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An Investigation on a Subclass of p-Valently Starlike Functions in the Unit Disc

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Abstract

Let $\mathcal{A}_p$ denote the class of functions of the form $f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \cdots$ which are regular and p-valent in the open unit disc $D = \{z : |z| < 1\}$. Let $M_p(\alpha)$ be the subclass of $\mathcal{A}_p$ consisting of functions $f(z)$ which satisfy $Re\left(z\frac{f'(z)}{f(z)}\right) < \alpha$, $(z \in D)$ for some real α $(\alpha > 1)$.

The aim of this paper is to give a representation theorem, a distortion theorem and a coefficient inequality for the class $M_p(\alpha)$.

Key Words: Starlike and convex functions, distortion theorem, coefficient inequality.

1. Introduction

Let Ω be the family of functions $w(z)$ which are regular in the open unit disc D and satisfy the conditions $w(0) = 0$, $|w(z)| < 1$ for $z \in D$.

Let, $\mathcal{A}_p$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \cdots, \tag{1}$$

which are analytic and p-valent in the open unit disc D.

Finally, let $M_p(\alpha)$ be the subclass of $\mathcal{A}_p$ consisting of functions $f(z)$ which satisfy the inequality

$$Re\left(z\frac{f'(z)}{f(z)}\right) < \alpha, \quad z \in D, \quad \alpha > p. \tag{2}$$

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This class was introduced by S. Owa and H. M. Sirivastava ([3], [4], [5]).

Definition 1 *([1]) Let $f(z)$ and $g(z)$ be analytic functions in the open unit disc D. Then we say that the function $f(z)$ is subordinate to $g(z)$, written $f \prec g$, if there exist as an analytic function $w(z)$ in the open unit disc D such that $w(0) = 0$, $|w(z)| < 1$ and $f(z) = g(w(z))$ for all $z \in D$.*

In particular, if $g(z)$ is univalent in D then $f \prec g$ if and only if $f(0) = g(0)$ and $f(D) \subseteq g(D)$.

The following lemma, known as the Jack's Lemma, is needed in the sequel.

Lemma 1 *([2]) Let $w(z)$ be a non-constant and analytic function in the unit disc D with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_0, then $z_0 w'(z_0) = k w(z_0)$ and $k \geq 1$.*

2. Main Results

In this section we derive a representation theorem, a distortion theorem and a coefficient inequality for the class $M_p(\alpha)$.

Lemma 2

$$f(z) \in M_p(\alpha) \Leftrightarrow \left(z\frac{f'(z)}{f(z)} \right) \prec \frac{p - (2\alpha - p)z}{1 - z}.$$

Proof. Let us define the function $p(z)$ by

$$p(z) = \frac{\alpha - z\frac{f'(z)}{f(z)}}{\alpha - p}$$

for $f(z) \in M_p(\alpha)$. Then $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is analytic in D, $p(0) = 1$ and $Re\, p(z) > 0$ $(z \in D)$. Hence we have

$$\frac{1 + w(z)}{1 - w(z)} = p(z) = \frac{\alpha - z\frac{f'(z)}{f(z)}}{\alpha - p}$$

or

$$z\frac{f'(z)}{f(z)} = \frac{p - (2\alpha - p)w(z)}{1 - w(z)} \Rightarrow z\frac{f'(z)}{f(z)} \prec \frac{p - (2\alpha - p)z}{1 - z}.$$

Conversely

$$z\frac{f'(z)}{f(z)} \prec \frac{p-(2\alpha-p)z}{1-z} \Rightarrow \frac{\alpha - z\frac{f'(z)}{f(z)}}{\alpha - p} = \frac{1+w(z)}{1-w(z)} = p(z) \Rightarrow$$

$$Re\left(z\frac{f'(z)}{f(z)}\right) < \alpha.$$

□

Theorem 1 *If $f(z) \in \mathcal{A}_p$ satisfies*

$$\left(z\frac{f'(z)}{f(z)} - p\right) \prec \frac{2(p-\alpha)z}{1-z}, \tag{3}$$

then $f(z) \in M_p(\alpha)$.

Proof. The linear transformation

$$w_1 = h(z) = \frac{2(p-\alpha)z}{1-z}$$

maps $|z| = r$ onto the circle centered at $C(r) = \frac{2(p-\alpha)r^2}{1-r^2}$ and having the radius $\rho(r) = \frac{2(\alpha-p)r}{1-r^2}$. Therefore $h(D)$ is contained in the closed disc centered at $C(r)$ with radius $\rho(r)$.

On the other hand, if we define the function $w(z)$ by

$$\frac{f(z)}{z^p} = (1-w(z))^{-2(p-\alpha)}, \tag{4}$$

where $(1-w(z))^{-2(p-\alpha)}$ has the value 1 at the origin, then $w(z)$ is analytic in D, and $w(0) = 0$. If we take the logarithmic derivate of equality (4), simple calculations yield

$$z\frac{f'(z)}{f(z)} - p = \frac{2(p-\alpha)zw'(z)}{1-w(z)}. \tag{5}$$

Now it is easy to realize that the subordination (3) is equivalent to $|w(z)| < 1$ for all $z \in D$. Indeed, assume the contrary. There exists $z_0 \in D$ such that $|w(z_0)| = 1$. Then by Jack's lemma, $z_0 w'(z_0) = kw(z_0)$ and $k \geq 1$, for such $z_0 \in D$, we have

$$z_0\frac{f'(z_0)}{f(z_0)} - p = \frac{2(p-\alpha)kw(z_0)}{1-w(z_0)} = h(w(z_0)) \notin h(D)$$

because $|w(z_0)| = 1$ and $k \geq 1$. But this contradicts condition (3) of this theorem and so $|w(z_0)| < 1$ for all $z \in D$. The sharpness of the result follows from the fact that $\frac{f(z)}{z^p} = (1 - w(z))^{-2(p-\alpha)}$ implies $z\frac{f'(z)}{f(z)} - p = \frac{2(p-\alpha)w(z)}{1-w(z)} = h(z)$.

On the other hand we have

$$\left(z\frac{f'(z)}{f(z)} - p\right) \prec \frac{2(p-\alpha)z}{1-z} \Rightarrow z\frac{f'(z)}{f(z)} = \frac{p - (2\alpha - p)w(z)}{1 - w(z)},$$

which shows that $f(z) \in M_p(\alpha)$ (by using Lemma 2). The sharpness of the result follows from the fact that $\frac{f(z)}{z^p} = (1 - z)^{-2(p-\alpha)}$ implies $\left(z\frac{f'(z)}{f(z)} - p\right) \prec \frac{2(p-\alpha)z}{1-z}$. □

Corollary 1 *If $f(z) \in M_p(\alpha)$ then $f(z)$ can be written in the form $f(z) = z^p(1 - w(z))^{-2(p-\alpha)}$, $w(z) \in \Omega$. Therefore the function $f_*(z) = \frac{z^p}{(1-z)^{2(p-\alpha)}}$ belongs to the class $M_p(\alpha)$.*

Theorem 2 *If $f(z) \in M_p(\alpha)$, then*

$$r^p(1 - r)^{2(\alpha-p)} \leq |f(z)| \leq r^p(1 + r)^{2(\alpha-p)}. \tag{6}$$

This result is sharp, since the extremal function is $f_*(z) = \frac{z^p}{(1-z)^{2(p-\alpha)}}$.

Proof. The linear transformation

$$w_2 = w_2(z) = \frac{p - (2\alpha - p)z}{1 - z}$$

maps $|z| = r$ into the circle

$$\left|w_2 - \frac{p - (2\alpha - p)r^2}{1 - r^2}\right| \leq \frac{2(\alpha - p)r}{1 - r^2}. \tag{7}$$

On the other hand, by using Lemma 2, the definition of subordination and (7), we get

$$\left|z\frac{f'(z)}{f(z)} - \frac{p - (2\alpha - p)r^2}{1 - r^2}\right| \leq \frac{2(\alpha - p)r}{1 - r^2}. \tag{8}$$

The inequality (8) can be written in the form

$$\frac{p - 2(\alpha - p)r - (2\alpha - p)r^2}{1 - r^2} \leq Re\left(z\frac{f'(z)}{f(z)}\right) \leq \frac{p + 2(\alpha - p)r - (2\alpha - p)r^2}{1 - r^2}. \tag{9}$$

Since

$$Re\left(z\frac{f'(z)}{f(z)}\right) = r\frac{\partial}{\partial r}\log|f(z)|\,, \quad |z| = r,$$

and by (9), we obtain

$$\frac{p-2(\alpha-p)r-(2\alpha-p)r^2}{r(1+r)(1-r)} \leq \frac{\partial}{\partial r}\log|f(z)| \leq \frac{p+2(\alpha-p)r-(2\alpha-p)r^2}{r(1+r)(1-r)}.$$

Integrating both sides of this inequalities from 0 to r we obtain (6). □

Corollary 2 *The radius of starlikeness of the class $M_p(\alpha)$ is*

$$r_s = \frac{p}{2\alpha - p}.$$

This radius is sharp because the extremal function is $f_*(z) = \frac{z^p}{(1-z)^{2(p-\alpha)}}$.

Proof. By using the inequality (9), we get

$$Re\left(z\frac{f'(z)}{f(z)}\right) \geq \frac{p-2(\alpha-p)r-(2\alpha-p)r^2}{1-r^2}. \tag{10}$$

Since $r < r_s$, the left hand side of the preceding inequality is positive, which implies that

$$r_s = \frac{p}{2\alpha - p}.$$

Also note that inequality (10) becomes an equality for the function

$$f_*(z) = \frac{z^p}{(1-z)^{2(p-\alpha)}}.$$

It follows that

$$r_s = \frac{p}{2\alpha - p}$$

and the proof is complete. □

Theorem 3 *If* $f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \cdots$ *belongs to* $M_p(\alpha)$, *then*

$$|a_{p+k}| \leq \frac{1}{k!}\prod_{m=0}^{k-1} 2(\alpha - p) + m. \tag{11}$$

Proof. Using Lemma 2, we get

$$p(z) = \frac{\alpha - z\frac{f'(z)}{f(z)}}{\alpha - p} \Leftrightarrow zf'(z) = \alpha f(z) + (p - \alpha)f(z)p(z) \tag{12}$$

$$\begin{cases} pz^p + (p+1)a_{p+1}z^{p+1} + (p+2)a_{p+2}z^{p+2} + \cdots + (p+k)a_{p+k}z^{p+k} + \cdots = \\ \left(\alpha z^p + \alpha a_{p+1}z^{p+1} + \alpha a_{p+2}z^{p+2} + \cdots + \alpha a_{p+k}z^{p+k} + \cdots\right) + \\ (p-\alpha)\left(z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \cdots + a_{p+k}z^{p+k} + \cdots\right) \cdot \\ \left(1 + p_1 z + p_2 z^2 + \cdots + p_k z^k + \cdots\right). \end{cases} \tag{13}$$

Evaluating the coefficient of z^{p+k} in both sides (13) gives

$$(p+k)a_{p+k} = \begin{cases} \alpha a_{p+k} + p(a_{p+k} + p_1 a_{p+k-1} + p_2 a_{p+k-2} + \cdots + p_{k-1}a_{p+1} + p_k) \\ -\alpha(a_{p+k} + p_1 a_{p+k-1} + p_2 a_{p+k-2} + \cdots + p_{k-1}a_{p+1} + p_k). \end{cases} \tag{14}$$

On the other hand, we have that

$$|p_n| \leq 2. \tag{15}$$

If we consider the relations (14) and (15) together, then we obtain

$$|a_{p+k}| \leq \frac{2(\alpha - p)}{k}\sum_{m=1}^{k} |a_{p+m-1}|, |a_p| = 1. \tag{16}$$

To prove (11) we will use the induction principle.

Now, consider inequality (16) and the following inequality:

$$|a_{p+k}| \leq \frac{1}{k!}\prod_{m=0}^{k-1} 2(\alpha - p) + m. \tag{17}$$

The right hand sides of these inequalities are the same, because

For $k = 1$:

$$|a_{p+1}| \leq \frac{1}{1!}\prod_{m=0}^{0} 2(\alpha - p) + m = 2(\alpha - p).$$

$$|a_{p+1}| \leq \frac{2(\alpha - p)}{1}\sum_{m=1}^{1} |a_{p+m-1}| = 2(\alpha - p);$$

For $k = 2$:

$$|a_{p+2}| \leq \frac{1}{2!}\prod_{m=0}^{1} 2(\alpha - p) + m = \frac{1}{2!}2(\alpha - p)(2(\alpha - p) + 1).$$

$$|a_{p+2}| \leq \frac{2(\alpha - p)}{2}\sum_{m=1}^{2} |a_{p+m-1}| = \frac{2(\alpha - p)}{2}(|a_p| + |a_{p+1}|) = \frac{1}{2!}2(\alpha - p)(2(\alpha - p) + 1).$$

Now, suppose that this result is true for $k = t$. Then we have

$$|a_{p+t}| \leq \frac{1}{t!}\prod_{m=0}^{t-1} x + m \Rightarrow$$

$$|a_{p+t}| \leq \frac{1}{t!}(x)(x + 1)(x + 2)\cdots(x + (t - 1)). \tag{18}$$

$$|a_{p+t}| \leq \frac{x}{t}\sum_{m=1}^{t} |a_{p+t-1}| \Rightarrow$$

$$|a_{p+t}| \leq \frac{x}{t}\left(1 + |a_{p+1}| + |a_{p+2}| + \cdots + |a_{p+t-1}|\right), \tag{19}$$

where $x = 2(\alpha - p)$.

From (18), (19) and the induction hypothesis, we get

$$\frac{x}{t!}(x + 1)(x + 2)\cdots(x + (t - 1)) = \frac{x}{t}\left(1 + |a_{p+1}| + |a_{p+2}| + \cdots + |a_{p+t-1}|\right) \Rightarrow$$

$$\frac{1}{t!}(x + 1)(x + 2)\cdots(x + (t - 1)) = \frac{1}{t}\left(1 + |a_{p+1}| + |a_{p+2}| + \cdots + |a_{p+t-1}|\right) \Rightarrow$$

$$\frac{x + t}{t + 1}\cdot\frac{1}{t!}(x+1)(x+2)\cdots(x+(t-1)) = \frac{x + t}{t + 1}\cdot\frac{1}{t}\left(1 + |a_{p+1}| + |a_{p+2}| + \cdots + |a_{p+t-1}|\right) \Rightarrow$$

$$\frac{1}{(t+1)!}(x+1)(x+2)\cdots(x+t) = \begin{cases} \frac{1}{t+1}[\frac{x}{t}(1+|a_{p+1}|+|a_{p+2}|+\cdots+|a_{p+t-1}|) \\ +(1+|a_{p+1}|+|a_{p+2}|+\cdots+|a_{p+t-1}|)] \Rightarrow \end{cases}$$

$$\frac{1}{(t+1)!}(x+1)(x+2)\cdots(x+t) = \frac{|a_{p+t}|}{t+1} + \frac{1}{t+1}(1+|a_{p+1}|+|a_{p+2}|+\cdots+|a_{p+t-1}|) \Rightarrow$$

$$\frac{1}{(t+1)!}(x+1)(x+2)\cdots(x+t) = \frac{1}{t+1}(1+|a_{p+1}|+|a_{p+2}|+\cdots+|a_{p+t-1}|+|a_{p+t}|) \Rightarrow$$

$$\frac{x}{(t+1)!}(x+1)(x+2)\cdots(x+t) = \frac{x}{t+1}(1+|a_{p+1}|+|a_{p+2}|+\cdots+|a_{p+t}|) \tag{20}$$

Equality (20) shows that the result is valid for $k = t+1$. Therefore, we have (11) and the proof is complete. □

References

[1] Goodman, A.W.: An invitation to the study of univalent and multivalent functions. Internal. J. Math. and Math. Sci. 2, 163-186 (1979).

[2] Jack, I.S.: Functions starlike and convex of order α. J. London Math. Soc. 3, 469-474 (1971).

[3] Nunokawa, M.: A sufficient condition for univalence and starlikeness, Proc. Japan, Acad. Ser a Math. Sci. 65, 163-164 (1989).

[4] Owa, S. and Srivastava, H.M.: Some generalized convolution properties associated with certain subclasses of analytic functions, J. Ineg. Pure Appl. Math. 3(3) (2002).

[5] Owa, S. and Srivastava, H.M.: Current topics in analytic function theory, Singapore, World Scientific, 1972.

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