Turk J Math
31 (2007) , 95 – 109.

On Dimension of Modules

S. Karimzadeh, R. Nekooei

Abstract

In this paper we prove the lying over and going down theorems for modules. Finally, we apply the above theorems and prove some results on the dimension of a module and its submodule.

Key Words: Prime Submodule, Multiplication module, Dimension of a module.

Introduction

Throughout this note, all rings are commutative with identity and all modules are unital. For R-modules M and M', we denote all R-module homomorphisms of M into M' by $Hom_R(M, M')$. For any submodule N of an R-module M, we define

$$(N : M) = \{r \in R : rM \subseteq N\}$$

and denote $(O : M)$ by $Ann_R(M)$.

A submodule P of M is called prime if $P \neq M$, and whenever $r \in R$, $m \in M$ and $rm \in P$, then $m \in P$ or $r \in (P : M)$ [see 8]. It is easy to show that, if P is a prime submodule of an R-module M, then $(P : M)$ is a prime ideal of R. The sets of all prime submodules and proper maximal submodules of M are respectively denoted by $Spec(M)$ and $Max(M)$. Following [4], we denote the intersection of all prime submodules of an R-module M by $rad_M(0)$ and the intersections of all proper maximal submodules by $Rad(M)$. The radicals of R and an ideal I of R are denoted by $N(R)$ and $\sqrt{I}$, respectively.

2000 *Mathematics Subject Classification:* 13C13, 13C99.

An R-module M is called a multiplication module if for any submodule N of M there exists an ideal I of R such that $N = IM$. It is easy to check that M is a multiplication module if and only if $N = (N : M)M$ for every submodule N of M (See [7]).

Let R be a principal ideal domain (PID) and m and n be positive integers. Let $A = (a_{ij}) \in M_{m\times n}(R)$ and F be the free R-module $R^{(n)}$. We shall use the notation $\langle A\rangle$ for the submodule N of F generated by the rows of A, and the notation $(r_1, \dots, r_m)A$, $r_i \in R$, for an element of N.

In this paper we shall first prove the lying-over and going-down theorem for modules, and then prove results on the dimension of a module and its submodule.

1. Lying over Theorem for Modules

The following Proposition is used widely in the sequel.

Proposition 1.1 *Let $\varphi \in Hom_R(M, M')$, N and N' be submodules of M and M' respectively. Then we have:*

(i) If $\varphi^{-1}(N') \subseteq N$ then there exists a submodule P' of M' containing N' which is maximal with respect to $\varphi^{-1}(P') \subseteq N$. Furthermore, $\varphi^{-1}(P') = N$;

(ii) If $N \in Spec(M)$ and $rm' \in P'$, for $r \in R$ and $m' \in M'$, then $m' \in P'$ or $r \in (N : M)$.

Proof. (*i*) Put $T = \{L' \leq M' | N' \subseteq L'$ and $\varphi^{-1}(L') \subseteq N\}$. Since $N' \in T$, $T \neq \emptyset$. By Zorn's Lemma, T has a maximal element P'. Suppose that $\varphi^{-1}(P') \subset N$. Then there exists $n \in N$ such that $\varphi(n) \notin P'$. Hence $P' \subset P' + \langle\varphi(n)\rangle \notin T$ and so there exists $m \in \varphi^{-1}(P' + \langle\varphi(n)\rangle)$ such that $m \notin N$. Therefore $\varphi(m - rn) \in P'$ for some $r \in R$, and hence $m \in N$, which is a contradiction. Thus $\varphi^{-1}(P') = N$.

(*ii*) Let $rm' \in P'$ and $m' \notin P'$. Hence $P' \subset P' + Rm' \notin T$ and so there exists $m \in \varphi^{-1}(P' + Rm')$ such that $m \notin N$. Therefore $r\varphi(m) \in P'$ and so $rm \in N$. Since $N \in Spec(M)$ and $m \notin N$, hence $r \in (N : M)$. □

Lemma 1.2 *Let $\varphi \in Hom_R(M, M')$. If $N' \in Spec(M')$ and $\varphi(M) \not\subseteq N'$, then*

(i) $\varphi^{-1}(N') \in Spec(M)$;

(ii) $(\varphi^{-1}(N') : M) = (N' : M')$.

Proof. (i) By [6, Proposition 1.2].

(ii) Suppose that $r \in (N' : M')$ and $m \in M$. Hence $r\varphi(m) \in N'$ and so $rm \in \varphi^{-1}(N')$. Therefore $(N' : M') \subseteq (\varphi^{-1}(N') : M)$. Now let $r \in (\varphi^{-1}(N') : M)$ and $m \in M \backslash \varphi^{-1}(N')$. Since $N' \in Spec(M')$ and $\varphi(m) \notin N'$, $r \in (N' : M')$, hence $(\varphi^{-1}(N') : M) \subseteq (N' : M')$ and the proof is complete. □

Definition. *Let M and M' be R-modules. We said to be lying over (or simply, LO) holds for (M, M') if $M \subseteq M'$ and for any $P \in Spec(M)$ there exists $P' \in Spec(M')$ with $P' \cap M = P$.*

Example. Let V be a vector space over a field F with $dim_F V \geq 2$. Let W be a proper subspace of V. Since every proper subspace of a vector space is prime, hence LO holds for (V, W).

Proposition 1.3 *Let $\varphi \in Hom_R(M, M')$. Suppose that, for every $m' \in M'$ and $P \in Spec(M)$, there exists $s \in R \backslash (P : M)$ such that $sm' \in \varphi(M)$.*
If $Ker\varphi \subseteq rad_M(0)$, then for any $P \in Spec(M)$ there exists $P' \in Spec(M')$ with $\varphi^{-1}(P') = P$.

Proof. Suppose that $P \in Spec(M)$. Since $\varphi^{-1}(\{0\}) = Ker\varphi \subseteq P$, by Proposition 1.1 (i), there exists a submodule P' of M' which is maximal with respect to $\varphi^{-1}(P') = P$. Now we show that $P' \in Spec(M')$. It is clear that P' is a proper submodule of M'. Suppose that $r \in R$, $m' \in M'$ and $rm' \in P'$. If $m' \notin P'$, then by Proposition 1.1 (ii), $r \in (P : M)$. Assume that $rM' \nsubseteq P'$, hence there exists $m'_1 \in M'$ such that $rm'_1 \notin P'$. By assumption, there exists $s \in R \backslash (P : M)$ such that $rsm'_1 \in \varphi((P : M)M)$. Hence $srm'_1 \in P'$. Again by Proposition 1.1 (ii), $s \in (P : M)$, which is a contradiction. Therefore $rM' \subseteq P'$ and $P' \in Spec(M')$. □

Example. Let R be a commutative ring with identity . Let M be a flat R-module which is not faithfully flat (for example, **Q** as **Z**-module).
By [9, Proposition 2.11.24] ,$mM = M$ for some maximal ideal m of R. Define the monomorphism $\varphi \in Hom_R(M, M \oplus R/m)$ by $\varphi(x) = (x, 0), x \in M$. Let P be a prime submodule of M. Since $mM = M$ and $P \neq M$, hence $(P : M) \neq m$. Therefore there exists $s \in (R \backslash (P : M)) \cap m$. Now for any $(x, r + m) \in (M \oplus R/m)$, we have

$s(x, r+m) = (sx, 0) \in \varphi(M)$. By Proposition 1.3, we conclude that for any $P \in Spec(M)$ there exists $P' \in Spec(M \oplus R/m)$ with $\varphi^{-1}(P') = P$.

Theorem 1.4 *(Lying Over) Let $M \subseteq M'$ be R-modules.*

If for each $m' \in M'$, $P \in Spec(M)$, there exists $s \in R\backslash(P : M)$ such that $sm' \in M$, then LO holds for (M, M').

Proof. This follows by Proposition 1.3. □

A ring R is called Von Neumann regular ring if for every $a \in R$ there exists an element $b \in R$ such that $a = aba$.

Theorem 1.5 *Let $M \subseteq M'$ be R-modules and $P \in Spec(M)$. Then there exists $P' \in Spec(M')$ with $P' \cap M = P$, if one of the following conditions hold.*

(i) For each $m' \in M'$ and $r \in (P : M)$ there exists $s \in R\backslash(P : M)$ such that $rsm' \in P$;

(ii) R is Von Neumann regular ring and $(P : M) \subseteq \sqrt{P : M'}$.

Proof. By Proposition 1.1, there exists a submodule P' of M' that is maximal with respect to $P' \cap M = P$. Now we show that $P' \in Spec(M')$. Let $re \in P'$ and $e \notin P'$, where $r \in R$ and $e \in M'$. By Proposition 1.1 (ii), $r \in (P : M)$. Suppose that (i) holds. If $rM' \not\subseteq P'$ then there exists $m' \in M'$ such that $rm' \notin P'$. By assumption there is $s \in R\backslash(P : M)$ such that $rsm' \in P$. Thus $rsm' \in P'$ and so by Proposition 1.1, $s \in (P : M)$, which is a contradiction. Now suppose that (ii) holds. Since $r \in (P : M)$, hence there is $n \in \mathbb{N}$ such that $r^n \in (P : M') \subseteq (P' : M')$. Therefore P' is a primary submodule of M' and so $P' \in Spec(M')$. □

Proposition 1.6 *Let $\varphi \in Hom_R(M, M')$ and M' be a multiplication module. If $Ker\varphi \subseteq rad_M(0)$, then for every $P \in Spec(M)$, there exists $P' \in Spec(M')$ with $\varphi^{-1}(P') = P$ if and only if $(\varphi(M) : M') \not\subseteq (P : M)$.*

Proof. Suppose that there exists $P' \in Spec(M')$ such that $\varphi^{-1}(P') = P$. By Lemma 1.2, $(P' : M') = (P : M)$. If $(\varphi(M) : M') \subseteq (P : M)$ then $\varphi(M) \subseteq P'$, because M' is a multiplication module. Hence $P = M$, which is a contradiction. Conversely, suppose that $(\varphi(M) : M') \not\subseteq (P : M)$. Then there exists $s \in (\varphi(M) : M')\backslash(P : M)$. Let $m' \in M'$ and $r \in (P : M)$. Hence $rsm' \in \varphi((P : M)M)$ and so by Proposition 1.3, there exists

$P' \in Spec(M')$ such that $\varphi^{-1}(P') = P$. □

Corollary 1.7 *Let $M \subseteq M'$ be R-modules. If $\sqrt{(M : M')} + Ann_R(M) = R$ then LO holds for (M, M').*

Proof. Suppose that $P \in Spec(M)$. We have $Ann_R(M) \subseteq (P : M)$. Since $\sqrt{(M : M')} + Ann_R(M) = R$, hence $(M : M') \nsubseteq (P : M)$.Therefore LO holds. □

Corollary 1.8 *Let $M \subseteq M'$ be R-modules and M' be a multiplication module. LO holds for (M, M') if and only if for every $P \in Spec(M)$, $(M : M') \nsubseteq (P : M)$.*

Definition. *(See [2]) Let R be a principal ideal domain (PID). Let $J = \{j_1, \ldots, j_\alpha\}$ be a subset of the integer between 1 and n and let $p \in R$ be a prime element. A matrix $A \in M_n(R)$, $A = (a_{ij})$, is said to be a prime matrix (or simply prime), if A satisfies the following conditions:*

(i) A is a upper triangular;

(ii) For all i, $1 \leq i \leq n$, $a_{ii} = p$ if $i \in J$ and $a_{ii} = 1$ if $i \notin J$;

(iii) For all i, $1 \leq i \leq j \leq n$, $a_{ij} = 0$ except possibly when $i \notin J$ and $j \in J$.

Sometimes we call J the set of integers associated with A and denote it by J_A.

By (i) and (ii) it's clear that $det(A) = p^\alpha$.

Theorem 1.9 *Let R be a PID.*

(i) Let M be a free R-module of rank m such that $M = \langle A_{m\times n}\rangle$ $(m \leq n)$ and $N = \langle CA\rangle$ for some $C \in M_{\ell\times m}(R)$. Then N is a prime submodule of M if and only if $\langle C\rangle$ is a prime submodule of R^m.

(ii) Let M be a free R-module such that $M = \langle A_{n\times n}\rangle$ $(detA \neq 0)$ and $N = \langle CA\rangle$ for some $C \in M_n(R)$ then $detC \in (N : M)$.

(iii) Let M be a free R-module such that $M = \langle A_{n\times n}\rangle$ $(detA \neq 0)$. If N is a prime submodule of M and $rankN = n$, then there exists a prime matrix $C_{n\times n}$ such that $N = \langle CA\rangle$.

Proof. (i) Suppose that $N \in Spec(M)$. If $r(x_1, \ldots, x_m) \in \langle C\rangle$ for $r, x_i \in R$, $1 \leq i \leq n$, then there exists $(d_1, \ldots, d_\ell) \in R^\ell$ such that $r(x_1, \ldots, x_m) = (d_1, \ldots, d_\ell)C$ and so $r(x_1,\ldots,x_m)A=(d_1,\ldots,d_\ell)CA$. Since $r((x_1,\ldots,x_m)A)\in N$, hence $r\in(N:M)$ or $(x_1,\ldots,x_m)A\in N$.

Case 1. Let $(x_1,\ldots,x_m)A\in N$. There exists $(b_1,\ldots,b_\ell)\in R^\ell$ such that $(x_1,\ldots,x_m)A$ $=(b_1,\ldots,b_\ell)CA$. Put $(c_1,\ldots,c_m)=(b_1,\ldots,b_\ell)C$. Suppose that $a_1,\ldots,a_m$ are rows of A, then $x_1a_1+\cdots+x_ma_m=c_1a_1+\cdots+c_ma_m$. Since $rankM=m$, we have $(x_1,\ldots,x_m)=(c_1,\ldots,c_m)$. This implies that $(x_1,\ldots,x_m)=(b_1,\ldots,b_\ell)C$ and therefore $(x_1,\ldots,x_m)\in\langle C\rangle$.

Case 2. Now let $r\in(N:M)$. Suppose that j, $1\le j\le n$, is fixed. Since $(0,\ldots,0,r,0,\ldots,0)A\in N$, there exists $(d_1,\ldots,d_\ell)\in R^\ell$ such that $(0,\ldots,0,r,0,\ldots,0)A=(d_1,\ldots,d_\ell)CA$. Put $(c_1,\ldots,c_m)=(d_1,\ldots,d_\ell)C$. Suppose that $a_1,\ldots,a_m$ are rows of A. We have $(0,\ldots,0,r,0,\ldots,0)A=(c_1,\ldots,c_m)A$ and so $ra_j=c_1a_1+\cdots+c_ma_m$. Therefore $c_j=r$, $c_i=0$ $(i\ne j)$. Then $(c_1,\ldots,c_m)=(0,\ldots,0,r,0,\ldots,0)=(d_1,\ldots,d_\ell)C$ and hence $r\in(\langle C\rangle:R^m)$.

Conversely, suppose that $\langle C\rangle\in Spec(R^m)$. If $r(x_1,\ldots,x_n)\in N$, for $r\in R$ and $(x_1,\ldots,x_n)\in M$, there exist $(y_1,\ldots,y_m)\in R^m$ and $(d_1,\ldots,d_\ell)\in R^\ell$ such that $r(x_1,\ldots,x_n)=(d_1,\ldots,d_\ell)CA$ and $(x_1,\ldots,x_n)=(y_1,\ldots,y_m)A$. Suppose that $(b_1,\ldots,b_m)$ $=(d_1,\ldots,d_\ell)C$ such that $(b_1,\ldots,b_m)\in R^m$ and $a_1,\ldots,a_m$ are rows of A. Then $ry_1a_1+\cdots+ry_ma_m=b_1a_1+\cdots+b_ma_m$ and therefore $r(y_1,\ldots,y_m)=(b_1,\ldots,b_m)$. We have $r(y_1,\ldots,y_m)=(b_1,\ldots,b_m)=(d_1,\ldots,d_\ell)C$. Hence $r(y_1,\ldots,y_m)\in\langle C\rangle$ and $\langle C\rangle\in Spec(R^m)$, and this implies that $r\in(\langle C\rangle:R^m)$ or $(y_1,\ldots,y_m)\in\langle C\rangle$.

Case 1. Suppose that $r\in(\langle C\rangle:R^m)$ and j, $1\le j\le m$. We have $(0,\ldots,0,r,0,\ldots,0)\in\langle C\rangle$. There exists $(d_1,\ldots,d_\ell)\in R^\ell$ such that $(0,\ldots,0,r,0,\ldots,0)=(d_1,\ldots,d_\ell)C$. Also $(0,\ldots,0,r,0,\ldots,0)A=(d_1,\ldots,d_\ell)CA$; hence $r\in(N:M)$.

Case 2. Suppose that $(y_1,\ldots,y_m)\in\langle C\rangle$. There exists $(d_1,\ldots,d_\ell)\in R^\ell$ such that $(y_1,\ldots,y_m)=(d_1,\ldots,d_\ell)C$. Hence $(x_1,\ldots,x_n)=(y_1,\ldots,y_m)A=(d_1,\ldots,d_\ell)CA$, therefore $(x_1,\ldots,x_n)\in N$.

(ii) Suppose that $detC\ne0$ and $A'=(a'_{ij})$, $C'=(c'_{ij})$ are the adjoint matrices of A and C respectively. If $(x_1,\ldots,x_n)\in M$, then by [2, Lemma 1.2] $detA|\sum_{i=1}^{n}x_ia'_{ij}$, for every j, $1\le j\le n$. But $(detC)(detA)=det(CA)|\sum_{i=1}^{n}(detC)x_i(\sum_{k=1}^{n}a'_{ik}c'_{kj})$, hence by [2, Lemma 1.2] we have $((detC)x_1,\ldots,(detC)x_n)\in N$.

(iii) By (i), we have $\langle C\rangle \in Spec(R^n)$. Now there exists a prime matrix $B_{n\times n}$ such that $\langle C\rangle = \langle B\rangle$ and by [2, Theorem 2.5], we have $\langle CA\rangle = \langle BA\rangle$. □

Theorem 1.10 *Let R be a PID and M be a free R-module such that $M = \langle A\rangle$, $A \in M_n(R)$ $(detA \neq 0)$.*

(i) Let $N \in Spec(M)$, $N = \langle BA\rangle$ for some $B \in M_n(R)$. If $(detB, detA) = 1$ then there exists $N' \in Spec(R^n)$ such that $N' \cap M = N$.

(ii) Let $N \in Spec(M)$ and $N = \langle BA\rangle$ such that A is a diagonal matrix and B is a prime matrix. If there exists a prime element $p \in R$ such that $p|(detB, detA)$ then there exists $N' \in Spec(R^n)$ such that $N' \cap M = N$ if and only if for $I = \{i : p|a_{ii}, 1 \leq i \leq n\}$, $A = (a_{ij})$ and $B = (b_{ij})$ we have (1) for all $i \in I$, $b_{ii} = 1$ and (2) if $i_0 = \min I$, $j > i_0$, $b_{jj} = p$ then $p|b_{ij}$, $\forall i \in I$.

(iii) If $N \in Spec(M)$ and $rankN < rankM$ then there exists $N' \in Spec(R^n)$ such that $N' \cap M = N$.

Proof. (i) Suppose that $N \in Spec(M)$. By [2], $detB = up^{\alpha}$ $(\alpha \geq 1)$ such that p is prime and u is a unit element of R. By theorem 1.9(ii), $up^{\alpha} \in (N : M)$ and therefore $p \in (N : M)$. Also, $detA \in (M : R^n)$. Since $(detB, detA) = 1$, $detA \notin (N : M)$. By theorem 1.4, there exists $N' \in Spec(R^n)$ such that $N' \cap M = N$.

(ii) Let $P = (p_{ij})$ be a diagonal matrix such that $p_{ii} = p$, $1 \leq i \leq n$. We show that $\langle P\rangle \cap M \subseteq N$.

If $m \in \langle P\rangle \cap M$ then $m = (d_1a_{11}, \ldots, d_na_{nn})$ such that $d_j \in R$ and $p|d_ja_{jj}$ for all $j, 1 \leq j \leq n$. Let i be the smallest integer such that $p \not| d_i$ and therefore $p|a_{ii}$ and $i \in I$. Now $(d_1a_{11}, \ldots, d_{i-1}a_{i-1i-1}, 0, \ldots, 0) \in N$, since by Theorem 1.9.(ii), $detB \in (N : M)$. But,

$$BA = \begin{pmatrix} b_{11}a_{11} & \ldots & \ldots & b_{1n}a_{nn} \\ 0 & b_{22}a_{22} & \ldots & b_{2n}a_{nn} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & \ldots & b_{nn}a_{nn} \end{pmatrix}.$$

It is enough to show that $(0, \ldots, 0, d_ia_{ii}, \ldots, d_na_{nn}) = (r_1, \ldots, r_n)BA$, for some $r_j \in R$.

Put $r_1 = \cdots = r_{i-1} = 0$, thus $d_i a_{ii} = r_i b_{ii} a_{ii}$, hence $d_i = r_i b_{ii}$. By (1), $b_{ii} = 1$. So $r_i = d_i$. We will show that the equation $d_i b_{i\ i+1} a_{i+1\ i+1} + r_{i+1} b_{i+1\ i+1} a_{i+1\ i+1} = d_{i+1} a_{i+1\ i+1}$ or equivalently, $d_i b_{i\ i+1} + r_{i+1} b_{i+1\ i+1} = d_{i+1}$ has a solution.

Case 1: If $b_{i+1\ i+1} = 1$ then $r_{i+1} = d_{i+1} - d_i b_{ii+1}$.

Case 2: If $b_{i+1\ i+1} = p$ then $i+1 \notin I$.

Since $i+1 \notin I$ by hypothesis $p \not| a_{i+1\ i+1}$, which implies that $p|d_{i+1}$.
Now $d_{i+1} = pd'_{i+1}$ and by(2), $b_{ii+1} = pb'_{ii+1}$ and hence $r_{i+1} = d'_{i+1} - d_i b'_{i+1}$.

Suppose that for every $j < k$ the equation $d_i b_{ij} + r_{i+1} b_{i+1j} + \cdots + r_{j-1} b_{j-1j} + r_j b_{jj} = d_j$ has a solution, we shall find $r_k \in R$ such that $d_i b_{ik} + r_{i+1} b_{i+1k} + \cdots + r_k b_{kk} = d_k$.

For $b_{kk} = 1$,

$$r_k = d_k - (d_i b_{ik} + \cdots + r_{k-1} b_{k-1k}).$$

If $b_{kk} = p$ then $k \notin I$. Hence $p \not| a_{kk}$ and therefore $p|d_k$, by (2). If $1 \leq j < k$ and $j \in I$ by hypothesis we have $p|b_{jk}$. It follows that $p|r_j b_{jk}$.
If $j \notin I$ we have two cases for b_{jj}:
If $b_{jj} = p$, since B is a prime matrix, $b_{jk} = 0$ $(k < j)$, which implies that $p|r_j b_{jk} (j < k)$.
If $b_{jj} = 1$, since B is a prime matrix, $b_{\ell j} = 0$ for $1 \leq \ell < j$ and so $r_j = d_j$, $p|d_j$ since $j \notin I$. Hence in any case we have $p|r_j b_{jk}, 1 \leq j < k$, and so the equation has a solution. Therefore $\langle P \rangle \cap M \subseteq N$. Put $T = \{L \leq R^n | \langle P \rangle \subseteq L \text{ and } L \cap M \subseteq N\}$. Since $\langle P \rangle \in T$, $T \neq \emptyset$. By Zorn's Lemma T has a maximal element N'. It is clear that $N' \cap M = N$. Since $\langle p \rangle = (\langle P \rangle : R^n) \subseteq (N' : R^n) \subseteq (N : M) = \langle p \rangle$, we have $(N' : R^n) = (N : M)$. Therefore $N' \in Spec(R^n)$.

Conversely, let there exist $N' \in Spec(R^n)$ such that $N' \cap M = N$. By Lemma 1.2, $\langle p \rangle = (N : M) = (N' : R^n)$. Let P be as above. Since $\langle P \rangle \subseteq N'$, hence $\langle P \rangle \cap M \subseteq N$. If $i \in I$ then $p|a_{ii}$ and hence $(0, \ldots, 0, a_{ii}, 0, \ldots, 0) \in \langle P \rangle$. But $(0, \ldots, 0, a_{ii}, 0, \ldots, 0) \in M$, implies that $(0, \ldots, 0, a_{ii}, 0, \ldots, 0) \in N$. Thus $(0, \ldots, 0, a_{ii}, 0, \ldots, 0) = (r_1, \ldots, r_n)BA$, for some $r_j \in R$. Hence $r_1 = \cdots = r_{i-1} = 0$ and therefore $r_i b_{ii} a_{ii} = a_{ii}$. So $r_i b_{ii} = 1$ and hence $b_{ii} = 1$. Suppose that $k > i_0$ and $b_{kk} = p$, so $p \not| a_{kk}$. If $k-1 \in I$ we have $p|a_{k-1k-1}$. But

$$(0, \ldots, 0, a_{k-1k-1}, pa_{kk}, 0, \ldots, 0) \in \langle P \rangle \cap M$$

and hence $b_{k-1\ k} + r_k p = p$, for some $r_k \in R$, so $p|b_{k-1\ k}$. In general, if $i \in I$ then $p|a_{ii}$ and

$$(0, \ldots, 0, a_{ii}, 0, \ldots, 0, pa_{kk}, 0, \ldots, 0) \in \langle P \rangle \cap M.$$

Thus

$$(0,\dots,0,a_{ii},0,\dots,0,pa_{kk},0,\dots,0)=(r_1,\dots,r_n)BA$$

for some $r_j \in R$. Hence $r_1 = r_2 = \cdots = r_{i-1} = 0$. Now $r_i b_{ii} a_{ii} = a_{ii}$, $i \in I$. It follows that $b_{ii} = 1$, which implies that $r_i = 1$. Since $(b_{ik} + r_{i+1}b_{i+1\ k} + \cdots + r_k b_{kk})a_{kk} = pa_{kk}$, so $b_{ik} + r_{i+1}b_{i+1\ k} + \cdots + r_k b_{kk} = p$. We now show that $r_j b_{jk} = 0$, $i+1 \le j < k$.
If $b_{jj} = p$,since B is a prime matrix, hence $b_{jk} = 0$.
If $b_{jj} = 1$. Since $b_{ij} + r_{i+1}b_{i+1\ j} + \cdots + r_j b_{jj} = 0$, and B is a prime matrix, hence $b_{\ell j} = 0$ for every $i \le \ell \le j-1$. It follows that $r_j = 0$. Hence $r_j b_{jk} = 0$. We have $b_{ik} + rp = p$,for some $r \in R$. Thus $p|b_{ik}$.

(*iii*) Put $T = \{N' \le R^n | N' \cap M = N\}$. Since $N \in T$, $T \ne \emptyset$. By Zorn's Lemma T has a maximal element N'. Hence $N' \cap M = N$ and $(N' : R^n) = (N : M)$, we have $N' \in Spec(R^n)$.□

2. Going down and incomparability theorems for modules

Definition. Let M and M' be R-modules.

(i) We say *to be going* down (or simply GD) holds for (M, M'), if $M \subseteq M'$, and given any $P'_1 \in Spec(M')$ and $P_0 \subseteq P_1$ in $Spec(M)$ with $P'_1 \cap M = P_1$, we have $P'_0 \in Spec(M')$ such that $P'_0 \subseteq P'_1$ and $P'_0 \cap M = P_0$.

(*ii*) INC="Incomparability" means that, if $M \subseteq M'$ and given P'_1 and P'_2 in $Spec(M')$ with $P'_1 \cap M = P'_2 \cap M \ne M$, we have $P'_1 = P'_2$.

Examples. Let $M = 2\mathbf{Z}$ and $M' = \mathbf{Z}$ be $\mathbf{Z}$-modules. Let p_1 and p'_1 be prime numbers. Put $P'_1 = p'_1\mathbf{Z} \in Spec(M')$ and $P_0 = \{0\} \subseteq P_1 = 2p_1\mathbf{Z} \in Spec(M)$. Since $p'_1\mathbf{Z} \cap 2\mathbf{Z} = 2p_1\mathbf{Z}$, hence $2 \ne p'_1 = p_1$. We choose $P'_0 = \{0\} \in Spec(M')$. Thus we have $P'_0 \cap 2\mathbf{Z} = P_0$ and therefore GD holds for (M, M'). It is easy to see that $4\mathbf{Z} \in Spec(M)$ has no lying over and so LO does not hold for (M, M'). In this example, we show that INC holds for (M, M'). Let $p'_1 \ne 2$ and $p'_2 \ne 2$ be prime numbers. If $P'_1 = p'_1\mathbf{Z}$ and $P'_2 = p'_2\mathbf{Z}$, then $P'_1 \cap 2\mathbf{Z} = 2p'_1\mathbf{Z}$ and $P'_2 \cap 2\mathbf{Z} = 2p'_2\mathbf{Z}$. We must have $2p'_1\mathbf{Z} = 2p'_2\mathbf{Z} \ne 2\mathbf{Z}$ and so $p'_1 = p'_2$. Therefore $P'_1 = P'_2$ and INC holds for (M, M'). Also it is clear that GD holds for (V, W), where V is a vector space with $dim_F V \ge 2$ and proper subspace W.

Theorem 2.1 *Let $\varphi \in Hom_R(M, M')$ and $Ker\varphi \subseteq rad_M(0)$. If for each $m' \in M'$,*

$P \in Spec(M)$ and $r \in (P : M)$ there exists $s \in R\backslash(P : M)$ such that $rsm' \in \varphi(rad_M(0))$, then GD holds for (M, M').

Proof. Suppose $P_0 \subseteq P_1$ in $Spec(M)$ and $Q_1 \in Spec(M')$ such that $\varphi^{-1}(Q_1) = P_1$. Put $T = \{L' \leq M' | L' \subseteq Q_1$ and $\varphi^{-1}(L') \subseteq P_0\}$. Since $\{0\} \in T$, $T \neq \emptyset$, it follows by Zorn's Lemma that T has a maximal element Q_0. Now we show that $\varphi^{-1}(Q_0) = P_0$. Assume that $\varphi^{-1}(Q_0) \subset P_0$. Hence there exists $p_0 \in P_0$ such that $\varphi(p_0) \notin Q_0$. Since $Q_0 \subset Q_0 + \langle\varphi(p_0)\rangle \subseteq Q_1$, there exists $m \in \varphi^{-1}(Q_0 + \langle\varphi(p_0)\rangle)$ such that $m \notin P_0$. Hence there exists $q \in Q_0$ and $r \in R$ such that $\varphi(m) - r\varphi(p_0) = q$, and so $m \in P_0$, which is a contradiction. Therefore $\varphi^{-1}(Q_0) = P_0$.

Now we show that $Q_0 \in Spec(M')$. Let $rm' \in Q_0$, $r \in R$, $m' \in M'$ and $m' \notin Q_0$. Suppose that $Q_0 + Rm' \subseteq Q_1$ and hence there exists $m \in \varphi^{-1}(Q_0 + Rm')$ such that $m \notin P_0$. Therefore $\varphi(m) = q + tm'$, where $q \in Q_0$ and $t \in R$. Thus $r\varphi(m) = rq + rtm'$ and so $rm \in P_0$. Since $P_0 \in Spec(M)$, and $m \notin P_0$, hence $r \in (P_0 : M)$. By Lemma 1.2 (ii), $r \in (P_1 : M) = (Q_1 : M')$.

Now if $Q_0 + Rm' \nsubseteq Q_1$. Then there exists $x \in (Q_0 + Rm')\backslash Q_1$. Hence there exists $q \in Q_0$ and $s \in R$ such that $rx = rq + rsm'$, and so $rx \in Q_1$. Since $Q_1 \in Spec(M')$ and $x \notin Q_1$, hence $r \in (Q_1 : M')$. Assume that $r \notin (Q_0 : M')$. Hence there exists $m'_1 \in M'$ such that $rm'_1 \notin Q_0$. By assumption, there exists $s \in R\backslash(P_1 : M)$ such that $rsm'_1 \in \varphi(rad_M(0))$. Since $rad_M(0) \subseteq P_0$, hence $\varphi(rad_M(0)) \subseteq \varphi(P_0) \subseteq Q_0$ and so $rsm'_1 \in Q_0$. Since $rm'_1 \notin Q_0$, hence $Q_0 \subset Q_0 + \langle rm'_1\rangle \subseteq Q_1$ and by a proof similar to the above, $s \in (P_0 : M)$ and so $s \in (P_1 : M)$; which is a contradiction. We conclude that $Q_0 \in Spec(M')$ and the proof is complete. □

Example. Let M be an R-module such that $Ann_R(M) = m$, where $m \in Max(R)$ (for example, vector spaces). Define the monomorphism $\varphi \in Hom_R(M, M \oplus R/m)$ by $\varphi(x) = (x, 0), x \in M$. Let P be a prime submodule of M. It is clear that $(P : M) = m$. For any $s \in R\backslash(P : M)$, $r \in (P : M)$ and $(x, t+m) \in (M \oplus R/m)$, we have $sr(x, t+m) = s(rx, rt + m) = (0, 0) \in \varphi(rad_M(0))$. Therefore by theorem 2.1, GD holds for (M, M').

Lemma 2.2 *Let $M \subset M'$ be R-modules. The INC holds if one of the following conditions holds.*

(i) For each $m' \in M'$ and $P \in Spec(M)$ there exists $s \in R\backslash(P : M)$ such that $sm' \in M$;

(ii) M' is a multiplication R-module.

(iii) $\sqrt{M : M'} + Ann_R(M) = R$.

Proof. Let P'_0 and P'_1 in $Spec(M')$ and $P'_0 \cap M = P'_1 \cap M \neq M$. Assume that (i) holds and there exists $p'_0 \in P'_0 \backslash P'_1$. By assumption and Lemma 1.2 (ii), there exists $s \in R \backslash (P'_1 : M')$ such that $sp'_0 \in M$ and hence $sp'_0 \in P'_1$. Since $P'_1 \in Spec(M')$ and $p'_0 \notin P'_1$, so $s \in (P'_1 : M')$, which is a contradiction. Now assume that (ii) holds. By Lemma 1.2 (ii), we have

$$(P'_1 : M') = (P'_1 \cap M : M) = (P_0 \cap M : M) = (P'_0 : M).$$

Hence $P'_0 = (P'_0 : M)M' = (P'_1 : M')M' = P'_1$.

(iii) The proof is obvious by part (i). □

In the following we will show that LO and GD are local properties.

Lemma 2.3 *Let $M \subseteq M'$ be R-modules then the following conditions are equivalent.*

(i) LO holds for $M \subseteq M'$.

(ii) LO holds for $M_P \subseteq M'_P$, for all $P \in Spec(R)$.

(iii) LO holds for $M_Q \subseteq M'_Q$, for all $Q \in Max(R)$.

Proof. $(i) \to (ii)$ Let $P \in Spec(R)$, $S = R \backslash P$ and $N'_1 \in Spec(M_P)$. There exists $N_1 \in Spec(M)$ such that $S^{-1}N_1 = N'_1$, by [5, Proposition 1]. By (i), there exists $N_2 \in Spec(M')$ such that $N_2 \cap M = N_1$. Thus $S^{-1}(N_2) \cap M_P = S^{-1}(N_2 \cap M) = S^{-1}N_1 = N'_1$.

$(ii) \to (iii)$ The proof is obvious.

$(iii) \to (i)$ Let $N \in Spec(M)$ there exists $Q \in Max(R)$ such that $(N : M) \subseteq Q$. Put $S = R \backslash Q$, and so $S^{-1}N \in Spec(M_Q)$, by [5, Corollary 3]. By (iii), there exists $S^{-1}N' \in Spec(M_Q)$ such that $S^{-1}N' \cap M_Q = S^{-1}N$ and hence $N = N' \cap M$. □

Lemma 2.4 *Let $M \subseteq M'$ be R-modules. Then the following conditions are equivalent.*

(i) GD holds for $M \subseteq M'$,

(ii) GD holds for $M_P \subseteq M'_P$, for all $P \in Spec(R)$.

(iii) GD holds for $M_Q \subseteq M'_Q$, for all $Q \in Max(R)$.

Proof. The proof is similar to the Lemma 2.3. □

3. On the dimension of a module

Let R be a ring and M be an R-module. Let N be a prime submodule of M. Then we define the height of N to be the maximal positive integer k, if it exists, such that there exists a chain of prime submodules of M as follows:

$$N = N_0 \supset N_1 \supset \cdots \supset N_k.$$

We shall denote the height of N in M by $ht_M(N)$ (see [6]). Suppose that M is an R-module and P be a prime ideal of R. Put $S = R \backslash P$ and define the distinguished submodule $PM(S_P) = \{x \in M : sx \in PM$, for some $s \in S\}$ of M. We define [see 1] the dimension of M to be the maximal positive integer k, if such exists, such that there exists a chain of prime distinguished submodules of M as

$$N_0 \subset N_1 \subset \cdots \subset N_k.$$

Lemma 3.1 *Let $M \subseteq M'$ be R-modules. Assume that for every $m' \in M'$ and $P \in Spec(M)$ there exists $s \in R \backslash (P : M)$ such that $sm' \in M$. If $PM(S_P) \neq M$ then $PM'(S_P) \cap M = PM(S_P)$.*

Proof. Since $PM(S_P) \neq M$, by [1, Proposition 1.1], $PM(S_P) \in Spec(M)$. It is clear that $PM(S_P) \subseteq M \cap PM'(S_P)$. Suppose that $PM'(S_P) \cap M \not\subseteq PM(S_P)$. Hence there exists $m \in PM'(S_P) \cap M$ such that $m \notin PM(S_P)$. Thus there exists $s \in R \backslash P$ such that $sm = \sum_{i=1}^{n} p_i m'_i$, where $p_i \in P$ and $m'_i \in M'$. By assumption there exists $s_i \in R \backslash P$ such that $s_i m'_i \in M$. Since $(\prod_{j=1}^{n} s_i) sm = \sum_{i=1}^{n} p_i (\prod_{j=1}^{n} s_i) m'_i$, hence $m \in PM(S_P)$, which is a contradiction. □

Lemma 3.2 *Let $M \subseteq M'$ be R-modules and assume that GD holds for (M, M'). Suppose $P \in Spec(M)$ and $ht_M(P) = k$. If $P' \in Spec(M')$ such that $P' \cap M = P$ then $ht_{M'}(P') \geq k$.*

Proof. Since $ht_M(P) = k$, there exists a chain of prime submodules of M as follows

$$P = P_0 \supset P_1 \supset \cdots \supset P_k.$$

Since GD holds for (M, M') and so there exists a chain of prime submodules of M' as follows

$$P' = P'_0 \supset P'_1 \supset \cdots \supset P'_k.$$

Hence $ht_{M'}(P') \geq k$. □

Corollary 3.3 *Let $M \subseteq M'$ be R-modules. Let M' be multiplication module and for each $m' \in M'$, $P \in Spec(M)$ and $r \in (P : M)$, there exists $s \in R\backslash(P : M)$ such that $rsm' \in rad_M(0)$. Suppose that $ht_M(P) = k$. Then there exists $P' \in Spec(M')$ such that $P' \cap M = P$ and $ht_{M'}(P') = k$.*

Proof. By Theorem 1.5, there exists $P' \in Spec(M')$ such that $P' \cap M = P$ and by Lemma 3.2, $ht_{M'}(P') \geq k$. Suppose that $ht_{M'}(P') = n$, so there exists a chain of prime submodules of M' as follows:

$$P' = P'_0 \supset P'_1 \supset \cdots \supset P'_n.$$

By Theorem 2.1 and Lemma 2.2, since GD and INC hold, we have the following chain of prime submodules of M:

$$P = P' \cap M \supset P'_1 \cap M \supset \cdots \supset P'_n \cap M.$$

Therefore $ht_M(P) \geq n$ and so $ht_{M'}(P') = k$. □

Proposition 3.4 *Let $M \subseteq M'$ be R-modules and M' be a finitely generated module such that $Ann_R(M') \subseteq N(R)$. If for all P_1, P_2 in $Spec(M')$ we have $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$ then $dimM \leq dimM'$.*

Proof. Let $dimM = n$. Hence there exists a chain of distinguished submodules of M as follows:

$$N_0 \subset N_1 \subset \cdots \subset N_n.$$

Put $(N_i : M) = P_i$, for all i. Since M' is finitely generated and $Ann_R(M') \subseteq N(R)$, hence $P_iM'(S_{P_i}) \neq M'$ by [1, Corollary 1.2]. By assumption $P_iM'(S_{P_i}) \subseteq P_{i+1}(M')(S_{P_{i+1}})$ or

$P_{i+1}M'(S_{P_{i+1}}) \subseteq P_iM'(S_{P_i})$. Since $P_i \subseteq P_{i+1}$ implies that $P_iM'(S_{P_i}) \subseteq P_{i+1}M'(S_{P_{i+1}})$, we have the chain

$$P_0M'(S_{P_0}) \subset P_1M'(S_{P_1}) \subset \cdots \subset P_nM'(S_{P_n}).$$

Therefore $dimM \leq dimM'$. □

Proposition 3.5 *Let $M \subseteq M'$ be R-modules and M be a finitely generated module such that $Ann_R(M) \subseteq N(R)$. If for all P_1, P_2 in Spec(M) we have $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$, then $dimM' \leq dimM$.*

Proof. It is similar to the proof of Proposition 3.4. □

Proposition 3.6 *Let $M \subseteq M'$ be R-modules. Suppose that for every $m' \in M'$ and $P \in Spec(M)$ there exists $s \in R\backslash(P : M)$ such that $sm' \in M$. If $dimM = n$ then $dimM = dimM'$.*

Proof. Since $dimM = n$, there exists a chain of distinguished submodules of M as follows

$$N_0 \subset N_1 \subset \cdots \subset N_n.$$

Let $(N_i : M) = P_i$, for all i. Then by Lemma 3.1, we have the following chain of distinguished submodules of M'

$$P_0M'(S_{P_0}) \subset P_1M'(S_{P_1}) \subset \cdots \subset P_nM'(S_{P_n}).$$

This implies $dimM \leq dimM'$. Now let there exist a chain of distinguished submodules of M' as follows

$$P_0M'(S_{P_0}) \subset P_1M'(S_{P_1}) \subset \cdots \subset P_kM'(S_{P_k}).$$

We show that $M \not\subset P_kM'(S_{P_k})$. Suppose that $M \subset P_kM'(S_{P_k})$. Since $P_kM'(S_{P_k}) \neq M'$, there exists $m' \in M'\backslash P_kM'(S_{P_k})$. By assumption there exists $s \in R\backslash P_k$ such that $sm' \in M \subset P_kM'(S_{P_k})$. Since $m' \notin P_kM'(S_{P_k})$, hence $s \in P_k$, which is a contradiction. Therefore $P_iM(S_{P_i}) \neq M$, for all i, and so we have the following chain of distinguished submodules of M

$$P_0M(S_{P_0}) \subset \cdots \subset P_kM(S_{P_k})$$

hence $dimM \geq k$. □

Acknowledgement

The authors would like to thank the referee for his/her useful suggestions that improved the presentation of this paper.

References

[1] Abu-Saymeh, S.: On dimensions of finitely generated modules, Comm. Algebra 23, 1131-1144 (1995).

[2] Hedayat, S., Nekooei, R.: Prime and radical submodules of free modules over a PID, Houston Journal of Mathematics, 32, 2,355-367 (2006).

[3] Lu, C.P.: Prime submodules of modules, Comment. Math. Univ. Sancti Pauli, 33, 1, 61-69 (1984).

[4] Lu, C.P.: M-Radicals of submodules in modules, Math. Japonica 34, 2, 211-219 (1989).

[5] Lu, C.P.: Spectra of modules, Comm. Alg. 23, 3471-3752 (1995).

[6] McCasland, R.L., Smith, P.F.: Prime submodules of Neotherian modules, Rocky mountain J. Math. 23, 1041-1062 (1993).

[7] Nekooei, R.: On finitely generated multiplication modules, Czechoslovak Mathematical Journal, 55, 130, 503-510 (2005).

[8] Pusat-Yilmaz, D.: On prime submodules of finitely generated free modules, Turkish Journal of Mathematics, 27, 329- 342 (2003).

[9] Rowen, L.: Ring Theory, Vol. I, Academic Press, San Diego, 1988.

[10] Sharp, D.W., Vamos, P.: Injective Modules, Cambridge University Press, London, 1972.

S. KARIMZADEH and R. NEKOOEI
Department of Mathematics,
Shahid Bahonar University of Kerman
Kerman-IRAN
e-mail: rnekooei@mail.uk.ac.ir

Received 27.09.2005