehlerian manifold, Tachibana operator.

1. Introduction

Let M be an almost Hermitian manifold with almost complex structure φ and hybrid Riemannian metric tensor field g. Then

$$\varphi^2 = -I, \quad g(\varphi X, \varphi Y) = g(X, Y) \tag{1}$$

for any vector field X and Y on M. We denote by ∇ the operator of covariant differentiation with respect to g in M. If the almost complex structure φ of M satisfies

$$(\nabla_X \varphi)Y + (\nabla_Y \varphi)X = 0$$

for any vector field X and Y on M, then the manifold M is called a nearly Kaehlerian manifold (Tachibana spaces). The condition above reduces to

$$(\nabla_X \varphi)X = 0.$$

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Let N be the Nijenhuis tensor field of φ defined by

$$N(X,Y)=[\varphi X,\varphi Y]-\varphi[X,\varphi Y]-\varphi[\varphi X,Y]-[X,Y]$$

any vector field X and Y on M. By a simple computation we have

$$N(X,Y)=-4\varphi(\nabla_X\varphi)Y.$$

Proposition: *If the Nijenhuis torsion N of a nearly Kaehlerian manifold vanishes, then M is a Keahlarian manifold.*

We define a Tachibana operator [3] (see also [2, 4]) $\Phi_\varphi\xi$ associated with an almost complex structure φ and an arbitrary $X\in\Im_0^1(M)$ and applied to a tensor $\xi\in\Im_2^0(M)$ as

$$\Phi_\varphi\xi(X,Z_1,Z_2)=(L_{\varphi X}\xi)(Z_1,Z_2)-L_X(\xi\circ\varphi)(Z_1,Z_2)$$
$$+\xi(Z_1,\varphi(L_XZ_2))-\xi(\varphi Z_1,L_XZ_2), \tag{2}$$

where L_X denotes the operator of Lie derivation with respect to X and $(\xi\circ\varphi)(Z_1,Z_2)=\xi(\varphi Z_1,Z_2)$. Expression (2) defines a tensor field $\Phi_\varphi\xi\in\Im_2^0(M)$ if and only if ξ as a pure tensor [4]. When

$$\Phi_\varphi\xi(X,Z_1,Z_2)=(L_{\varphi X}\xi)(Z_1,Z_2)-L_X(\xi\circ\varphi)(Z_1,Z_2)=0 \tag{3}$$

for a pure tensor ξ and for any $X,Z_1,Z_2\in\Im_0^1(M)$, M being a manifold with almost complex structure φ, ξ is said to be almost analytic [3].

2. Operator Φ Applied to a Hybrid Tensor

Let g be a hybrid Riemannian metric tensor. The following formulas are known (see [1]):

$$(L_Xg)(Y_1,Y_2)=X(g(Y_1,Y_2))-g([X,Y_1],Y_2)-g(Y_1,[X,Y_2]), \tag{4}$$

$$L_X Y = [X,Y] = \nabla_X Y - \nabla_Y X - T(X,Y) = \nabla_X Y - \nabla_Y X, \tag{5}$$

$$(\nabla K)(X_1, X_2, ..., X_s, X) = (\nabla_X K)(X_1, X_2, ..., X_s) = \nabla_X (K(X_1, X_2, ..., X_s)) \tag{6}$$
$$- \sum_{i=1}^{s} K(X, ..., \nabla_X X_i, ..., X_s), \quad K \in \Im_s^1(M),$$

where ∇ denotes the operator of the Riemannian covariant derivation. By virtue of (1), (4) and (5), from (2) we get

$$\begin{aligned}(\Phi_\varphi g)(X, Z_1, Z_2) &= \varphi(X)(g(Z_1, Z_2)) - g(\nabla_{\varphi X} Z_1 - \nabla_{Z_1}\varphi(X), Z_2) \\ &-g(Z_1, \nabla_{\varphi X} Z_2 - \nabla_{Z_2}\varphi(X)) - X(g(\varphi Z_1, Z_2)) + (g \circ \varphi)(\nabla_X Z_1 - \nabla_{Z_1} X, Z_2) \\ &+(g \circ \varphi)(Z_1, \nabla_X Z_2 - \nabla_{Z_2} X) + g(Z_1, \varphi(\nabla_X Z_2 - \nabla_{Z_2} X) - g(\varphi Z_1, \nabla_X Z_2 - \nabla_{Z_2} X) \\ &= \varphi(X)(g(Z_1, Z_2)) - X(g(\varphi Z_1, Z_2)) \\ &-g(\nabla_{\varphi X} Z_1, Z_2) + g(\nabla_{Z_1}\varphi(X), Z_2) - g(Z_1, \nabla_{\varphi X} Z_2) + g(Z_1, \nabla_{Z_2}\varphi(X)) \\ &+g(\varphi(\nabla_X Z_1), Z_2) - g(\varphi(\nabla_{Z_1} X), Z_2) + g(\varphi Z_1, \nabla_X Z_2) - g(\varphi Z_1, \nabla_{Z_2} X) \\ &+g(Z_1, \varphi(\nabla_X Z_2)) - g(Z_1, \varphi(\nabla_{Z_2} X)) - g(\varphi Z_1, \nabla_X Z_2) + g(\varphi Z_1, \nabla_{Z_2} X) \\ &= \varphi(X)(g(Z_1, Z_2)) - X(g(\varphi Z_1, Z_2)) - g(\nabla_{\varphi X} Z_1, Z_2) + g(\nabla_{Z_1}\varphi(X), Z_2) \\ &-g(Z_1, \nabla_{\varphi X} Z_2) + g(Z_1, \nabla_{Z_2}\varphi(X)) + g(\varphi(\nabla_X Z_1), Z_2) - g(\varphi(\nabla_{Z_1} X), Z_2) \\ &+g(\varphi Z_1, \nabla_X Z_2) + g(Z_1, \varphi(\nabla_X Z_2)) + g(Z_1, \varphi(\nabla_X Z_2)) - g(Z_1, \varphi(\nabla_{Z_2} X))\end{aligned} \tag{7}$$

and making use of (6), we have

$$g(\nabla_{Z_1}\varphi(X), Z_2) - g(\varphi(\nabla_{Z_1} X), Z_2) + g(Z_1, \nabla_{Z_2}\varphi(X)) - g(Z_1, \varphi(\nabla_{Z_2} X)) \tag{8}$$
$$= g((\nabla\varphi)(X, Z_1), Z_2) + g(Z_1, (\nabla\varphi)(X, Z_2)).$$

Substitution (8) into (7) may be written as

$$(\Phi_\varphi g)(X, Z_1, Z_2) = \varphi(X)(g(Z_1, Z_2)) - X(g(\varphi Z_1, Z_2)) + g((\nabla\varphi)(X, Z_1), Z_2) \quad (9)$$
$$+g(Z_1, (\nabla\varphi)(X, Z_2)) - g(\nabla_{\varphi X} Z_1, Z_2) - g(Z_1, \nabla_{\varphi X} Z_2)$$
$$+g(\varphi(\nabla_X Z_1), Z_2) + g(\varphi Z_1, \nabla_X Z_2) + g(Z_1, \varphi(\nabla_X Z_2))$$
$$+g(Z_1, \varphi(\nabla_X, Z_2)).$$

On the other hand, with respect to the Riemannian connection, we have

$$\varphi(X)(g(Z_1, Z_2)) - g(\nabla_{\varphi X} Z_1, Z_2) - g(Z_1, \nabla_{\varphi X} Z_2) = (\nabla_{\varphi X} g)(Z_1, Z_2) = 0 \quad (10)$$

and

$$X(g(\varphi Z_1, Z_2)) - g(\nabla_X \varphi(Z_1), Z_2) - g(\varphi Z_1, \nabla_X Z_2) = (\nabla_X g)(\varphi Z_1, Z_2) = 0 \quad (11)$$
$$\Rightarrow -X(g(\varphi Z_1, Z_2)) + g(\varphi Z_1, \nabla_X Z_2) = -g(\nabla_X \varphi(Z_1), Z_2).$$

By virtue of (6), (9), (10) and (11) reduces to

$$(\Phi_\varphi g)(X, Z_1, Z_2) = -g(\nabla_X \varphi(Z_1), Z_2) + g(\varphi(\nabla_X Z_1), Z_2) + g((\nabla_{Z_1}\varphi)(X), Z_2) \quad (12)$$
$$+ g(Z_1, (\nabla_{Z_2}\varphi)(X)) + g(Z_1, \varphi(\nabla_X Z_2)) + g(Z_1, \varphi(\nabla_X Z_2))$$
$$= g(\varphi(\nabla_X Z_1) - \nabla_X \varphi(Z_1), Z_2) + g(Z_1 \varphi(X), Z_2)$$
$$+ g(Z_1, (\nabla_{Z_2}\varphi)(X)) + g(Z_1, \varphi(\nabla_X Z_2)) + g(Z_1, \varphi(\nabla_X Z_2))$$
$$= -g(\nabla_X \varphi(Z_1), Z_2) + g((\nabla_{Z_1}\varphi)(X), Z_2) + g(Z_1, (\nabla_{Z_2}\varphi)(X))$$
$$+ g(Z_1, \varphi(\nabla_X Z_2)) + g(Z_1, \varphi(\nabla_X Z_2)).$$

The analogue to (12) is

$$(\Phi_\varphi g)(Z_2, Z_1, X) = -g(\nabla_{Z_2} \varphi(Z_1), X) + g((\nabla_{Z_1}\varphi)(Z_2), X) + g(Z_1, (\nabla_X \varphi)(Z_2)) \quad (13)$$
$$+g(Z_1, \varphi(\nabla_{Z_2} X)) + g(Z_1, \varphi(\nabla_{Z_2} X)).$$

Lemma: *If a Riemannian metric tensor g is hybrid, then we have*

$$g((\nabla_Y \varphi)(Z), X) = -g(Z, (\nabla_Y \varphi)(X)), \tag{14}$$

where ∇ *denotes the operator of the Riemannian covariant derivative with respect to* g.

Proof. By virtue of (1) and

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

we have

$$Yg(\varphi Z, X) = -Yg(Z, \varphi X),$$

$$g(\nabla_Y \varphi(Z), X) + g(\varphi Z, \nabla_Y X) = -g(\nabla_Y Z, \varphi X) - g(Z, \nabla_Y \varphi(X))$$

or

$$-g(\nabla_Y Z, \varphi X) - g(\nabla_Y \varphi(Z), X) = g(\varphi Z, \nabla_Y X) + g(Z, \nabla_Y \varphi(X))$$

$$g(\varphi(\nabla_Y Z) - \nabla_Y \varphi(Z), X) = -g(Z, \varphi(\nabla_Y X) + \nabla_Y \varphi(X))$$

and therefore, by (6), the proof is completed. □

We have

$$\begin{aligned}(\Phi_\varphi g)(X, Z_1, Z_2) - (\Phi_\varphi g)(Z_2, Z_1, X) = & \; g(Z_1, (\nabla_X \varphi)(Z_2)) - g(Z_1, (\nabla_X \varphi)(Z_2)) \\ & -g(X, (\nabla_{Z_1} \varphi)(Z_2)) - g(X, (\nabla_{Z_1} \varphi)(Z_2)) \\ & +g(Z_1, (\nabla_{Z_2} \varphi)(X)) - g(Z_1, (\nabla_{Z_2} \varphi)(X)) \\ & +g(Z_1, \varphi(\nabla_X Z_2)) + g(Z_1, \varphi(\nabla_X Z_2)) \\ & -g(Z_1, \varphi(\nabla_{Z_2} X)) - g(Z_1, \varphi(\nabla_{Z_2} X))\end{aligned}$$

$$= -2g(X, (\nabla_{Z_1}\varphi)(Z_2)) + 2g(Z_1, \varphi(\nabla_X Z_2))$$
$$-2g(Z_1, \varphi(\nabla_{Z_2} X))$$
$$= -2g(X, (\nabla_{Z_1}\varphi)(Z_2)) + 2g(Z_1, \varphi(L_X Z_2))$$

or

$$(\psi_\varphi g)(X, Z_1, Z_2) - (\psi_\varphi g)(Z_2, Z_1, X) = -2g(X, (\nabla_{Z_1}\varphi)(Z_2)) \tag{15}$$

where

$$(\psi_\varphi g)(X, Z_1, Z_2) = (\Phi_\varphi g)(X, Z_1, Z_2) - g(Z_1, \varphi(L_X Z_2))$$
$$= (L_{\varphi X} g)(Z_1, Z_2) - (L_X(g \circ \varphi))(Z_1, Z_2) - g(\varphi Z_1, L_X Z_2),$$

$$(\psi_\varphi g)(Z_2, Z_1, X) = (\Phi_\varphi g)(Z_2, Z_1, X) - g(Z_1, \varphi(L_{Z_2} X))$$
$$= (L_{\varphi Z_2} g)(Z_1, X) - (L_{Z_2}(g \circ \varphi))(Z_1, X) - g(\varphi Z_1, L_{Z_2} X).$$

From (15) we have

$$(\psi_\varphi g)(X, Z_1, Z_2) - (\psi_\varphi g)(Z_2, Z_1, X) + (\psi_\varphi g)(X, Z_2, Z_1) - (\psi_\varphi g)(Z_1, Z_2, X)$$
$$= -2g(X, (\nabla_{Z_1}\varphi)(Z_2) + (\nabla_{Z_2}\varphi)(Z_1)).$$

Thus we have the following theorem.

Theorem A necessary and sufficient condition that an almost Hermitian manifold to be nearly Kahlerian is that

$$\underset{X,Z_2}{Alt}(\psi_\varphi g)(X, Z_1, Z_2) + \underset{X,Z_1}{Alt}(\psi_\varphi g)(X, Z_2, Z_1) = 0.$$

References

[1] Kobayashi, S., Nomizu, K.: Foundation of Differential Geometry, Interscience Publishers, New York, London, 1963.

[2] Salimov, A.A.: Generalized Yano-Ako operator and the complete lift of tensor fields, Tensor N.S. 55 (2), 142-146, (1994).

[3] Tachibana, S.: Analytic tensor and its generalization, Tohoku Math. J. 12, 208-221, (1960).

[4] Yano, K., Ako, M.: On certain operators associate with tensor fields , Kodai Math. Sem. Rep 20 (4), 414-436, (1968).

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