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Slant submanifolds of Kaehler Product Manifolds

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Abstract

In this paper, we study slant submanifolds of a Kaehler product manifold. We show that an F-invariant slant submanifold of Kaehler product manifold is a product manifold. We also obtain some curvature inequalities in terms of scalar curvature and Ricci tensor.

Key Words: Slant Submanifold, Kaehler Manifold, Ricci tensor.

1. Introduction

Submanifolds of a Kaehler manifold are defined with respect to the behaviour of complex structure J. More precisely, a real submanifold M of a Kaehler manifold is called invariant if $J(TM) = TM$, where TM denotes the tangent bundle of M. M is called totally real if $J(TM) \subset TM^{\perp}$ and M is called CR-submanifold [1] if there are orthogonal complement two distributions $D^{\perp}$, D such that D is invariant and $D^{\perp}$ is totally real. Recently, B. Y. Chen introduced slant submanifolds as follows: Let M be a submanifold of a Kaehler manifold $\bar{M}$, for each non zero vector $X \in T_pM$, we denote the angle between JX and T_pM by $\theta(X)$. Then M is said to be slant ([2]) if the angle $\theta(X)$ is constant, i.e., it is independent of the choice of $p \in M$ and $X \in T_pM$. The angle θ of a slant immersion is called the slant angle of the immersion. Invariant and anti-invariant immersions are slant immersions with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A proper slant immersion is neither invariant nor anti-invariant.

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The geometry of submanifolds of a Kaehler manifold has been investigated by many authors. In [6], K.Yano and M. Kon studied the geometry of F-invariant and F-anti-invariant submanifolds of Kaehler product manifolds and showed that an Finvariant, invariant submanifold of a Kaehler product manifold is also a product manifold.

Same result was obtained for anti-invariant submanifold [6]. On the other hand, CR-submanifolds of Kaehler product manifolds were studied in [5] by M. H. Shahid.

In this paper, we consider slant submanifolds of Kaehler product manifolds. We show that an F-invariant, slant submanifold of a Kaehler product manifold $\bar{M} = \bar{M}^m \times \bar{M}^n$ is a product manifold $M_1 \times M_2$ and M_1 (resp. M_2) is also a slant submanifold of $\bar{M}^m$ (resp. $\bar{M}^n$). Also we obtain, if $M = M_1 \times M_2$ is a Kaehler slant submanifold of $\bar{M}$, then M_1 is a Kaehler slant submanifold of $\bar{M}^m$ and M_2 is a Kaehler slant submanifold of $\bar{M}^n$. In the last section we study scalar curvature and Ricci tensor of various submanifolds of a Kaehler product manifold $\bar{M} = \bar{M}^m(c_1) \times \bar{M}^n(c_2)$ and obtain several inequalities for slant, invariant and anti-invariant submanifolds of $\bar{M}$.

2. Preliminaries

Let $(\bar{M}, g)$ be a $2k$-dimensional Riemannian manifold with Riemannian metric g. An almost complex structure on $\bar{M}$ is a tensor field J of type (1,1) such that at every $p \in \bar{M}$ we have $J^2 = -I$, where I denotes the identity transformation of $T_p\bar{M}$. Then, $\bar{M}$ is called an almost complex manifold. The Nijenhuis torsion tensor N_J, of J, is defined by

$$N_J(X,Y) = [JX, JY] - [X,Y] - J[X, JY] - J[JX, Y],$$

$\forall X, Y \in \Gamma(T\bar{M})$, where $\Gamma(T\bar{M})$ is the module of differentiable sections of the tangent bundle $T\bar{M}$. If the torsion tensor N_J vanishes identically on $\bar{M}$ then J is complex structure on $\bar{M}$ which becomes a complex manifold. A Hermitian metric on $\bar{M}$ is a Riemannian metric g satisfying

$$g(X,Y) = g(JX, JY), \quad \forall X, Y \in \Gamma(T\bar{M}). \tag{2.1}$$

An almost complex manifold endowed with a Hermitian metric is called an almost Hermitian manifold, denoted by $(\bar{M}, g, J)$. Denote the Levi-Civita connection on $\bar{M}$ with respect to g by $\bar{\nabla}$. Then, $\bar{M}$ is called an Kaehler manifold if J is parallel with respect to $\bar{\nabla}$, i.e.,

$$(\bar{\nabla}_X J)Y = 0, \quad \forall X, Y \in \Gamma(T\bar{M}). \tag{2.2}$$

The Riemann curvature tensor field, denoted by $\bar{R}$, satisfies

$$\bar{R}(X,Y)J = J\bar{R}(X,Y) \quad \bar{R}(JX,JY) = \bar{R}(X,Y). \tag{2.3}$$

A complex space form is a connected Kaehler manifold of constant holomorphic sectional curvature c, denoted by $\bar{M}(c)$. The curvature tensor field of $\bar{M}(c)$ is given by

$$\begin{aligned} \bar{R}(X,Y)Z &= \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ &- g(JX,Z)JY + 2g(X,JY)JZ\}, \quad \forall X, Y \in \Gamma(T\bar{M}). \end{aligned} \tag{2.4}$$

We consider a Kaehler product manifold $\bar{M} = \bar{M}^m \times \bar{M}^n$. We denote by $\bar{P}$ and $\bar{Q}$ the projection operators of the tangent space of $\bar{M}$ to the tangent space of $\bar{M}^m$ and $\bar{M}^n$, respectively. Then we have

$$\bar{P}^2 = \bar{P},\ \bar{Q}^2 = \bar{Q}, \bar{P}\bar{Q} = \bar{Q}\bar{P} = 0.$$

Putting $F = \bar{P} - \bar{Q}$, we have $F^2 = I$. Thus F is an almost product structure on $\bar{M}$. Then we can define a Riemannian metric g on $\bar{M}$ by

$$g(X,Y) = g_m(\bar{P}X, \bar{P}Y) + g_n(\bar{Q}X, \bar{Q}Y)$$

for any vector field X, Y on $\bar{M}$. Thus it follows

$$g(FX,Y) = g(FY,X).$$

Now, consider $JX = J_m\bar{P}X + J_n\bar{Q}X$ for any vector field X of $\bar{M}$. Then it can be verified that the following are satisfied:

$$J_m\bar{P} = \bar{P}J,\ J_n\bar{Q} = \bar{Q}J, \qquad FJ = JF, J^2 = -I, \tag{2.5}$$

$$g(JX,JY) = g(X,Y), \qquad \bar{\nabla}_X J = 0, \tag{2.6}$$

where $\bar{\nabla}$ is the metric connection on $\bar{M}$. Thus $\bar{M}$ is a Kaehler manifold. If $\bar{M}^m(c_1)$ and $\bar{M}^n(c_2)$ are complex space forms with constant holomorphic sectional curvatures c_1 and

c_2, respectively, then the Riemannian curvature tensor $\bar{R}$ of a Kaehler product manifold $\bar{M}$ is given by [6]

$$\begin{aligned} g(\bar{R}(X,Y)Z,W) &= \frac{1}{16}(c_1+c_2)[g(Y,Z)g(X,W)-g(X,Z)g(Y,W) \\ &+ g(JY,Z)g(JX,W)-g(JX,Z)g(JY,W) \\ &+ 2g(X,JY)g(JZ,W)+2g(FY,Z)g(FX,W) \\ &- g(FX,Z)g(FY,W)+g(JY,FZ)g(JX,FW) \\ &- g(JX,FZ)g(JY,FW)+2g(FX,JY)g(JZ,FW)] \\ &+ \frac{1}{16}(c_1-c_2)[g(FY,Z)g(X,W)-g(FX,Z)g(Y,W) \\ &+ g(Y,Z)g(FX,W)-g(X,Z)g(FY,W) \\ &+ g(JY,FZ)g(JX,W)-g(JX,FZ)g(JY,W) \\ &+ g(JY,Z)g(JX,FW)-g(JX,Z)g(JY,FW) \\ &+ 2g(FX,JY)g(JZ,W)-2g(X,JY)g(FZ,JW) \end{aligned} \tag{2.7}$$

for any vector fields X,Y,Z and W of $\bar{M}$. □

Let $\bar{M}$ be a Riemannian manifold and M be a Riemannian manifold isometrically immersed in $\bar{M}$. Then the formulas of Gauss and Weingarten for M in $\bar{M}$ are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.8}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.9}$$

for any vector fields X,Y tangent to M and N normal to M, where $\bar{\nabla}$ denotes the Riemannian connection on $\bar{M}$, h is the second fundamental form, $\nabla^{\perp}$ is the normal connection and A is the shape operator of M in $\bar{M}$. Moreover, the second fundamental form and the shape operator are related by

$$g(A_N X,Y) = g(h\left(X,Y\right),N), \tag{2.10}$$

where g denotes the Riemannian metric on M as well as on $\bar{M}$.

The equations of Gauss and Codazzi are given by

$$\begin{aligned}\bar{R}(X,Y,Z,W) &= R(X,Y,Z,W)+g(h(X,Z),h(Y,W)) \\ &\quad -g(h(X,W),h(Y,Z)) \end{aligned} \tag{2.11}$$

$$\left(\bar{R}(X,Y)Z\right)^{\perp} = \left(\nabla_X h\right)(Y,Z) - \left(\nabla_Y h\right)(X,Z) \tag{2.12}$$

for any X, Y, Z and W tangent to M, where $\bar{R}$, R denote the curvature tensors of $\bar{M}$, M,respectively, and $\left(\bar{R}(X,Y)Z\right)^{\perp}$ denotes the normal component of $\bar{R}(X,Y)Z$.

3. Slant Submanifolds of a Kaehler Product Manifold

Let M be an F-invariant submanifold of a Kaehler product manifold $\bar{M}^m \times \bar{M}^n$. Then, it is known that M is a locally decomposable Riemannian manifold $M = M_1 \times M_2$, where M_1 is a submanifold of $\bar{M}^m$ and M_2 is a submanifold of $\bar{M}^n$. Moreover, if M is Kaehler submanifold of $\bar{M} = \bar{M}^m \times \bar{M}^n$, then M is a Kaehler product manifold $M = M_1 \times M_2$, ([6]).

Let M be an invariant submanifold of a Kaehler product manifold $\bar{M} = \bar{M}^m \times \bar{M}^n$. Then, $M = M_1 \times M_2$, where M_1 is a invariant submanifold of $\bar{M}^m$ and M_2 is an invariant submanifold of $\bar{M}^n$. Then we have

$$\begin{aligned} T\bar{M} &= T\bar{M}^m \oplus T\bar{M}^n \qquad (3.1) \\ &= \{TM_1 \oplus TM_2\} \oplus \{TM_1^{\perp} \oplus TM_2^{\perp}\}. \qquad (3.2) \end{aligned}$$

For any $X \in \Gamma(TM_1)$ and $N_1 \in \Gamma(TM_1^{\perp})$, we put

$$JX = J_m X = F_1 X + \omega_1 X, JN_1 = J_m N_1 = B_1 N_1 + C_1 N_1, \tag{3.3}$$

where $F_1 X, B_1 N_1 \in \Gamma(TM_1)$ and $C_1 N_1, \omega_1 X \in \Gamma(TM_1^{\perp})$ Similarly,for any $Y \in \Gamma(TM_2)$ and $N_2 \in \Gamma(TM_2^{\perp})$, we put

$$JY = J_n Y = F_2 Y + \omega_2 Y, JN_2 = J_n N_2 = B_2 N_2 + C_2 N_2, \tag{3.4}$$

where $F_2Y, B_2N_2 \in \Gamma(TM_2)$ and $C_2N_2, \omega_2 Y \in \Gamma(TM_2^{\perp})$.

Theorem 3.1 *Let M be an F-invariant submanifold of a Kaehler product manifold $\bar{M} = \bar{M}^m \times \bar{M}^n$. If M is a slant submanifold of $\bar{M}$, then M is a slant product manifold $M_1 \times M_2$, where M_1 is a slant submanifold of $\bar{M}^m$ and M_2 is a slant submanifold of $\bar{M}^n$.*

Proof. Let us assume that M is a slant submanifold of a Kaehler product manifold $\bar{M} = \bar{M}^m \times \bar{M}^n$. Then, the angle $\theta(X)$ between JX and the tangent space T_xM at $x \in M$ is constant for $X \in T_xM$, i.e, it is independent of the choice $x \in M$ and $X \in T_xM$. Then we have

$$cos\theta(X) = \frac{\bar{g}(JX, \phi X)}{\mid X \mid \mid \phi X \mid}$$

where ϕX is the tangential part of JX.

Thus for $X_1 \in \Gamma(TM_1)$, we have

$$\begin{aligned} \cos\theta(X_1) &= \frac{\bar{g}(JX_1, \phi X_1)}{\mid X \mid \mid \phi X \mid} \\ &= \frac{\bar{g}(J_m \bar{P} X_1 + J_n \bar{Q} X_1, \phi X_1)}{\mid X \mid \mid F_1 X_1 \mid} \\ &= \frac{\bar{g}(J_m X_1 + \bar{Q} JX, \phi X_1)}{\mid X \mid \mid F_1 X_1 \mid}. \end{aligned}$$

Hence, we have $cos\theta(X_1) = \frac{\bar{g}(J_m X_1, F_1 X_1)}{|X| |F_1 X_1|}$. This means that the angle $\theta(X_1)$ between $J_m X$ and the tangent space $T_x M_1$ is constant. Since M_1 is a submanifold in $\bar{M}^m$, we conclude that M_1 is slant submanifold of $\bar{M}^m$. Similarly M_2 is slant submanifold of $\bar{M}^n$.□

Now, we denote the Weingarten operators of M, M_1 and M_2 in $\bar{M}$, $\bar{M}^m$ and $\bar{M}^n$ by A, A^1 and A^2, respectively. Also denote the second fundamental forms of M, M_1 and M_2 in $\bar{M}$, $\bar{M}^m$ and $\bar{M}^n$ by h, h_1 and h_2, respectively. Then we have ([4])

$$h(X, Y) = h_1(X_1, Y_1) + h_2(X_2, Y_2), \tag{3.5}$$

where $X = X_1 + X_2, Y = Y_1 + Y_2 \in \Gamma(TM)$. We also note that, since M is a Riemannian product manifold, we have

$$TM_1 = \{X \in \Gamma(TM) \mid fX = X\}, TM_2 = \{X \in \Gamma(TM) \mid fX = -X\}. \tag{3.6}$$

Now, since $h(X_2, X_1) = 0$, using (2.10) we have $A_{N_1}X_2 \in \Gamma(TM_2)$. On the other hand, from (2.12) we have $g(A_{N_1}X_2, Y_2) = g(h_2(X_2, Y_2), N_1)$, thus from (3.1) we get $g(A_{N_1}X_2, Y_2) = 0$, hence $A_{N_1}X_2 \in \Gamma(TM_1)$. Thus we obtain

$$A_{N_1}X_2 = 0. \tag{3.7}$$

In similar way, we have

$$A_{N_2}X_1 = 0. \tag{3.8}$$

Thus from (3.7) and (3.8) we obtain

$$A_N X = A_{N_1}X_1 + A_{N_2}X_2, \tag{3.9}$$

where $X = X_1 + X_2 \in \Gamma(TM)$ and $N = N_1 + N_2 \in \Gamma(TM^{\perp})$. Moreover, using (2.10) we have

$$A_N X = A^1_{N_1}X_1 + A^2_{N_2}X_2. \tag{3.10}$$

Theorem 3.2 *Let M be an F-invariant submanifold of a Kehlerian product manifold $\bar{M} = \bar{M}^m \times \bar{M}^n$. If M is a Kaehler slant submanifold in $\bar{M}$, then M_1 is a Kaehler slant submanifold of $\bar{M}^m$ and M_2 is a Kaehler slant submanifold of $\bar{M}^n$.*

Proof. Let M be a Kaehler slant submanifold of $\bar{M}$. Then using (2.9), (2.8), (3.3), (3.4) and taking into account (3.2), we obtain

$$(\nabla_{\bar{P}X}F_1)\bar{P}Y = A^1_{\omega_1\bar{P}Y}\bar{P}X - B_1h_1(\bar{P}X, \bar{P}Y) \tag{3.11}$$

$$(\nabla_{\bar{Q}X}F_2)\bar{Q}Y = A^2_{\omega_2\bar{Q}Y}\bar{Q}X - B_2h_2(\bar{Q}X, \bar{Q}Y) \tag{3.12}$$

for $X, Y \in \Gamma(TM)$. In similar way,

$$(\nabla_X\phi)Y = A_{\omega Y}X - Bh(X, Y) \tag{3.13}$$

for $X, Y \in \Gamma(TM)$. Let $X_1, Y_1 \in \Gamma(TM_1)$ in (3.13), then if $(\nabla_X\phi) = 0$, i.e., M is Kaehler slant submanifold in $\bar{M}$, from (3.5) and (3.10) we have

$$A^1_{\omega_1 Y_1}X_1 - B_1h_1(X_1, Y_1) = 0,$$

hence $(\nabla_{\bar{P}X}F_1) = 0$. Similarly,$(\nabla_{\bar{Q}X}F_2)\bar{Q}Y = 0$. Then our assertion follows from Theorem 3.1 □

4. Curvature Inequalities

In this section, we give some inequalities for invariant, anti-invariant and slant submanifolds of a Kaehler product manifolds in terms of Ricci tensor and scalar curvature. First we need the following.

Lemma 4.1. *Let M be a $(k+l)$ -dimensional F-invariant submanifold of a Kaehler product manifold $\bar{M} = \bar{M}^m \times \bar{M}^n$. If M is an invariant submanifold of $\bar{M} = \bar{M}^m \times \bar{M}^n$, then M is an invariant product manifold $M = M_1 \times M_2$, where M_1 is minimal in $\bar{M}^m$ and M_2 is minimal in $\bar{M}^n$.*

Proof, Let M be an F-invariant submanifold of $\bar{M} = \bar{M}^m \times \bar{M}^n$, then we have ([4])

$$H = \frac{k}{k+l}H_1 + \frac{l}{k+l}H_2, \tag{4.1}$$

where H, H_1 and H_2 are mean curvature vector fields of M, M_1 and M_2. On the other hand, it is known that invariant submanifolds of Kaehler manifolds are minimal ([6]); thus, $H = 0$, $H_1 = 0$ and $H_2 = 0$.

Let M be an F-invariant submanifold of Kaehler product manifold $\bar{M} = \bar{M}^m \times \bar{M}^n$. Then from (2.10) and (2.11) we obtain

$$\begin{aligned}
g(R(X,Y)Z,W) &= \frac{1}{16}(c_1+c_2)[g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\
&+ g(JY,Z)g(JX,W) - g(JX,Z)g(JY,W) \\
&+ 2g(X,JY)g(JZ,W) + 2g(FY,Z)g(FX,W) \\
&- g(FX,Z)g(FY,W) + g(JY,FZ)g(JX,FW) \\
&- g(JX,FZ)g(JY,FW) + 2g(FX,JY)g(JZ,FW)] \\
&+ \frac{1}{16}(c_1-c_2)[g(FY,Z)g(X,W) - g(FX,Z)g(Y,W) \\
&+ g(Y,Z)g(FX,W) - g(X,Z)g(FY,W) \\
&+ g(JY,FZ)g(JX,W) - g(JX,FZ)g(JY,W) \\
&+ g(JY,Z)g(JX,FW) - g(JX,Z)g(JY,FW) \\
&+ 2g(FX,JY)g(JZ,W) - 2g(X,JY)g(FZ,JW) \\
&+ g(h(X,W),h(Y,Z)) - g(h(Y,Z),h(X,W)).
\end{aligned} \tag{4.2}$$

Let M be an F-invariant submanifold of a Kaehler product manifold $\bar{M} = \bar{M}^m \times \bar{M}^n$. Since M is also a product manifold $M = M_1 \times M_2$, we can choose an orthonormal basis $\{e_1, ..., e_k, \tilde{e}_1, ..., \tilde{e}_l\}$ for T_pM such that $\{e_1, ..., e_k\}$ is tangent to M_1 and $\{\tilde{e}_1, ..., \tilde{e}_l\}$ is tangent to M_2. Thus from (4.1) and (4.2), using (3.6) we obtain

$$\begin{aligned} 2\tau &= \frac{1}{16}c_1[5k^2 - 4k + l^2 + 12 \parallel P_1 \parallel^2] + \frac{1}{16}c_2[5l^2 - 4l + k^2 + 12 \parallel P_2 \parallel^2] \\ &+ k^2 \parallel H_1 \parallel^2 + l^2 \parallel H_2 \parallel^2 - \parallel h \parallel^2, \end{aligned} \tag{4.3}$$

where $\parallel h \parallel^2 = \sum_{a,b=1}^{(k+l)} g(h(e_a, e_b), h(e_a, e_b))$, $\parallel P_1 \parallel^2 = \sum_{i,j=1}^{k} g(e_i, F_1 e_j)^2$ and $\parallel P_2 \parallel^2 = \sum_{\alpha,\beta=1}^{l} g(\tilde{e}_\alpha, F_2 \tilde{e}_\beta)^2$ and τ is the scalar curvature of M. □

Thus (4.3) enables us to state the following theorem:

Theorem 4.1. *Let M be an F-invariant submanifold of a Kaehler product manifold $\bar{M} = \bar{M}^m \times \bar{M}^n$. Then the following statements are true:*

1. *We have*

$$\begin{aligned} \tau &\leq \frac{1}{32}c_1[5k^2 - 4k + l^2 + 12 \parallel P_1 \parallel^2] + \frac{1}{32}c_2[5l^2 - 4l + k^2 + 12 \parallel P_1 \parallel^2] \\ &+ \frac{k^2}{2} \parallel H_1 \parallel^2 + \frac{l^2}{2} \parallel H_2 \parallel^2 . \end{aligned} \tag{4.4}$$

2. *If M is θ slant submanifold, then*

$$\begin{aligned} \tau &\leq \frac{k^2}{2} \parallel H_1 \parallel^2 + \frac{k^2}{2} \parallel H_2 \parallel^2 + \frac{1}{32}c_1[5k^2 + l^2 - 4m(1 - 3cos^2\theta) \\ &+ \frac{1}{32}c_2[5l^2 + k^2 - 4l(1 - 3cos^2\theta). \end{aligned} \tag{4.5}$$

3. *If M is an invariant submanifold, then*

$$\tau \leq \frac{1}{32}c_1[5k^2 + 8k + l^2] + \frac{1}{32}c_2[5l^2 + 8l + k^2]. \tag{4.6}$$

4. *If M is an anti-invariant submanifold, then*

$$\begin{aligned} \tau &\leq \frac{k^2}{2} \parallel H_1 \parallel^2 + \frac{k^2}{2} \parallel H_2 \parallel^2 + \frac{1}{32} c_1 [5k^2 - 4k + l^2] \\ &+ \frac{1}{32} c_2 [5l^2 - 4l + k^2]. \end{aligned} \tag{4.7}$$

5. *Equality cases 1.,2.,3, and 4 hold if and only if M is totally geodesic.*

Proof. Let M be an F-invariant submanifold of Kaehler product manifold $\bar{M} = \bar{M}^m \times \bar{M}^n$. Then M is also a product manifold $M = M_1 \times M_2$. Thus (1) follows from (4.2) and (4.3). If M is slant submanifold, then M_1 and M_2 are slant submanifolds of $\bar{M}^m$ and $\bar{M}^n$. It is known that a slant submanifold of a Kaehler manifold satisfies the following

$$g(PX, PY) = cos^2\theta g(X, Y).$$

Hence we have

$$\parallel P_1 \parallel^2 = kcos^2\theta, \parallel P_2 \parallel^2 = lcos^2\theta. \tag{4.8}$$

Thus, using (4.8) in (4.4) we obtain (2). Putting $\theta = \frac{\pi}{2}$ in (4.5) we have (3). Since invariant M implies $H = 0$, from Lemma 4.1 we get (4). The last statement is clear from (4.3).□

Theorem 4.2. *Let M be a $(k + l)$-dimensional F-invariant submanifold of a Kaehler product manifold $\bar{M} = \bar{M}^m(c_1) \times \bar{M}^n(c_2)$. Then the following statements are true:*

(1) *We have*

$$\begin{aligned} S(X, X) &\leq \frac{1}{16} c_1 [(5k - 4) \parallel X_1 \parallel^2 + l \parallel X_2 \parallel^2 + 12 \parallel PX_1 \parallel^2] \\ &+ \frac{1}{16} c_2 [(5l - 4) \parallel X_1 \parallel^2 + k \parallel X_1 \parallel^2 + 12 \parallel PX_2 \parallel^2] \\ &+ kg(H_1, h_1(X_1, X_1)) + lg(H_2, h_2(X_2, X_2)), \end{aligned} \tag{4.9}$$

where S is the Ricci tensor, $X = X_1 + X_2$, $\parallel X_1 \parallel^2 = \sum_{i=1}^{k} g(e_i, JX_1)^2$ and $\parallel X_1 \parallel^2 = \sum_{\alpha=1}^{l} g(\tilde{e}_\alpha, JX_2)^2$.

(2) *If M is $\theta-$ slant submanifold, then*

$$\begin{aligned} S(X,X) &\leq \frac{1}{16}c_1[(5k-4) \parallel X_1 \parallel^2 + l \parallel X_2 \parallel^2 + 12cos^2\theta \parallel X_1 \parallel^2] \\ &+ \frac{1}{16}c_2[(5l-4) \parallel X_1 \parallel^2 + k \parallel X_1 \parallel^2 + 12cos^2\theta \parallel X_2 \parallel^2] \\ &+ kg(H_1, h_1(X_1,X_1)) + lg(H_2, h_2(X_2,X_2)). \end{aligned} \tag{4.10}$$

(3) *If M is invariant, then*

$$\begin{aligned} S(X,X) &\leq \frac{1}{16}c_1[(5k+8) \parallel X_1 \parallel^2 + l \parallel X_2 \parallel^2] \\ &+ \frac{1}{16}c_2[(5l+8) \parallel X_1 \parallel^2 + k \parallel X_1 \parallel^2]. \end{aligned} \tag{4.11}$$

(4) *If M is anti-invariant*

$$\begin{aligned} S(X,X) &\leq \frac{1}{16}c_1[(5k-4) \parallel X_1 \parallel^2 + l \parallel X_2 \parallel^2] \\ &+ \frac{1}{16}c_2[(5l-4) \parallel X_1 \parallel^2 + k \parallel X_1 \parallel^2] \\ &+ kg(H_1, h_1(X_1,X_1)) + lg(H_2, h_2(X_2,X_2)). \end{aligned} \tag{4.12}$$

Proof. Let M be an F-invariant submanifold of a Kaehler product manifold $\bar{M} = \bar{M}^m \times \bar{M}^n$. Then M is a product manifold $M = M_1 \times M_2$. We choose an orthonormal basis $\{e_1, ..., e_k, \tilde{e}_1, ..., \tilde{e}_l\}$ for T_pM such that $\{e_1, ..., e_k\}$ is tangent to M_1 and $\{\tilde{e}_1, ..., \tilde{e}_l\}$ is tangent to M_2. Then , from (4.3) and (3.6), we obtain,

$$\begin{aligned} \sum_{i=1}^{k} g(R(e_i, X_1)Y_1), e_i) &= \frac{1}{16}c_1(5k-4)g(X_1,Y_1) + \frac{1}{16}c_2kg(X_1,Y_1) \\ &+ \sum_{i=1}^{k}\{12g(e_i, JX_1)g(e_i, JY_1) \\ &- g(h_1(e_i, X_1), h_1(Y_1, e_i))\} \\ &+ kg(H_1, h_1(X_1, Y_1)) \end{aligned} \tag{4.13}$$

and

$$\begin{aligned}\sum_{\alpha=1}^{l} g(R(\tilde{e}_\alpha, X_2)Y_2), \tilde{e}_\alpha) &= \frac{1}{16}c_2(5l-4)g(X_2,Y_2) + \frac{1}{16}c_2 lg(X_2,Y_2)\\ &+ \sum_{\alpha=1}^{l}\{12g(\tilde{e}_\alpha, JX_2)g(\tilde{e}_\alpha, JY_2)\\ &- g(h_2(\tilde{e}_\alpha, X_2), h_2(Y_2,\tilde{e}_\alpha))\}\\ &+ lg(H_2, h_2(X_2,Y_2)). \qquad (4.14)\end{aligned}$$

Thus, from (4.13) and (4.14), we have

$$\begin{aligned}S(X,X) &= \frac{1}{16}c_1[(5k-4)\parallel X_1\parallel^2 + l\parallel X_2\parallel^2]\\ &+ \frac{1}{16}c_2[(5l-4)\parallel X_2\parallel^2 + k\parallel X_1\parallel^2]\\ &+ \sum_{i=1}^{k} 12g(e_i, JX_1)^2 - g(h_1(e_i,X_1), h_1(Y_1,e_i))\\ &+ \sum_{\alpha=1}^{l} 12g(\tilde{e}_\alpha, JX_2)^2 - g(h_2(\tilde{e}_\alpha, X_2), h_2(Y_2,\tilde{e}_\alpha))\\ &+ kg(H_1, h_1(X_1,Y_1)) + lg(H_2, h_2(X_2,Y_2)). \qquad (4.15)\end{aligned}$$

Then (1) follows from (4.15). The proof of the other assertions are similar to the assertions of Theorem 4.1. □

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