Turk J Math
31 (2007) , 163 – 170.


# Extreme Points of Certain Subsets of Hermitian Elements in Banach Algebras

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**Abstract**

We consider the real Banach spaces $\mathcal{H}(\mathcal{A})$ of all hermitian elements of a complex Banach algebra $\mathcal{A}$. We prove that if an even power of $a \in \mathcal{H}(\mathcal{A})$ is hermitian, then $a$ is an extreme point of the unit ball of $\mathcal{H}(\mathcal{A})$ if and only if $a^2 = \mathbf{1}$. Moreover, if an odd power of $a \in \mathcal{H}(\mathcal{A})$ is hermitian and $a$ is an extreme point of the unit ball of $\mathcal{H}(\mathcal{A})$, then $a^3 = a$.

**Key words and phrases:** Extreme points, hermitian elements.

## 1. Terminology and Introduction

Let $K$ be a subset of a (real or complex) vector space $X$. A point $x \in K$ is called an *extreme point* of $K$ if $y,\ z \in K$, $0 < \mu < 1$ and

$$x = \mu y + (1 - \mu) z$$

imply that $x = y = z$. We denote the set of extreme points of $K$ by $ext(K)$.

Throughout this paper $\mathcal{A}$ denotes a complex unital Banach algebra with unit $\mathbf{1}$. If $\mathcal{B}$ is a subset of $\mathcal{A}$ we write $(\mathcal{B})_1$ for the set

$$(\mathcal{B})_1 = \{a \in \mathcal{B} : \|a\| \leq 1\}.$$

2000 *AMS Mathematics Subject Classification:* 46H05.

For $a \in \mathcal{A}$ we denote by $\sigma(a)$ the spectrum of $a$ and by $r(a)$ the spectral radius of $a$. Let $\mathcal{A}'$ denote the topological dual space of $\mathcal{A}$. Given $a \in \mathcal{A}$ the set

$$V(a) = \{\varphi(a) : \varphi \in \mathcal{A}',\, \varphi(\mathbf{1}) = 1 = \|\varphi\|\}$$

is called the *numerical range* of $a$.

An element $a \in \mathcal{A}$ is called *hermitian* if $V(a) \subseteq \mathbb{R}$. The set of all hermitian elements in $\mathcal{A}$ is denoted by $\mathcal{H}(\mathcal{A})$. Observe that if $\mathcal{A}$ is a $B^*$-algebra, then $a \in \mathcal{H}(\mathcal{A})$ if and only if $a^* = a$ (see [1, page 47]).

**Proposition 1.1**

(1) *If* $a \in \mathcal{A}$, *then* $\sigma(a) \subseteq V(a)$*;*

(2) *If* $a \in \mathcal{H}(\mathcal{A})$, *then* $\sigma(a) \subseteq \mathbb{R}$*;*

(3) $\mathcal{H}(\mathcal{A})$ *is a real Banach space;*

(4) *If* $a \in \mathcal{H}(\mathcal{A})$, *then* $r(a) = \|a\|$*; and*

(5) $a \in \mathcal{H}(\mathcal{A}) \iff \|\exp(ita)\| = 1$ *for all* $t \in \mathbb{R}$.

**Proof.** (1) [1, Theorem 2.6]. (2) follows from (1). (3) [1, Lemma 5.4]. (4) [5]. (5) [1, Lemma 5.2]. □

The following result is well-known. For the convenience of the reader we include a proof.

**Proposition 1.2** *Let* $a \in \mathcal{A}$ *and* $\mathcal{M} \subseteq \mathcal{A}$.

(1) *Suppose that* $a \in (\mathcal{A})_1$, *is invertible and that* $a^{-1} \in (\mathcal{A})_1$. *Then* $a \in ext((\mathcal{A})_1)$.

(2) *If* $a \in (\mathcal{M})_1$ *is invertible and* $a^{-1} \in (\mathcal{A})_1$, *then* $a \in ext((\mathcal{M})_1)$.

**Proof.** (1) Let $a = \mu b + (1-\mu)c$ with $b,\, c \in (\mathcal{A})_1$ and $0 < \mu < 1$. Then

$$\mathbf{1} = \mu b a^{-1} + (1-\mu) c a^{-1}\,.$$

For $\varphi \in \mathcal{A}'$ with $\varphi(\mathbf{1}) = 1 = \|\varphi\|$ it follows that

$$1 = \mu\varphi(ba^{-1}) + (1-\mu)\varphi(ca^{-1})\,.$$

Since $b, c, a^{-1} \in (\mathcal{A})_1$ we have $|\varphi(ba^{-1})| \leq 1$ and $|\varphi(ca^{-1})| \leq 1$. Since 1 is an extreme point of the closed unit disc in $\mathbb{C}$, we get $\varphi(ba^{-1}) = \varphi(ca^{-1}) = 1$. This shows that

$$V(ba^{-1}) = V(ca^{-1}) = \{1\}\,.$$

Hence $ba^{-1}$, $ca^{-1} \in \mathcal{H}(\mathcal{A})$. Proposition 1.1 (1) gives $\sigma(ba^{-1}) = \sigma(ca^{-1}) = \{1\}$. By Proposition 1.1 (3), $ba^{-1} - \mathbf{1}$, $ca^{-1} - \mathbf{1} \in \mathcal{H}(\mathcal{A})$. From $\sigma(ba^{-1} - \mathbf{1}) = \sigma(ca^{-1} - \mathbf{1}) = \{0\}$ we see (Proposition 1.1 (4)) that

$$\|ba^{-1} - \mathbf{1}\| = \|ca^{-1} - \mathbf{1}\| = 0\,,$$

thus $a = b = c$.
(2) If $a = \mu b + (1-\mu)c$ with $b$, $c \in (\mathcal{M})_1$ and $0 < \mu < 1$, then $b$, $c \in (\mathcal{A})_1$. It follows from (1) that $a = b = c$, thus $a \in ext((\mathcal{M})_1)$. □

**Proposition 1.3** *Let $a \in (\mathcal{A})_1$ and suppose that $\{-1, 1\} \subseteq \sigma(a)$. Let*

$$\langle \mathbf{1}, a\rangle_{\mathbb{R}} = \{\alpha\mathbf{1} + \beta a : \alpha,\, \beta \in \mathbb{R}\} \quad \textit{and} \quad \langle \mathbf{1},\, a\rangle_{\mathbb{C}} = \{\alpha\mathbf{1} + \beta a : \alpha,\, \beta \in \mathbb{C}\}\,.$$

*Then $a \in ext((\langle \mathbf{1},\, a\rangle_{\mathbb{R}})_1)$ and $a \in ext((\langle \mathbf{1},\, a\rangle_{\mathbb{C}})_1)$.*

**Proof.** Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. First we show that $\mathbf{1}$ and $a$ are linearly independent. To this end assume that $\alpha$, $\beta \in \mathbb{K}$ and $0 = \alpha\mathbf{1} + \beta a$. Since $1, -1 \in \sigma(a)$, the spectral mapping theorem gives $\alpha + \beta = 0 = \alpha - \beta$. Thus $\alpha = \beta = 0$.
Now let $a = \mu b + (1-\mu)c$ with $b$, $c \in (\langle \mathbf{1},\, a\rangle_{\mathbb{K}})_1$ and $0 < \mu < 1$. There are $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{K}$ such that $b = \alpha_1\mathbf{1} + \alpha_2 a$ and $c = \beta_1\mathbf{1} + \beta_2 a$, hence

$$a = (\mu\alpha_1 + (1-\mu)\beta_1)\mathbf{1} + (\mu\alpha_2 + (1-\mu)\beta_2)a\,.$$

Since $\mathbf{1}$ and $a$ are linearly independent,

$$\mu\alpha_1 + (1-\mu)\beta_1 = 0$$

and

$$\mu\alpha_2 + (1-\mu)\beta_2 = 1\,.$$

It follows that

$$\mu(\alpha_2 + \alpha_1) + (1-\mu)(\beta_2 + \beta_1) = 1 \tag{1.1}$$

and

$$\mu(\alpha_2 - \alpha_1) + (1 - \mu)(\beta_2 - \beta_1) = 1\,. \tag{1.2}$$

From $-1,\, 1 \in \sigma(a)$ we get $\alpha_1+\alpha_2,\ \alpha_1-\alpha_2 \in \sigma(b)$. Since $\|b\| \leq 1,\ |\alpha_1+\alpha_2|,\ |\alpha_1-\alpha_2| \leq 1$. Similarly, $|\beta_1+\beta_2|,\ |\beta_1-\beta_2| \leq 1$. From (1.1) and (1.2) we now see that $\alpha_1+\alpha_2 = \beta_1+\beta_2 = 1 = \alpha_2 - \alpha_1 = \beta_2 - \beta_1$, hence $\alpha_2 = \beta_2 = 1$ and $\alpha_1 = \beta_1 = 0$, thus $a = b = c$. □

## 2. Extreme points of subsets of $\mathcal{H}(\mathcal{A})$

We first give an example, due to M. J. Crabb, which shows the existence of a hermitian element of which the square is not hermitian.

**Example 2.1** *([1, page 57]).*
*Let $\mathcal{A} = \mathbb{C}^3$ with pointwise multiplications, and let $p : \mathbb{C}^3 \to [0,\infty)$ be defined by*

$$p(\alpha,\, \beta,\, \gamma) = \sup\{|\lambda^{-1}\alpha + \beta + \lambda\gamma| : \lambda \in \mathbb{C},\ |\lambda| = 1\}\,.$$

*Define the norm $\|\cdot\|$ on $\mathcal{A}$ by*

$$\|a\| = \sup\{p(xa) : x \in \mathcal{A},\ p(x) = 1\}\,.$$

*Then $\mathcal{A}$ is a complex unital (commutative) Banach algebra with respect to $\|\cdot\|$.*
*Let $a = (-1, 0, 1)$. The following properties are shown in [1]:*

(1) $a \in \mathcal{H}(\mathcal{A}),\ a^2 \notin \mathcal{H}(\mathcal{A}),\ \sigma(a) = \{-1, 0, 1\}$;

(2) $\mathcal{A} = \{\alpha\mathbf{1} + \beta a + \gamma a^2 : \alpha,\, \beta,\, \gamma \in \mathbb{C}\}$;

(3) $\mathcal{H}(\mathcal{A}) = \{\alpha\mathbf{1} + \beta a : \alpha, \beta \in \mathbb{R}\}$.

*We have $a^3 = a$ and, by Proposition 1.3 and (3), $a \in ext((\mathcal{H}(\mathcal{A}))_1)$. This is not an accident, as we see in what follows.*

Before we state the main results of this paper, we need the following lemma.

**Lemma 2.2** *Let the functions* $f_1, f_2, f_3, f_4 : [-1,1] \to \mathbb{R}$ *be defined by*

$$f_1(t) = t + \frac{1}{2k}(1 - t^{2k}),\ f_2(t) = t - \frac{1}{2k}(1 - t^{2k}),$$
$$f_3(t) = t + \frac{1}{2k+1}(t - t^{2k+1}) \quad \text{and} \quad f_4(t) = t - \frac{1}{2k+1}(t - t^{2k+1}),$$

*where* $k \in \mathbb{N}$*. Then*

$$f_j([-1,1]) = [-1,1] \quad (j = 1, \dots, 4).$$

**Proof.** Routine. □

**Theorem 2.3** *Suppose that* $a \in (\mathcal{H}(\mathcal{A}))_1$*, and that* $a^{2k} \in \mathcal{H}(\mathcal{A})$ *for some* $k \in \mathbb{N}$*. Then the following assertions are equivalent:*

(1) $a \in ext((\mathcal{H}(\mathcal{A}))_1)$.

(2) $a^2 = \mathbf{1}$.

**Proof.** (1) $\Rightarrow$ (2): Let $h = \frac{1}{2k}(\mathbf{1} - a^{2k})$. Then $h, a+h, a-h \in \mathcal{H}(\mathcal{A})$. Let $\lambda \in \sigma(a+h)$. By the spectral mapping theorem, $\lambda = \alpha + \frac{1}{2k}(1 - \alpha^{2k})$ for some $\alpha \in \sigma(a)$. Since $\sigma(a) \subseteq [-1,1]$, $\lambda = f_1(\alpha)$ with $\alpha \in [-1,1]$, where $f_1$ is as in Lemma 2.2. Hence $\lambda \in [-1,1]$. This shows that $\sigma(a+h) \subseteq [-1,1]$. Hence, by Proposition 1.1 (4), $\|a+h\| = r(a+h) \le 1$. Thus $a+h \in (\mathcal{H}(\mathcal{A}))_1$. A similar argument (use the function $f_2$ in Lemma 2.2) shows that $a - h \in (\mathcal{H}(\mathcal{A}))_1$. We have

$$a = \frac{1}{2}(a+h) + \frac{1}{2}(a-h).$$

Since $a$ is an extreme point of $(\mathcal{H}(\mathcal{A}))_1$, $a = a+h = a-h$, hence $h = 0$. Therefore $a^{2k} = \mathbf{1}$. If the entire function $g$ is defined by $g(z) = z^{2k} - 1$, then $g$ has only simple zeros and $g(a) = 0$. It follows from [2, Proposition 8.11] that

$$\sigma(a) = \{\lambda_1, \dots, \lambda_r\}, \quad g(\lambda_j) = 0 \quad (j = 1, \dots, r)$$

and

$$(a - \lambda_1\mathbf{1})(a - \lambda_2\mathbf{1}) \cdot \ldots \cdot (a - \lambda_r\mathbf{1}) = 0.$$

Since $|\lambda_j| = 1\,(j = 1, \ldots, r)$ and $\sigma(a) \subseteq [-1, 1]$, it follows that $\sigma(a) \subseteq \{-1, 1\}$, hence $a^2 = \mathbf{1}$.
(2) $\Rightarrow$ (1): Since $a \in \mathcal{H}(\mathcal{A})$ and $a = a^{-1}$, $1 = r(a^2) = r(a)^2 = \|a\|^2$, hence $a \in (\mathcal{H}(\mathcal{A}))_1$. With $\mathcal{M} = \mathcal{H}(\mathcal{A})$ we see that $a \in ext((\mathcal{M})_1)$, by Proposition 1.2 (2). □

**Theorem 2.4** *If $a \in (\mathcal{H}(\mathcal{A}))_1$ is an extreme point of $(\mathcal{H}(\mathcal{A}))_1$ and if $a^{2k+1} \in \mathcal{H}(\mathcal{A})$ for some $k \in \mathbb{N}$, then $a^3 = a$.*

**Proof.** Let $h = \frac{1}{2k+1}(a - a^{2k+1})$. Then $h$, $a + h$, $a - h \in \mathcal{H}(\mathcal{A})$. As in the proof of Theorem 2.3 we see that $\sigma(a \pm h) \subseteq [-1, 1]$ (use the functions $f_3$ and $f_4$ of Lemma 2.2). Thus $\|a \pm h\| = r(a \pm h) \leq 1$, therefore $a \pm h \in (\mathcal{H}(\mathcal{A}))_1$. Since $a$ is an extreme point of $(\mathcal{H}(\mathcal{A}))_1$ and since

$$a = \frac{1}{2}(a + h) + \frac{1}{2}(a - h)\,,$$

it follows that $a = a + h = a - h$, and so $h = 0$. If the entire function $g$ is defined by $g(z) = z - z^{2k+1}$, then $g$ has only simple zeros and $g(a) = 0$. As in the proof of Theorem 2.3 we derive $\sigma(a) \subseteq \{0, 1, -1\}$ and $a^3 = a$. □

**Corollary 2.5** *If $a \in (\mathcal{H}(\mathcal{A}))_1$ is an extreme point of $(\mathcal{H}(\mathcal{A}))_1$ and if $a^n \in \mathcal{H}(\mathcal{A})$ for some $n \in \mathbb{N}$, $n \geq 2$, then $a^3 = a$.*

We say that $a \in \mathcal{H}(\mathcal{A})$ is *positive* if $\sigma(a) \subseteq [0, \infty)$. We denote by $pos(\mathcal{A})$ the set of all positive elements of $\mathcal{A}$.

**Corollary 2.6** *If $p \in (pos(\mathcal{A}))_1$ and $p^2 \in \mathcal{H}(\mathcal{A})$, then the following assertions are equivalent:*

(1) $p \in ext((pos(\mathcal{A}))_1)$.

(2) $p^2 = p$.

**Proof.** Since

$$p \in (pos(\mathcal{A}))_1 \quad \Longleftrightarrow \quad 2\left(p - \frac{1}{2}\mathbf{1}\right) \in (\mathcal{H}(\mathcal{A}))_1$$

and

$$p^2 \in \mathcal{H}(\mathcal{A}) \quad \Longleftrightarrow \quad \left(2\left(p - \frac{1}{2}\mathbf{1}\right)\right)^2 \in \mathcal{H}(\mathcal{A}),$$

it follows from Theorem 2.3 that $p$ is an extreme point of $(pos(\mathcal{A}))_1$ if and only if $(2(p - \frac{1}{2}\mathbf{1}))^2 = \mathbf{1}$. □

Now let $H$ denote a complex Hilbert space and consider the $B^*$-algebra $\mathcal{A} = \mathcal{B}(H)$, the Banach algebra of all bounded linear operators on $H$. Then

$$\mathcal{H}(\mathcal{A}) = \{A \in \mathcal{B}(H) : A \text{ is selfadjoint}\}$$

and

$$pos(\mathcal{A}) = \{A \in \mathcal{H}(\mathcal{A}) : A \geq 0\},$$

thus $(pos(\mathcal{A}))_1 = \{A \in \mathcal{H}(\mathcal{A}) : 0 \leq A \leq I\}$, where $I$ denotes the identity operator on $H$. Observe that if $A \in \mathcal{H}(\mathcal{A})$, then $A^n \in \mathcal{H}(\mathcal{A})$ for all $n \in \mathbb{N}$.

As an immediate consequence of Theorem 2.3 and Corollary 2.6, we get the following well-know results (see [4, 2.5.6]):

**Corollary 2.7** *Let $H$ and $\mathcal{A}$ be as above.*

(1) $A \in (\mathcal{H}(\mathcal{A}))_1$ *is an extreme point of* $(\mathcal{H}(\mathcal{A}))_1$, *if and only if* $A^2 = I$.

(2) $P \in (pos(\mathcal{A}))_1$ *is an extreme point of* $(pos(\mathcal{A}))_1$ *if and only if* $P^2 = P$.

For a characterisation of the extreme points of the unit ball of a general $B^*$-algebra see [3, Theorem 9.5.16].

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