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On the Normalizer of the Congruence Subgroup $H_0^5(I)$ of the Hecke Group H^5

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Abstract

Let $\lambda = 2\cos\frac{\pi}{5}$ and let H^5 be the Hecke group associated to λ. In this paper, the normalizers of the congruence subgroups $H_0^5(I)$ in $PSL(2,\mathbb{Z}[\lambda])$ are studied in the case where $I = (2)^{\alpha}I'$, $(2, I') = 1$ and I' is a prime ideal.

Key Words: Normalizer, Congruence subgroup, Hecke group.

1. Introduction

The congruence subgroups of the Hecke group H^q $(q = 3, 4, 6)$ and the normalizers of these groups in H^q, in $PSL(2,\mathbb{Z}[\lambda_q])$ and in $PSL(2,\mathbb{R})$ were studied by various authors (see [1], [2], [4], [5], [8], [9], [16]). The normalizers of the congruence subgroups of the Hecke group H^5 in $PSL(2,\mathbb{R})$ were given for prime ideals (see [11],[12]). In this paper, we investigate the normalizer of the congruence subgroup $H_0^5(I)$ of the Hecke group H^5 in $PSL(2,\mathbb{Z}[\lambda])$. Furthermore, in [8], it is conjectured that the normalizer of $H_0^5(I)$ in H^5 is $H_0^5((2)^{\alpha'}I')$, where $I = (2)^{\alpha}I'$ is an ideal of $\mathbb{Z}[\lambda]$, $(2, I') = 1$ and $\alpha' = \alpha - \min\left(2, \left[\left|\frac{\alpha}{2}\right|\right]\right)$. We give a proof to the conjecture in the case where I' is a prime ideal.

We start by recalling definitions, notations, and some preliminary results of these concepts. By a Hecke group we mean a discrete subgroup of $PSL(2,\mathbb{R})$ generated by T and U_q, where T and U_q are the Möbius transformations given by $T(z) = -\frac{1}{z}$, and $U_q(z) = z + \lambda_q$. Hecke [6] showed that these groups are discrete if and only if $\lambda_q = 2\cos\frac{\pi}{q}$

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or $\lambda_q > 2$. This group is denoted by H^q. It is known that a presentation for H^q is

$$\langle T, U_q \rangle = \left\langle T, S_q \mid T^2 = S_q^q = I \right\rangle,$$

where $S_q = TU_q$, and so H^q is isomorphic to the free product $C_2 * C_q$.

We have the following table of the values of λ_q for small q:

q	3	4	5	6
λ_q	1	$\sqrt{2}$	$\frac{1+\sqrt{5}}{2}$	$\sqrt{3}$

The best known example is when $q = 3$, and H^3 is the modular group $\Gamma = PSL(2, \mathbb{Z})$ so the above can be thought of as a natural generalization of Γ. Furthermore, we have the following geometric interpretation: the modular group Γ is the triangle group $(2, 3, \infty)$ and the Hecke group H^q is the triangle group $(2, q, \infty)$.

Let $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ and $\widehat{\mathbb{H}} := \mathbb{H} \cup \mathbb{Q}(\lambda_q) \cup \{\infty\}$. Then the Hecke group H^q, namely a subgroup of $SL_2(\mathbb{Z}[\lambda_q])/\{\pm I\}$, acts on $\widehat{\mathbb{H}}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \wedge z = \frac{az + b}{cz + d}.$$

As usual, we denote an element of H^q as a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ remembering to identify any such matrix with its negative.

Let I be an ideal of $\mathbb{Z}[\lambda_q]$. The principal congruence subgroup of level I is

$$H^q(I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H^q \mid a - 1, b, c, d - 1 \equiv 0 (\mathrm{mod} I) \right\}$$

and any subgroup Λ^q of H^q containing $H^q(I)$ is called a congruence subgroup of level I. The two most important of these are

$$H_0^q(I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H^q \mid c \equiv 0 (\mathrm{mod} I) \right\}$$

and

$$H_1^q(I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H^q \mid a - 1, c, d - 1 \equiv 0 (\mathrm{mod} I) \right\}.$$

It is easy to see that we have the inclusions $H^q(I) \leq H_1^q(I) \leq H_0^q(I) \leq H^q$ and we can also see that $H^q(I)$ is normal in H^q and $H_1^q(I)$ is normal in $H_0^q(I)$.

Again, $H_0^q(I)$ is a natural generalization of the congruence subgroups $\Gamma_0(n)$ of Γ. It works because the elements of H^q sit naturally in the ring $\mathbb{Z}[\lambda_q]$.

Recall that H^q is commensurable with $PSL(2,\mathbb{Z})$ if and only if $q = 4$ and 6. The elements of such groups are completely known (see [17]).

Suppose H^q is not commensurable with $PSL(2,\mathbb{Z})$. By the result of A. Leutbecher ([14],[15]), $\mathbb{Q}(\lambda) \cup \{\infty\}$ is the set of cusps of H^q if and only if $q = 5$. Also, 5 is the only q other than 4, 6 for which $\mathbb{Q}(\lambda)$ is a quadratic field. For all other q's, the degree is > 2. As a consequence, $q = 5$ is the next most workable and interesting q. Some of the classical results on the modular group can be generalized to H^5 (see [3], [9], [10], [11]).

From now on, q will be 5, so $\lambda := \lambda_5$, then $\mathbb{Z}[\lambda] = \mathbb{Z}[\lambda_5]$ and $U := U_5$, or $U = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$.

The main facts used in our proofs :

(a) $\mathbb{Z}[\lambda]$ is a principal ideal domain. The norm of any element $u+v\lambda$ of $\mathbb{Z}[\lambda]$ is defined by $Nor(u+v\lambda) = u^2 - v^2 + uv$. Let I be a non-zero ideal of $\mathbb{Z}[\lambda]$. For an element $a \in \mathbb{Z}[\lambda]$, we say that a and I are relatively prime if there exist elements $x \in \mathbb{Z}[\lambda]$ and $b \in I$ such that $ax + b = 1$, and this is denoted by $(a, I) = 1$.

Let $a, b \in \mathbb{Z}[\lambda]$. The element a is said to be congruent to b modulo I (denoted by $a \equiv b (\text{mod} I)$) if $a - b \in I$.

(b) The set of cusps of H^5 is $\mathbb{Q}(\lambda) \cup \{\infty\}$ ([14],[15]). Furthermore, if $x \in \mathbb{Q}(\lambda)$ is a cusp, x has a unique reduced form $x = \frac{a}{c}$ ([13]). By definition, this means that a, $c \in \mathbb{Z}[\lambda]$ with $c \geq 0$ and there exists b, $d \in \mathbb{Z}[\lambda]$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H^5$. Clearly, $(a, c) = 1$ so that if $x = \frac{a'}{c'}$ with $(a', c') = 1$, then $a = \mu a'$, $c = \mu c'$ where μ is a unit in $\mathbb{Z}[\lambda]$.

(c) (Corollary 5 of [13]) $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in H^5$ if and only if $b = m\lambda$, $m \in \mathbb{Z}$. Similarly, $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \in H^5$ if and only if $c = n\lambda$, $n \in \mathbb{Z}$.

(d) (Proposition 6 of [13]) Suppose x_i, x_j are H^5-rationals with reduced form $\frac{a_i}{c_i}$ and $\frac{b_j}{d_j}$, respectively, and suppose that $x_i < x_j$. Then the following statements are equivalent:

(i) $\begin{pmatrix} a_i & b_j \\ c_i & d_j \end{pmatrix} \in H^5$;

(ii) (x_i, x_j) is an even line, that is, it is the image of the complete hyperbolic geodesic with ends at 0 and ∞ under the action of some $A \in H^5$;

(iii) $a_i d_j - b_j c_i = 1$.

(e) $A \in H$ if and only if it is a finite word in the generators T and U. The word can be written as

$$A = U^{r_0} T U^{r_1} T \dots T U^{r_{n+1}} \tag{1.1}$$

where r_i are non zero integers except r_0 and r_{n+1} which may be 0. The word in turn gives rise to the matrix A. By judicious applications of the generators the word can be made unique ([18]).

(f) (Lemma 1 of [3]) If I is a non-zero ideal of $\mathbb{Z}[\lambda_5]$, then

$$[H^5 : H_0^5(I)] = N(I) \prod_{P|I} \left(1 + \frac{1}{N(P)}\right),$$

where the product is over the set of all prime ideals P which divide I. Here, for a non-zero ideal I of $\mathbb{Z}[\lambda]$, $N(I)$ denotes the absolute norm of I.

(g) (Theorem 4.5 of [7]) If K, H, G are groups with $K < H < G$, then $[G : K] = [G : H][H : K]$. If any two of these indices are finite, then so is the third.

(h) (Corollary 2 of [10]) The indices of the congruence subgroups of H^5 of level $I = (2)$ are $[H^5 : H^5(I)] = 10$, and $[H^5 : H_1^5(I)] = [H^5 : H_0^5(I)] = 5$.

The rest of this paper is organized as follows. In the next section, we give some results concerning congruence subgroup $H_0^5(I)$, where $I = (2)^\alpha$, $(\alpha = 1, 2)$ or I is a prime ideal, which will be needed later. In section 3, we find the normalizer of the congruence subgroup $H_0^5(I)$ in $PSL(2, \mathbb{Z}[\lambda])$, where $I = (2)^\alpha I'$, $(2, I') = 1$, and the proof of the conjecture in [8] for this case is given in Corollary 17.

2. Congruence subgroup $H_0^5(I)$

Lemma 1. *Let the ideal $I = (2) = 2\mathbb{Z}[\lambda]$ and let $A \in H^5$. Then*

$$A \in H_0^5(I) \text{ if and only if } A \equiv \pm \begin{pmatrix} 1 & r\lambda \\ 0 & 1 \end{pmatrix} (mod I),$$

where $r = 0, 1$.

Proof. By using (g) and (h), we have that

$$[H_0^5(I) : H^5(I)] = 2.$$

Then, since $U = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \notin H^5(I)$, the partition of $H_0^5(I)$ associated to the subgroup $H^5(I)$ is

$$H_0^5(I) = H^5(I) \cup UH^5(I). \tag{2.2}$$

Thus, for every matrix A in $H_0^5(I)$, from (2.2), there are two cases as follows.

Case 1. If $A \in H^5(I)$, then we get

$$A \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\text{mod} I).$$

Case 2. If $A \in UH^5(I)$, then we have

$$A \equiv \pm \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} (\text{mod} I).$$

This completes the proof of the lemma. □

Corollary 2. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be an element of* $H_0^5(I)$. *In this case,*

(*i*) *if* $I = (2)$, *then,* $a - d \equiv 0(mod(2))$

(*ii*) *if* $I = (4) = (2)^2$, *then,* $a - d \equiv 0(mod(4))$.

Proof. (*i*) Since $a^2 - 1 = (a-1)(a+1)$, by Lemma 1, we have

$$a^2 - 1 \equiv 0(\text{mod}(2)^2). \tag{2.3}$$

Since $A \in H_0^5(2)$ *and* $ad - bc = 1$, we have

$$ad \equiv 1(\text{mod}(2)). \tag{2.4}$$

Suppose that $a-d\equiv u(\mathrm{mod}(2))$ for some $u\in\mathbb{Z}[\lambda]$. Multiplying by a, one has $a^2-ad\equiv au(\mathrm{mod}(2))$. In this case, since $a\not\equiv 0(\mathrm{mod}(2))$, by (2.3) and (2.4), we have $u\equiv 0(\mathrm{mod}(2))$. This implies that $a-d\equiv 0(\mathrm{mod}(2))$.

(ii) Since $A\in H_0^5(4)$ *and* $ad-bc=1$, it is clear that $ad\equiv 1(\mathrm{mod}(4))$. Here, using a similar argument as in the proof of (i), we have $a-d\equiv 0(\mathrm{mod}(4))$. □

Remark 3. *Let* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be an element of* $H_0^5(2)$. *Then* $a-d\equiv 0(\mathrm{mod}(2)^2)$ *is not necessarily true.*

Example 4. *From (1.1),*

$$A=UTU^2T=\begin{pmatrix} 1+2\lambda & -\lambda \\ 2\lambda & -1 \end{pmatrix}\in H_0^5(2).$$

Then, for the matrix A, we have $a-d=2(1+\lambda)\not\equiv 0(mod(2)^2)$.

Remark 5. *Let* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be an element of* $H_0^5(4)$. *Then* $a-d\equiv 0(\mathrm{mod}(2)^3)$ *is not necessarily true.*

Example 6. *If we take* $A=\begin{pmatrix} 1+2\lambda & -\lambda \\ 2\lambda & -1 \end{pmatrix}$, *then, for matrices*

$$A^2=\begin{pmatrix} 3+6\lambda & -2-2\lambda \\ 4(1+\lambda) & -1-2\lambda \end{pmatrix}\in H_0^5((2)^2) \text{ and } A^4=\begin{pmatrix} 29+48\lambda & -4(3+5\lambda) \\ 8(3+5\lambda) & -11-16\lambda \end{pmatrix}$$

$\in H_0^5((2)^3)$, *we have* $a-d=4(1+\lambda)\not\equiv 0(mod(2)^3)$ *and* $a-d=4(10+16\lambda)\not\equiv 0(mod(2)^3)$, *respectively.*

Remark 7. *If the ideal* $I\neq(2)$, *then Corollary 2 (i) and (ii) are not true.*

Example 8. *Let* $I=(3)$. *From (1.1),*

$$B=UTU^4TU^2TUTU^{-1}T=\begin{pmatrix} 25+40\lambda & 10+17\lambda \\ 9(2+3\lambda) & 8+11\lambda \end{pmatrix}\in H_0^5(3).$$

For the matrix B*,* $a = 25+40\lambda$*. Then it is easily seen that* $a^2-1 \not\equiv 0(mod(3))$*. It follows that* $a-d \not\equiv 0(mod(3))$*.*

Example 9. *Let* $I = (2+\lambda)$*. From (1.1),*

$$C = TU^2TU^3TU^2TU^{-2}T = \begin{pmatrix} 12+25\lambda & 5+6\lambda \\ -(2+\lambda)^2(4+5\lambda) & -4(3+5\lambda) \end{pmatrix} \in H_0^5(2+\lambda).$$

For the matrix C*,* $a = 12+25\lambda$ *and* $a^2-1 = (11+25\lambda)(13+25\lambda)$*. From (a), we have* $Nor(a^2-1) = 131.229$ *and* $Nor(2+\lambda) = 5$*. In this case, since* $Nor(a^2-1) \equiv 4 \ mod\ 5$*, we obtain* $a^2-1 \not\equiv 0(mod(2+\lambda))$*. This implies that* $a-d \not\equiv 0(mod(2+\lambda))$*.*

Example 10. *Let* $I = (4-\lambda)$*. From (1.1),*

$$D = TU^7TU^{-5}T = \begin{pmatrix} 5\lambda & 1 \\ -(4-\lambda)^3(2+3\lambda) & -7\lambda \end{pmatrix} \in H_0^5(4-\lambda).$$

For the matrix D*,* $a = 5\lambda$*. Using a similar argument as in Example(9), we have* $a^2-1 \not\equiv 0(mod(4-\lambda))$*. This implies that* $a-d \not\equiv 0(mod(4-\lambda))$*.*

Corollary 11. $H_0^5(2) = H_1^5(2)$*.*

Lemma 12. *Let* $I = (\tau)$ *be a prime ideal of* $\mathbb{Z}[\lambda]$*. Let* p *be the positive rational prime which lies below* τ*. Then*

(i) $(p) = (\tau)$ *if and only if* $H_0^5(p) = H_0^5(\tau)$*.*

(ii) $(p) \neq (\tau)$ *if and only if* $H_0^5(p) \lneqq H_0^5(\tau)$*.*

Proof. (i) If $(p) = (\tau)$, then it is easily seen that $H_0^5(p) = H_0^5(\tau)$. Suppose that $H_0^5(p) = H_0^5(\tau)$. Let $x = \frac{1}{\tau} \in \mathbb{Q}(\lambda)$. By Leutbecher's Theorem ([14], [15]), x is a cusp of H^5. By (b), the reduced form for x is of the form $\frac{c}{c\tau}$, where c is a unit in $\mathbb{Z}[\lambda]$. Thus, by (d), $H_0^5(\tau)$ contains an element of the form

$$A_c = \begin{pmatrix} c & b \\ c\tau & d \end{pmatrix}. \tag{2.5}$$

In this case, since $H_0^5(p) = H_0^5(\tau)$, it follows that $\tau = c^{-1}pu$, where $u \in \mathbb{Z}[\lambda]$. Thus we have $(p) = (\tau)$.

(ii) Let $(p) \neq (\tau)$. Since $(p) \subset (\tau)$, it is clear that $H_0^5(p) < H_0^5(\tau)$. By (2.5), $A_c \notin H_0^5(p)$. This implies that $H_0^5(p) \lneqq H_0^5(\tau)$.

Conversely, from (2.5), we have $(p) \neq (\tau)$.

3. Upper bound for $N(H_0^5(2^\alpha\tau))$

Let I' be an ideal of $\mathbb{Z}[\lambda]$. Since $\mathbb{Z}[\lambda]$ is a principal ideal domain, $I' = (\tau)$ for some τ. Note that we may assume that τ is positive.

From now on, we take the ideal

$$I = (2)^\alpha I' = (2)^\alpha \cap I' = (2^\alpha\tau), \tag{3.6}$$

where $(2, I') = 1$ and $I' = (\tau)$ is a prime ideal. Denote by $N(H_0^5(2^\alpha\tau))$ the normalizer of $H_0^5(2^\alpha\tau)$ in $PSL(2,\mathbb{Z}[\lambda])$. Let $X = \begin{pmatrix} x & z \\ y & t \end{pmatrix} \in N(H_0^5(2^\alpha\tau))$, and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H_0^5(2^\alpha\tau)$. Then, we have that

$$XAX^{-1} = \begin{pmatrix} atx - bxy - dyz + ctz & -(a-d)xz + bx^2 - cz^2 \\ (a-d)ty - by^2 + ct^2 & -ayz + byx + dxt - ctz \end{pmatrix} \in H_0^5(2^\alpha\tau) \tag{3.7}$$

$$X^{-1}AX = \begin{pmatrix} * & * \\ -(a-d)xy - by^2 + cx^2 & * \end{pmatrix} \in H_0^5(2^\alpha\tau). \tag{3.8}$$

If we take $A = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$, then,

$$XAX^{-1} = \begin{pmatrix} 1 - xy\lambda & x^2\lambda \\ -y^2\lambda & 1 + xy\lambda \end{pmatrix} \in H_0^5(2^\alpha\tau) \tag{3.9}$$

$$X^{-1}AX = \begin{pmatrix} 1 + ty\lambda & t^2\lambda \\ -y^2\lambda & 1 - ty\lambda \end{pmatrix} \in H_0^5(2^\alpha\tau). \tag{3.10}$$

Lemma 13. *Let* $X = \begin{pmatrix} x & z \\ y & t \end{pmatrix} \in N(H_0^5(2^\alpha\tau))$. *Then*

$$y \equiv 0 (mod\ (2^{\alpha'}\tau)),$$

where $\alpha' = \alpha - \min\left(2, \left[\left|\frac{\alpha}{2}\right|\right]\right)$.

Proof. Since λ is a unit in $\mathbb{Z}[\lambda]$, by (3.9) and (3.10), then

$$y^2 \equiv 0(mod\ (2^\alpha \tau)). \tag{3.11}$$

Since $ad - bc = 1$ and $xt - yz = 1$, from (3.8), we have

$$(a^2 - 1)y \equiv 0(mod\ (2^\alpha \tau)). \tag{3.12}$$

Here, since $I' = (\tau)$ *and* (2) are prime ideals, by (3.6), (3.11) and (3.12), we obtain

$$y \equiv 0(mod\ (2)) \tag{3.13}$$

$$(a^2 - 1)y \equiv 0(mod\ (2)). \tag{3.14}$$

and

$$y \equiv 0(mod\ (\tau)) \tag{3.15}$$

$$(a^2 - 1)y \equiv 0(mod\ (\tau)). \tag{3.16}$$

By (3.13), there exists $\alpha' \in \mathbb{Z}_+$ such that

$$y \equiv 0(mod\ (2)^{\alpha'})\ and\ y \not\equiv 0(mod\ (2)^{\alpha'+1}). \tag{3.17}$$

This implies that

$$y^2 \equiv 0(mod\ (2)^{2\alpha'}) \tag{3.18}$$

For α and α', there are two cases:

Case1. Let $\alpha < \alpha'$. Then (3.11) and (3.12) are always true.

Case2. Let $\alpha \geq \alpha'$. From (3.11) and (3.18), we obtain

$$\alpha \leq 2\alpha' \Rightarrow \frac{\alpha}{2} \leq \alpha' \leq \alpha. \tag{3.19}$$

From (3) and (3.17), we get

$$(a^2 - 1)y \equiv 0(mod\ (2)^{\alpha'+2}). \tag{3.20}$$

By using (3.12) and (3.20), we have that

$$\alpha \leq \alpha' + 2 \Rightarrow \; 0 \leq \alpha - \alpha' \leq 2. \tag{3.21}$$

Thus, the smallest element $\alpha' \in \mathbb{Z}_+$ which satisfies (3.19) and (3.21) must be found.

(i) For $\alpha = 1, 2$ and 3, by (3.19) and (3.21), we have that $\alpha' = 1, 1$ and 2, respectively.

(ii) For $\alpha \geq 4$, there exists an element $\beta \in \mathbb{N}$ such that $\alpha = \beta + 4$. In this case, by (3.19) and (3.21),

$$2 \leq \alpha' \text{ and } \beta + 2 \leq \alpha'.$$

Since α' is smallest, it follows that $\alpha' = \beta + 2$. Thus, for every $\alpha \in \mathbb{Z}_+$ such that $\alpha \geq 4$, we obtain $\alpha' = \alpha - 2$. Consequently, from (i) and (ii), we have $y \equiv 0(mod\ (2^{\alpha'}\tau))$, where $\alpha' = \alpha - \min\left(2, \left[\left|\frac{\alpha}{2}\right|\right]\right)$.

This completes the proof of the lemma. □

Lemma 14. *(Lemma 1 of [12]) If I is a prime ideal of $\mathbb{Z}[\lambda]$, then*

$$N(H_0^5(I)) = H_0^5(I).$$

Remark 15. *If I is not a prime ideal of $\mathbb{Z}[\lambda]$, then Lemma 14 is not necessarily true as in the following theorem.*

Theorem 16. *Let the ideal $I = (2^\alpha \tau)$ as in (6). Then*

$$N(H_0^5(2^\alpha \tau)) = H_0^5(2^{\alpha'} \tau)$$

where $\alpha' = \alpha - \min\left(2, \left[\left|\frac{\alpha}{2}\right|\right]\right)$.

Proof. By Corollary 2 (i) and (ii), it is clear that

$$H_0^5(2^{\alpha'}\tau) \leq N(H_0^5(2^\alpha \tau)) \tag{3.22}$$

where $\alpha' = \alpha - \min\left(2, \left[\left|\frac{\alpha}{2}\right|\right]\right)$. Now we prove the converse inclusion, that is,

$$H_0^5(2^{\alpha'}\tau) \geq N(H_0^5(2^\alpha \tau))$$

where $\alpha' = \alpha - \min\left(2, \left[\left|\frac{\alpha}{2}\right|\right]\right)$. Let $X = \begin{pmatrix} x & z \\ y & t \end{pmatrix} \in N(H_0^5(2^\alpha\tau))$. Then, by Lemma 13 and (3.10), it is clear that

$$y \equiv 0(\bmod(2^{\alpha'}\tau)).$$

This implies that $y = c2^{\alpha'}\tau$ for some $c \in \mathbb{Z}[\lambda]$. Suppose $c \neq 0$. Recall that $a = \frac{x}{c2^{\alpha'}\tau} \in \mathbb{Q}(\lambda)$ is a cusp of H^5 as in (b). Let $a = \frac{x'}{y'}$ be the reduced form for a. Then H^5 contains an element of the form

$$Y = \begin{pmatrix} x' & z' \\ y' & t' \end{pmatrix}.$$

Since $(x, c2^{\alpha'}\tau) = 1$, $y' = \mu c2^{\alpha'}\tau$ where μ is a unit of $\mathbb{Z}[\lambda]$. Hence y' is a multiple of $2^{\alpha'}\tau$. This implies that $Y \in H_0^5(2^{\alpha'}\tau) \leq N(H_0^5(2^\alpha\tau))$. Since $X\infty = Y\infty$, it follows that

$$Y^{-1}X = \begin{pmatrix} u & v \\ 0 & u^{-1} \end{pmatrix} \in N(H_0^5(2^\alpha\tau)),$$

where u, $v \in \mathbb{Z}[\lambda]$. Applying (9) and (10) to $Y^{-1}X$, we have that

$$\begin{pmatrix} 1 & u^2\lambda \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & u^{-2}\lambda \\ 0 & 1 \end{pmatrix}$$

are elements of $H_0^5(2^\alpha\tau)$. By (c), $u = \pm 1$. Multiplying $Y^{-1}X$ by $-I$ if necessary, we may assume that $u = 1$ and

$$Y^{-1}X = \begin{pmatrix} 1 & x + y\lambda \\ 0 & 1 \end{pmatrix},$$

where x, $y \in \mathbb{Z}$. Note that

$$\begin{pmatrix} 1 & y\lambda \\ 0 & 1 \end{pmatrix} \in N(H_0^5(2^\alpha\tau)).$$

As a consequence,

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in N(H_0^5(2^\alpha\tau)).$$

Suppose that $x \neq 0$. Since $\lambda \in \mathbb{R} \setminus \mathbb{Q}$, for any $\epsilon > 0$, there exist k and l such that

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^k \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}^l = \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix} = \sigma \in N(H_0^5(2^\alpha \tau)),$$

where $0 < |\delta| < \epsilon$. As a consequence,

$$\sigma \begin{pmatrix} 1 & 0 \\ 2^\alpha p\lambda & 1 \end{pmatrix} \sigma^{-1} = \begin{pmatrix} 1+2^\alpha \delta p & 2^\alpha \delta^2 p \\ 0 & 1-2^\alpha \delta p \end{pmatrix} \in H_0^5(2^\alpha \tau),$$

where p is the positive rational prime which lies below τ. This implies that $H_0^5(2^\alpha \tau)$ is not discrete, giving a contradiction. Hence $x = 0$ and $Y^{-1}X \in H_0^5(2^{\alpha'}\tau)$. Since $Y \in H_0^5(2^{\alpha'}\tau)$, then we obtain $X \in H_0^5(2^{\alpha'}\tau)$.

Suppose $y = 0$. From the above argument, we have that $X \in H_0^5(2^{\alpha'}\tau)$. Consequently,

$$N(H_0^5(2^\alpha \tau)) \leq H_0^5(2^{\alpha'}\tau),$$

where $\alpha' = \alpha - \min\left(2, \left[\left|\frac{\alpha}{2}\right|\right]\right)$.

This completes the proof of the theorem. □

Corollary 17. *Let $I = (2)^\alpha I'$ be an ideal of $\mathbb{Z}[\lambda]$, where I' is a prime ideal of $\mathbb{Z}[\lambda]$ and $(2, I') = 1$. Then the normalizer of $H_0^5(I)$ in H^5 is $H_0^5((2)^{\alpha'} I')$, where $\alpha' = \alpha - \min\left(2, \left[\left|\frac{\alpha}{2}\right|\right]\right)$.*

Proof. From Theorem 16, it is clear that

$$N(H_0^5(I)) \cap H^5 = H_0^5((2)^{\alpha'} I'),$$

where $\alpha' = \alpha - \min\left(2, \left[\left|\frac{\alpha}{2}\right|\right]\right)$. □

Theorem 18. *Let $I = (2)^\alpha I'$ be an ideal of $\mathbb{Z}[\lambda]$, and $(2, I') = 1$. Then*

$$[H_0^5((2)^{\alpha'} I') : H_0^5((2)^\alpha I')] = \begin{cases} 1\,, & \alpha = 1 \\ 4\,, & \alpha = 2,3 \\ 16, & \alpha \geq 4 \end{cases}$$

where $\alpha' = \alpha - \min\left(2, \left[\left|\frac{\alpha}{2}\right|\right]\right)$.

Proof. By using (f) and (g), we have that

$$[H_0^5((2)^{\alpha'}I') : H_0^5((2)^{\alpha}I')] = \frac{[H^5 : H_0^5((2)^{\alpha}I')]}{[H^5 : H_0^5((2)^{\alpha'}I')]} = \begin{cases} 1, & \alpha = 1 \\ 4, & \alpha = 2,3 \\ 16, & \alpha \geq 4 \end{cases},$$

where $\alpha' = \alpha - \min\left(2, \left[\left|\frac{\alpha}{2}\right|\right]\right)$. □

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