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The Construction of Maximum Independent Set of Matrices via Clifford Algebras

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Abstract

In [1], [2] and [6] the maximum number of some special type $n \times n$ matrices with elements in F whose nontrivial linear combinations with real coefficients are nonsingular is studied where F is the real field $\mathbb{R}$, the complex field $\mathbb{C}$ or the skew field $\mathbb{H}$ of quaternions. In this work we construct such matrices explicitly by using representations of Clifford algebras. At the end we give some analogues of the celebrated theorem of Radon-Hurwitz.

Key Words: Hurwitz Theorem, Clifford algebras, maximum independent set of matrices.

1. Introduction

We shall write $F(n)$ for (resp. $F_x(n)$, $n \times n$ matrices with property x) the maximum number of $n \times n$ matrices with elements in F whose nontrivial linear combinations with real coefficients are non-singular and x will stand for hermitian (h), skew-hermitian (sk-h), symmetric (s), or skew-symmetric (sk-s). If n is a positive integer we write $n = (2a+1)\,2^b$ where $b = c{+}4d$ and a, b, c, d are non-negative integers with $0 \leq c < 4$, and Radon-Hurwitz function ρ of n as $\rho(n) = 2^c + 8d$.

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In [1] and [2] $F(n)$ and $F_h(n)$ are calculated as follows: (Note that $\mathbb{R}_h(n) = \mathbb{R}_s(n)$)

$$\begin{aligned}
\mathbb{R}(n) &= \rho(n) \\
\mathbb{R}_s(n) &= \rho(\frac{n}{2}) + 1 \\
\mathbb{C}(n) &= 2b + 2 \\
\mathbb{C}_h(n) &= 2b + 1 \\
\mathbb{H}(n) &= \rho(\frac{n}{2}) + 4 \\
\mathbb{H}_h(n) &= \rho(\frac{n}{4}) + 5
\end{aligned}$$

and in [6] $\mathbb{R}_{sk-s}(n)$, $F_s(n)$ and $F_{sk-h}(n)$ are calculated as follows:

$$\begin{aligned}
\mathbb{R}_{sk-s}(n) &= \rho(n) - 1 \\
\mathbb{C}_s(n) &= \rho(\frac{n}{2}) + 2 \\
\mathbb{C}_{sk-h}(n) &= 2b + 1 \\
\mathbb{H}_s(n) &= \rho(\frac{n}{2}) + 4 \\
\mathbb{H}_{sk-h}(n) &= \mathbb{H}(n) - 1 = \rho(\frac{n}{2}) + 3
\end{aligned}$$

The goal of this work is to construct the maximum number of matrices for each case.

2. Construction of Matrices

Most of our statements in this work are related to the Radon-Hurwitz Theorem.

Theorem 2.1 *(Radon-Hurwitz) The maximum number of $n \times n$ real orthogonal matrices $\{A_1, A_2, \cdots A_k\}$ satisfying the relations $A_i^2 = -I$, $A_iA_j + A_jA_i = 0$ for $i \neq j$ is $\rho(n) - 1$.*

There are various applications of this theorem; (see [5]). If a family of matrices $\{A_1, A_2, \cdots A_k\}$ has the above properties then it is called a Radon-Hurwitz family. Such a family of matrices is given in [3] by using the representations of the real Clifford algebra $Cl_{n,0}$ where $Cl_{n,0}$ is the Clifford algebra on $\mathbb{R}^n$ with the quadratic form $q : \mathbb{R}^n \longrightarrow \mathbb{R}$, $q(x_1, x_2, \cdots, x_n) = \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)$; (see [4]).

2.1. Construction of Real Matrices

1. Let $\{A_1, A_2, ..., A_k\}$ be a family of Radon-Hurwitz matrices, where $k = \rho(n) - 1$. Note that if a matrix A has the property $A^2 = -I$ then A is orthogonal iff A is skew-symmetric. Consider the nontrivial linear combination $A = \sum_{i=1}^{k} \lambda_i A_i$ for some non-zero real numbers λ_i. We can calculate

$$A^2 = \left(-\lambda_1^2 - \lambda_2^2 - ... - \lambda_k^2\right) I.$$

Since some $\lambda_i \neq 0$, the *determinat* of A^2 is not equal to zero, so $\det A$ is not *zero*.

2. Let $\{A_1, A_2, ..., A_k\}$ be a family of Radon-Hurwitz matrices, where $k = \rho(n) - 1$ and consider the set of $n \times n$ matrices $\{A_1, A_2, ..., A_k, I\}$ where I is the $n \times n$ unit matrix. Then the non-trivial real linear combination of these matrices

$$A = \sum_{i=1}^{k} \lambda_i A_i + \lambda I$$

is non-singular. Since $A^t = -\sum_{i=1}^{k} \lambda_i A_i + \lambda I$, and

$$AA^t = -\left(\lambda_1^2 + \lambda_2^2 + ... + \lambda_k^2 + \lambda^2\right) I,$$

we get $\det(AA^t) \neq 0$, whence $\det A \neq 0$. Hence we obtain a collection of $\rho(n) = \mathbb{R}(n)$ $n \times n$ real matrices.

3. The construction of $n \times n$,$\mathbb{R}_s(n) = \rho\left(\frac{n}{2}\right) + 1$ real symmetric matrices:

Such matrices can be obtained using by a real representation of Clifford algebra $Cl_{0,k+2}$ where $k = \rho\left(\frac{n}{2}\right) - 1$ and $Cl_{0,k+2}$ is the Clifford algebra on $\mathbb{R}^{k+2}$ with the quadratic form $q : \mathbb{R}^{k+2} \longrightarrow \mathbb{R}$, $q(x_1, x_2, \cdots, x_{k+2}) = -\left(x_1^2 + x_2^2 + \cdots + x_{k+2}^2\right)$ (see [4]). Let $\{A_1, A_2, ..., A_k\}$ be a family of Radon-Hurwitz matrices of type $\left(\frac{n}{2}\right) \times \left(\frac{n}{2}\right)$. Then we can define the following map:

$$\begin{array}{rccc} \psi_k : & Cl_{k,0} & \longrightarrow & Mat\left(\frac{n}{2}, \mathbb{R}\right) \\ & e_i & \longmapsto & \psi_k(e_i) = A_i \end{array}$$

where $Cl_{k,0}$ is the Clifford algebra over $\mathbb{R}^k$. This map is a real representation of Clifford algebra $Cl_{k,0}$. Also the map

$$\begin{array}{rccc} \psi_{0,2}: & Cl_{0,2} & \longrightarrow & Mat\,(2,\mathbb{R}) \\ & \varepsilon_1 & \longmapsto & \psi_{0,2}(\varepsilon_1)=\sigma_1 \\ & \varepsilon_2 & \longmapsto & \psi_{0,2}(\varepsilon_2)=\sigma_2 \end{array}$$

is a representation of $Cl_{0,2}$ where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Moreover, $\psi_{0,2}$ is an algebra isomorphism. By using the maps $\psi_{k,0}$ and $\psi_{0,2}$ we can define a new map $\psi_{k,0} \otimes \psi_{0,2}$ which is called the tensor (or Kronecker) product of the maps $\psi_{k,0}$ and $\psi_{0,2}$:

$$\begin{gathered} \psi_{k,0} \otimes \psi_{0,2} : Cl_{k,0} \otimes Cl_{0,2} \longrightarrow Mat\left(\tfrac{n}{2},\mathbb{R}\right) \otimes Mat\,(2,\mathbb{R}) \cong Mat\,(n,\mathbb{R}) \\ \left(\psi_{k,0} \otimes \psi_{0,2}\right)(u \otimes v) = \psi_{k,0}\,(u) \otimes \psi_{0,2}\,(v)\,, \end{gathered}$$

where $Mat\,(n,\mathbb{R})$ is the set of all $n \times n$ real matrices. The map $\psi_{k,0} \otimes \psi_{0,2}$ is an algebra homomorphism. Since $Cl_{k,0} \otimes Cl_{0,2} \cong Cl_{0,k+2}$ we can get a real representation $\psi_{0,k+2}$ of the Clifford algebra $Cl_{0,k+2}$ which is defined on generators as follows:

$$\begin{array}{ccccc} Cl_{0,k+2} & \longrightarrow & Cl_{k,0} \otimes Cl_{0,2} & \longrightarrow & Mat\,(n,\mathbb{R}) \\ \varepsilon_1 & \longmapsto & 1 \otimes \varepsilon_1 & \longmapsto & I \otimes \sigma_1 \\ \varepsilon_2 & \longmapsto & 1 \otimes \varepsilon_2 & \longmapsto & I \otimes \sigma_2 \\ \varepsilon_3 & \longmapsto & e_1 \otimes \varepsilon_1\varepsilon_2 & \longmapsto & A_1 \otimes \sigma_1\sigma_2 \\ \vdots & & \vdots & & \vdots \\ \varepsilon_{k+2} & \longmapsto & e_k \otimes \varepsilon_1\varepsilon_2 & \longmapsto & A_k \otimes \sigma_1\sigma_2 \end{array}$$

The image of the generators ε_i, $1 \leq i \leq k+2$ of $Cl_{0,k+2}$ under the homomorphism $\psi_{0,k+2}$ are the matrices that we are looking for. Define

$$B_1 = I \otimes \sigma_1,\ B_2 = I \otimes \sigma_2 \text{ and } B_{j+2} = A_j \otimes \sigma_1\sigma_2,\ \text{for } j = 1,2,\cdots,k.$$

Note that these matrices are symmetric and $B_i^2 = I$, $B_iB_j + B_jB_i = 0$ for $i \neq j$. Let us consider a non-trivial real linear combination of the members of the family $\{B_1, B_2, ..., B_{k+2}\}$,

$$B = \sum_{i=1}^{k+2} \lambda_i B_i.$$

We calculate

$$B^2 = \left(\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_{k+2}^2\right) I.$$

Since some $\lambda_i \neq 0$, the *determinant* of B^2 is not equal to zero, so $\det B \neq 0$.

2.2. The construction of complex matrices

In this case, before giving $\mathbb{C}(n) = 2b+2$ complex matrices of type $n \times n$, we obtain $\mathbb{C}_{sk-h}(n) = 2b+1$ complex skew-hermitian matrices of type $n \times n$ by using a skew-hermitian representation of the complex Clifford algebra $\mathbb{C}l_m$, where $n = (2a+1)2^b$. $\mathbb{C}l_m$ is the complex Clifford algebra over the complex vector space $\mathbb{C}^m$ with the quadratic form $Q : \mathbb{C}^m \longrightarrow \mathbb{C}$, $Q(z_1, z_2, \cdots, z_m) = \left(z_1^2 + z_2^2 + \cdots + z_m^2\right)$; (see [4]).

First, irreducible skew-hermitian representation of complex Clifford algebra $\mathbb{C}l_m$ will be obtained for all m.

To do this we use following representations in dimension 1 and 2. The map

$$\begin{array}{cccc} \phi_1 : & \mathbb{C}l_1 & \longrightarrow & Mat(1,\mathbb{C}) = \mathbb{C} \\ & e_1 & \mapsto & i \end{array}$$

is a skew-hermitian irreducible representation of complex Clifford algebra $\mathbb{C}l_1$, where $Mat(1,\mathbb{C})$ is the space of 1×1 matrices with complex entries. The map

$$\begin{array}{ccc} \phi_2 : \mathbb{C}l_2 & \longrightarrow & Mat(2,\mathbb{C}) \\ e_1 & \longmapsto & i\sigma_2 \\ e_2 & \longmapsto & \sigma_1\sigma_2 \end{array}$$

is a skew-hermitian irreducible representation of complex Clifford algebra $\mathbb{C}l_2$ where $Mat(2,\mathbb{C})$ is the space of 2×2 matrices with complex entries where $\sigma_1$and σ_2 are as in 2.1. It is known that there is a periodicity relation $\mathbb{C}l_{m+2} \cong \mathbb{C}l_m \otimes \mathbb{C}l_2$ between complex Clifford algebras and its also known that the complex tensor product of irreducible complex representations of $\mathbb{C}l_m$ and $\mathbb{C}l_2$ gives an irreducible complex representation of $\mathbb{C}l_{m+2} \cong \mathbb{C}l_m \otimes \mathbb{C}l_2$ (see [4]). From these datum, we can get skew-hermitian irreducible representation of $\mathbb{C}l_m$ for all $m \geq 3$ recursively.

In particular, we obtain such a representation for $\mathbb{C}l_3$:

$$\begin{array}{ccccc} \mathbb{C}l_3 & \longrightarrow & \mathbb{C}l_1 \otimes \mathbb{C}l_2 & \longrightarrow & Mat\,(1,\mathbb{C}) \otimes Mat\,(2,\mathbb{C}) \\ e_1 & \longmapsto & 1 \otimes e_1 & \longmapsto & 1 \otimes i\sigma_1 \\ e_2 & \longmapsto & 1 \otimes e_2 & \longmapsto & 1 \otimes i\sigma_1\sigma_2 \\ e_3 & \longmapsto & ie_1 \otimes e_1e_2 & \longmapsto & i\phi_1\,(e_1) \otimes \sigma_2 \end{array}$$

In generally, let $\phi_m : \mathbb{C}l_m \longrightarrow Mat\,(\mathbb{C}, 2^p)$ be irreducible complex representation of $\mathbb{C}l_m$ such that $(\rho_m\,(e_i))^* = -\rho_m\,(e_i)$, where $p = \frac{m}{2}$ if m is even and $p = \frac{m-1}{2}$ if m is odd. The irreducible skew-hermitian representation of Clifford algebra $\mathbb{C}l_{m+2}$ is obtained as the map

$$\begin{array}{ccccc} \mathbb{C}l_{m+2} & \longrightarrow & \mathbb{C}l_m \otimes \mathbb{C}l_2 & \longrightarrow & Mat\,(2^p,\mathbb{C}) \otimes Mat\,(2,\mathbb{C}) \\ e_1 & \longmapsto & 1 \otimes e_1 & \longmapsto & I \otimes i\sigma_1 \\ e_2 & \longmapsto & 1 \otimes e_2 & \longmapsto & I \otimes i\sigma_1\sigma_2 \\ e_3 & \longmapsto & ie_1 \otimes e_1e_2 & \longmapsto & i\phi_m\,(e_1) \otimes \sigma_2 \\ \vdots & & \vdots & & \vdots \\ e_{m+2} & \longmapsto & ie_m \otimes e_1e_2 & \longmapsto & i\phi_m\,(e_m) \otimes \sigma_2. \end{array}$$

Let us define

$$C_1 = I \otimes i\sigma_1, C_2 = I \otimes i\sigma_1\sigma_2, C_3 = i\phi_m\,(e_1) \otimes \sigma_2, \cdots, C_{m+2} = i\phi_m\,(e_m) \otimes \sigma_2.$$

These matrices satisfy

$$C_i^* = -C_i,\ C_i^2 = -I,\ C_iC_j + C_jC_i = 0, \text{ for } i \neq j.$$

1. Now we can find $\mathbb{C}_{sk-h}\,(n) = 2b+1$, $n \times n$ complex skew-hermitian matrices, where $n = (2a+1)2^b$ is as follows:

   Let $\{C_1, C_2, \cdots, C_{2b+1}\}$ be a family of $2^b \times 2^b$ type matrices which are obtained by irreducible, skew-hermitian representation of Clifford algebra $\mathbb{C}l_{2b+1}$ as above. Let I be the unit matrix of type $(2a+1) \times (2a+1)$, then

   $$D_i = I \otimes C_i, \qquad 1 \leq i \leq 2b+1$$

   are $n \times n$ matrices and they also satisfy

   $$D_i^* = -D_i,\ D_i^2 = -I,\ D_iD_j + D_jD_i = 0, \text{ for } i \neq j.$$

The non-trivial linear combinations with real coefficients of the family of matrices $\{D_1, D_2, ..., D_{2b+1}\}$ are non-singular: If

$$D = \sum_{i=1}^{2b+1} \beta_i D_i$$

then $D^2 = -\left(\beta_1^2 + \beta_2^2 + ... + \beta_{2b+1}^2\right) I$. Since $\det(D^2) \neq 0$, we get $\det D \neq 0$.

2. We can define a new family of matrices $\{E_1, E_2, ..., E_{2b+1}\}$ using the above family of the matrices by $E_1 = iD_1$, $E_2 = iD_2, \cdots, E_{2b+1} = iD_{2b+1}$. These matrices satisfy

$$E_i^* = E_i,\ E_i^2 = I,\ E_i E_j + E_j E_i = 0, \text{ for } i \neq j.$$

Then non-trivial linear combinations $E = \sum\limits_{i=1}^{k} \gamma_i E_i = i \sum\limits_{i=1}^{k} \gamma_i D_i$ with real coefficients of the family of matrices $\{E_1, E_2, \cdots, E_{2b+1}\}$ are non-singular. Therefore we obtained $\mathbb{C}_h(n) = 2b+1$, $n \times n$ hermitian matrices.

3. The construction of $\mathbb{C}(n) = 2b+2$, $n \times n$ type complex matrices:

Consider the family $\{E_1, E_2, ..., E_{2b+1}, iI\}$ where $E_1, E_2, ..., E_{2b+1}$ as above and iI is i times the $n \times n$ unit matrix. Let

$$E = \sum_{j=1}^{2b+1} \lambda_j E_j + \lambda i I$$

be a nontrivial linear combination with real coefficients. Then we can write

$$EE^* = \left(\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_{2b+1}^2 + \lambda^2\right) I.$$

Hence E is non-singular.

4. In order to find $\mathbb{C}_s(n) = \rho\left(\frac{n}{2}\right) + 2$ complex symmetric $n \times n$ matrices, we must add the symmetric matrix iI to the $\rho\left(\frac{n}{2}\right) + 1$ real symmetric $n \times n$ matrices, as in 1.2. Hence we obtain the family of matrices

$$\left\{B_1, B_2, ..., B_{\rho\left(\frac{n}{2}\right)+1}, iI\right\}.$$

Let $B=\sum\limits_{j=1}^{m}\lambda_j B_j+\lambda iI$, where $m=\rho\left(\frac{n}{2}\right)+1$. Since

$$\left(\sum_{j=1}^{m}\lambda_j B_j+\lambda iI\right)\left(\sum_{j=1}^{m}\lambda_j B_j-\lambda iI\right)=\left(\lambda_1^2+\lambda_2^2+\cdots+\lambda_m^2+\lambda^2\right)I,$$

the nontrivial linear combination B with real coefficients is non-singular.

2.3. Construction of Quaternionic matrices

1. $\mathbb{H}(n)=\mathbb{H}_s(n)=\rho\left(\frac{n}{2}\right)+4$, $n\times n$ quaternionic symmetric matrices can be found as follows. We know that there are $\mathbb{R}_s(n)=\rho\left(\frac{n}{2}\right)+1$, $n\times n$ real symmetric matrices given 2.1. Let $\left\{B_1,B_2,...,B_{\rho\left(\frac{n}{2}\right)+1}\right\}$ be real symmetric matrices. Let us add the symmetric matrices iI,jI,kI to this family where I is a $n\times n$ unit matrix. Then the family

$$\left\{B_1,B_2,...,B_{\rho\left(\frac{n}{2}\right)+1},iI,jI,kI\right\}$$

has $\rho\left(\frac{n}{2}\right)+4$ members and they are $n\times n$ symmetric quaternionic matrices. Take the non-trivial linear combination with real coefficient of these matrices:

$$B=\sum_{j=1}^{q}\lambda_j B_j+\alpha_1 iI+\alpha_2 jI+\alpha_3 kI,$$

where $q=\rho\left(\frac{n}{2}\right)+1$. Since

$$BB^*=\left(\lambda_1^2+\lambda_2^2+\cdots+\lambda_q^2+\alpha_1^2+\alpha_2^2+\alpha_3^2\right)I,$$

$\det\left(BB^*\right)\neq 0$. Hence B is non-singular.

2. $\mathbb{H}_h(n)=\mathbb{H}\left(\frac{n}{2}\right)+1=\rho\left(\frac{n}{4}\right)+5$ hermitian matrices of type $n\times n$ are obtained by using representation of Clifford algebra $Cl_{1,k}$. $Cl_{1,k}$ is the real Clifford algebra over the real vector space $\mathbb{R}^{1+k}$ with the quadratic form $q:\mathbb{R}^{1+k}\longrightarrow\mathbb{R}$, $q\left(x_1,x_2,\cdots,x_k,x_{k+1}\right)=\left(x_1^2+x_2^2+\cdots+x_k^2-x_{k+1}^2\right)$; (see [4]).

Let $\{B_1, B_2, ..., B_{k-1}\}$ be the family of matrices of type $\left(\frac{n}{2}\right) \times \left(\frac{n}{2}\right)$ as in 2.1. Then we can define the following map:

$$\begin{array}{rccc} \psi_{0,k-1}: & Cl_{0,k-1} & \longrightarrow & Mat\left(\frac{n}{2}, \mathbb{R}\right) \\ & \varepsilon_i & \longmapsto & \psi_k(\varepsilon_i) = B_i, \end{array}$$

where $k - 1 = \rho\left(\frac{n}{4}\right) + 1$. This map is a real representation of Clifford algebra $Cl_{0,k-1}$. Also the map

$$\begin{array}{rccc} \psi_{1,1}: & Cl_{1,1} & \longrightarrow & Mat(2, \mathbb{R}) \\ & \varepsilon_1 & \longmapsto & \psi_{1,1}(e_1) = \sigma_1 \\ & e_1 & \longmapsto & \psi_{1,1}(\varepsilon_1) = \sigma_1\sigma_2 \end{array}$$

is a representation of the Clifford algebra $Cl_{1,1}$ where σ_1 and σ_2 as in 2.1. Moreover, $\psi_{1,1}$ is an algebra isomorphism. By using the maps $\psi_{0,k-1}$ and $\psi_{1,1}$ we consider the tensor product map $\psi_{0,k-1} \otimes \psi_{1,1}$

$$\psi_{0,k-1} \otimes \psi_{1,1} : Cl_{0,k-1} \otimes Cl_{1,1} \longrightarrow Mat\left(\tfrac{n}{2}, \mathbb{R}\right) \otimes Mat(2, \mathbb{R}) \cong Mat(n, \mathbb{R})$$
$$(\psi_{0,k-1} \otimes \psi_{1,1})(u \otimes v) = \psi_{0,k-1}(u) \otimes \psi_{1,1}(v),$$

where $Mat(n, \mathbb{R})$ is the set of all $n \times n$ real matrices. The map $\psi_{0,k-1} \otimes \psi_{1,1}$ is an algebra homomorphism. It is known that $Cl_{1,k} \cong Cl_{0,k-1} \otimes Cl_{1,1}$. We can get a real representation $\psi_{1,k}$ of the Clifford algebra $Cl_{1,k}$ which is defined on generators as follows:

$$\begin{array}{llllll} Cl_{1,k} & \longrightarrow & Cl_{0,k-1} \otimes Cl_{1,1} & \longrightarrow & Mat(n, \mathbb{R}) \\ \varepsilon_1 & \longrightarrow & 1 \otimes \varepsilon_1 & \longrightarrow & I \otimes \sigma_1 \\ \varepsilon_2 & \longrightarrow & \varepsilon_1 \otimes \varepsilon_1 e_1 & \longrightarrow & B_1 \otimes \sigma_2 \\ \varepsilon_3 & \longrightarrow & \varepsilon_2 \otimes \varepsilon_1 e_1 & \longrightarrow & B_2 \otimes \sigma_2 \\ \varepsilon_4 & \longrightarrow & \varepsilon_3 \otimes \varepsilon_1 e_1 & \longrightarrow & B_3 \otimes \sigma_2 \\ \vdots & \longrightarrow & \vdots & \longrightarrow & \vdots \\ \varepsilon_k & \longrightarrow & \varepsilon_{k-1} \otimes \varepsilon_1 e_1 & \longrightarrow & B_{k-1} \otimes \sigma_2 \\ e_1 & \longrightarrow & 1 \otimes e_1 & \longrightarrow & I \otimes \sigma_1\sigma_2. \end{array}$$

The image of the generators ε_i, $1 \le i \le k$ and e_1 of $Cl_{1,k}$ under the above homomorphism are the matrices we are looking for. Note that $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k, e_1$

is a q-orthogonal basis for $\mathbb{R}^{1+k}$ such that $q(\varepsilon_j) = -1$ for all j and $q(e_1) = 1$ (see [4] on page 26). Define

$$F_1 = I \otimes \sigma_1,\ F_2 = B_1 \otimes \sigma_2,\ F_3 = B_2 \otimes \sigma_2,\ \cdots,\ F_k = B_{k-1} \otimes \sigma_2,\ F_{k+1} = I \otimes \sigma_1\sigma_2.$$

Consider the family

$$\{F_1, F_2, \cdots, F_k, iF_{k+1}, jF_{k+1}, kF_{k+1}\}.$$

Note that this family has $k+3 = \rho\left(\frac{n}{4}\right) + 5$ members. These matrices satisfy

$$F_i^2 = I \text{ for } 1 \leq i \leq k, F_{k+1}^2 = -I, F_iF_j + F_jF_i = 0, \text{ for } i \neq j.$$

Elements of this family are quaternionic hermitian and their nontrivial linear combination with real coefficients are non-singular. Let

$$F = \sum_{j=1}^{k} \lambda_j F_j + \beta_1 iF_{k+1} + \beta_2 jF_{k+1} + \beta_3 kF_{k+1}$$

be a non-trivial linear combination with real coefficients. Since

$$F^2 = \left(\sum_{j=1}^{k} \lambda_j^2 + \beta_1^2 + \beta_2^2 + \beta_3^2\right) I$$

F^2 is non-singular, so is F.

3. $\mathbb{H}_{sk-h}(n) = \mathbb{H}(n) - 1 = \rho\left(\frac{n}{2}\right) + 3$ skew-hermitian matrices of type $n \times n$ are obtained as follows. Representation of Clifford algebra $Cl_{k,1}$ will be used to obtain skew-hermitian matrices of type $n \times n$, where $k = \rho\left(\frac{n}{2}\right)$. $Cl_{k,1}$ is the real Clifford algebra over the real vector space $\mathbb{R}^{k+1}$ with the quadratic form $q : \mathbb{R}^{k+1} \longrightarrow \mathbb{R}$, $q(x_1, x_2, \cdots, x_k, x_{k+1}) = -\left(x_1^2 + x_2^2 + \cdots + x_k^2 - x_{k+1}^2\right)$; (see [4]).

   Let $\{A_1, A_2, ..., A_{k-1}\}$ be the Radon-Hutwitz family of matrices of type $\left(\frac{n}{2}\right) \times \left(\frac{n}{2}\right)$. Then we can define following map

$$\begin{array}{rccc} \psi_{k-1,0}: & Cl_{k-1,0} & \longrightarrow & Mat\left(\frac{n}{2}, \mathbb{R}\right) \\ & e_i & \longmapsto & \psi_k(e_i) = A_i. \end{array}$$

This map is a real representation of Clifford algebra $Cl_{k-1,0}$. Also consider the map $\psi_{1,1} : Cl_{1,1} \longrightarrow Mat\,(2,\mathbb{R})$. Consider the tensor product map $\psi_{k-1,0} \otimes \psi_{1,1}$ of $\psi_{k-1,0}$ and $\psi_{1,1}$.

The map $\psi_{k-1,0} \otimes \psi_{1,1}$ is an algebra homomorphism. Since $Cl_{k,1} \cong Cl_{k-1,0} \otimes Cl_{1,1}$ we can get a real representation $\psi_{k,1}$ of the Clifford algebra $Cl_{k,1}$ which is defined on generators as follows:

$$\begin{array}{ccccc}
Cl_{k,1} & \longrightarrow & Cl_{k-1,0} \otimes Cl_{1,1} & \longrightarrow & Mat\,(n,\mathbb{R}) \\
e_1 & \longrightarrow & 1 \otimes e_1 & \longrightarrow & I \otimes \sigma_1\sigma_2 \\
e_2 & \longrightarrow & e_1 \otimes \varepsilon_1 e_1 & \longrightarrow & A_1 \otimes \sigma_2 \\
e_3 & \longrightarrow & e_2 \otimes \varepsilon_1 e_1 & \longrightarrow & A_2 \otimes \sigma_2 \\
e_4 & \longrightarrow & e_3 \otimes \varepsilon_1 e_1 & \longrightarrow & A_3 \otimes \sigma_2 \\
\vdots & \longrightarrow & \vdots & \longrightarrow & \vdots \\
e_k & \longrightarrow & e_{k-1} \otimes \varepsilon_1 e_1 & \longrightarrow & A_{k-1} \otimes \sigma_2 \\
\varepsilon_1 & \longrightarrow & 1 \otimes \varepsilon_1 & \longrightarrow & I \otimes \sigma_1.
\end{array}$$

The image of the generators e_i, $1 \leq i \leq k$ and ε_1 of $Cl_{k,1}$ under the above homomorphism are the matrices we are looking for. Define

$$G_1 = I \otimes \sigma_1\sigma_2,\ G_2 = A_1 \otimes \sigma_2,$$
$$G_3 = A_2 \otimes \sigma_2,\ \cdots,\ G_k = A_{k-1} \otimes \sigma_2,\ G_{k+1} = I \otimes \sigma_1.$$

Let us consider the family

$$\{G_1, G_2, \cdots, G_k, iG_{k+1}, jG_{k+1}, kG_{k+1}\}.$$

Note that this family has $k+3 = \rho\left(\frac{n}{2}\right) + 3$ members. These matrices satisfy

$$G_i^2 = -I \text{ for } 1 \leq i \leq k, G_{k+1}^2 = I, G_iG_j + G_jG_i = 0 \text{ for } i \neq j.$$

Elements of this family are quaternionic skew-hermitian and nontrivial linear combination with real coefficients are non-singular. Let

$$G = \sum_{j=1}^{k} \lambda_j G_j + \beta_1 iG_{k+1} + \beta_2 jG_{k+1} + \beta_3 kG_{k+1}$$

be a non-trivial linear combination with real coefficients. Since

$$G^2 = -\left(\sum_{j=1}^{k} \lambda_j^2 + \beta_1^2 + \beta_2^2 + \beta_3^2\right) I,$$

G^2 is non-singular, so is G.

3. Some Analogues of Radon-Hurwitz Theorem:

As a result of the above calculations we can express some analogues of Radon-Hurwitz theorem as follows:

1. The maximum number of $n \times n$ real orthogonal matrices $\{B_1, B_2, ..., B_k\}$ satisfying the relations $B_i^2 = I$, $B_iB_j + B_jB_i = 0$ for $i \neq j$ is $\rho\left(\frac{n}{2}\right) + 1$.
2. The maximum number of $n \times n$ complex unitary matrices $\{D_1, D_2, ..., D_k\}$ satisfying the relations $D_i^2 = -I$, $D_iD_j + D_jD_i = 0$ for $i \neq j$ is $2b + 1$.
3. The maximum number of $n \times n$ complex unitary matrices $\{E_1, E_2, ..., E_k\}$ satisfying the relations $E_i^2 = I$, $E_iE_j + E_jE_i = 0$ for $i \neq j$ is $2b + 1$.
4. The maximum number of $n \times n$ quaternionic unitary matrices $\{Q_1, Q_2, ..., Q_k\}$ satisfying the relations $Q_i^2 = I$, $Q_iQ_j + Q_jQ_i = 0$ for $i \neq j$ is $\rho\left(\frac{n}{2}\right) + 4$.
5. The maximum number of $n \times n$ quaternionic unitary matrices $\{Q_1, Q_2, ..., Q_k\}$ satisfying the relations $Q_i^2 = -I$, $Q_iQ_j + Q_jQ_i = 0$ for $i \neq j$ is $\rho\left(\frac{n}{4}\right) + 5$.

References

[1] Adams, J. F., Lax, P. D. and Phillips, S.: On matrices whose real linear combinations are nonsingular. Proc. Amer. Math. Soc. 16, 318-322 (1965).

[2] Adams, J. F., Lax, P. D.and Phillips, S.: Correction to On matrices whose real linear combinations are nonsingular. Proc. Amer. Math. Soc. 17, 945-947 (1966).

[3] Değirmenci, N., Koçak, Ş.: Generalized self-duality of 2-forms. Advances in Applied Clifford Algebras. 13, 107-113 (2003).

[4] Lawson, H. B., Michelson, M. L.: Spin Geometry. Princeton University Press 1989.

[5] Prasolov, V. V.: Problems and Theorems in Linear Algebra. American Mathematical Society (Translations of Mathematical Monographs volume 134) 1994.

[6] Yik-Hoi Au-Yeung: On matrices whose real linear combinations are nonsingular. Proc. Amer. Math. Soc. 29, 17-22 (1971).

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