

Potts Model with two Competing Binary Interactions

Nasir Ganikhodjaev, Hasan Akın and Seyit Temir

Abstract

The Potts model on a Cayley tree in the presence of competing two binary interactions and magnetic field is considered. We exactly solve a problem of phase transitions for the model, namely we calculate critical surface such that there is a phase transition above it, and a single Gibbs state found elsewhere.

Key Words: Potts model, competing interactions, Gibbs measure.

1. Introduction

Lattice spin systems is a large class of systems considered in statistical mechanics. Some of them have a real physical meaning, others are studied as suitable simplified models of more complicated systems. The structure of the lattice plays an important role in investigation of spin systems. For example, in order to study a phase transition problem for a system on \mathbb{Z}^d and on a Cayley tree, there are two different methods: Pirogov-Sinai theory on \mathbb{Z}^d , Markov random field theory and recurrent equations of this theory on Cayley tree. In [2-9], Gibbs measures are described for several models on a Cayley tree, using the Markov random field theory. The Potts model was introduced as a generalization of the Ising model. The idea came from the representation of the Ising model as interacting spins which can be either parallel or antiparallel. An obvious generalization was to extend the number of directions of the spins. Such a model was proposed by C.Domb as a PhD thesis for his student R.Potts in 1952. At present, the Potts model encompasses a number of problems in statistical physics and lattice theory. It has been a subject of increasing intense research interest in recent years [10]. It includes the ice-rule vertex and bond percolation models as special cases. In this paper we consider

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a Potts model with competing two binary interactions and external magnetic field on the Cayley tree. The paper is organized as follows. In sections 2 we give definitions of the model, Cayley tree and Gibbs measures. In sections 3 we reduce the problem of describing limit Gibbs measures to the problem of solving a system of nonlinear functional equations. Last section is devoted to describe translation-invariant Gibbs measures.

2. Definitions

Cayley tree. The Cayley tree Γ^k (see [1]) of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, from each vertex of which exactly $k + 1$ edges issue. Let $\Gamma^k = (V, L, i)$ where V is the set of vertices of Γ^k , L is the set of edges of Γ^k and i is the incidence function associating each edge $\ell \in L$ with its ends points $x, y \in V$. If $i(\ell) = \{x, y\}$, then x and y are called nearest neighboring vertices and we write $\ell = \langle x, y \rangle$. The distance $d(x, y), x, y \in V$ on Cayley tree is defined by the formula

$$d(x, y) = \min\{d|x = x_0, x_1, x_2, \dots, x_{d-1}, x_d = y \in V \text{ such that the pairs } \langle x_0, x_1 \rangle, \dots, \langle x_{d-1}, x_d \rangle \text{ are neighboring vertices}\}.$$

For the fixed $x^0 \in V$ we set

$$W_n = \{x \in V : d(x^0, x) = n\},$$

$$V_n = \{x \in V : d(x^0, x) \leq n\},$$

$$L_n = \{\ell = \langle x, y \rangle \in L : x, y \in V_n\}.$$

A collection of the pairs $\langle x_0, x_1 \rangle, \dots, \langle x_{d-1}, y \rangle$ is called a path from x to y . We write $x < y$ if the path from x^0 to y goes through x . We call the vertex y a direct successor of x , if $y > x$ and x, y are nearest neighbors. The set of the direct successors of x is denoted by $S(x)$, i.e.,

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}, x \in W$$

We observe that, for any vertex $x \neq x^0$, x has k direct successors and x^0 has $k + 1$. The vertices x and y are called second neighbors which is denoted by $> x, y <$, if there exists a vertex $z \in V$ such that x, z and y, z are nearest neighbors.

We consider a semi-infinite Cayley tree J^k of order $k \geq 2$, i. e., a graph without cycles with $(k + 1)$ edges issuing from each vertex except x^0 and with k edges issuing from the

vertex x^0 , which is called the tree root. According to well known theorems, this can be reconstituted as a Cayley tree [4]. The second neighbors $> x, y <$ is called one-level neighbors, if vertices x and y belong to W_n for some n , that is if they are situated on the same level. We will consider only one-level second neighbors.

In the Potts model, spin variables $\sigma(x)$ take their values on a discrete set $\Phi = \{1, 2, \dots, q\}$, $q > 2$ which are associated with each vertex of the tree J^k . The Potts model with competing two binary interactions, is defined by the following Hamiltonian:

$$H(\sigma) = -J \sum_{\langle x,y \rangle} \delta_{\sigma(x)\sigma(y)} - J_1 \sum_{>x,y<} \delta_{\sigma(x)\sigma(y)} - h \sum_{x \in V} \delta_{1\sigma(x)}, \quad (2.1)$$

where the sum in the first term ranges all nearest neighbors, second sum ranges all one-level second neighbors, δ is the Kroneker's symbol and $J, J_1, h \in \mathbb{R}$ are constants.

3. Recursive Equations

There are several approaches to derive equation or system equations describing limiting Gibbs measure for lattice models on Cayley tree. One approach is based on properties of Markov random fields on Bethe lattices [6–9]. Another approach is based on recursive equations for partition functions (for example [5]). Naturally, both approaches lead to the same equation (see [3]). The second approach is more suitable for models with competing interactions.

Let Λ be a finite subset of V . We will denote by $\sigma(\Lambda)$ the restriction of σ to Λ . Let $\bar{\sigma}(V \setminus \Lambda)$ be a fixed boundary configuration. The total energy of $\sigma(\Lambda)$ under condition $\bar{\sigma}(V \setminus \Lambda)$ is defined as

$$\begin{aligned} H_\Lambda(\sigma(\Lambda) | \bar{\sigma}(V \setminus \Lambda)) = & - J \sum_{\langle x,y \rangle : x,y \in \Lambda} \delta_{\sigma(x)\sigma(y)} - J_1 \sum_{>x,y< : x,y \in \Lambda} \delta_{\sigma(x)\sigma(y)} \\ & - J \sum_{\langle x,y \rangle : x \in \Lambda, y \notin \Lambda} \delta_{\sigma(x)\sigma(y)} - J_1 \sum_{>x,y< : x \in \Lambda, y \notin \Lambda} \delta_{\sigma(x)\sigma(y)} \\ & - h \sum_{x \in \Lambda} \delta_{1\sigma(x)}. \end{aligned} \quad (3.1)$$

Then partition function $Z_\Lambda(\bar{\sigma}(V \setminus \Lambda))$ in volume Λ under boundary condition $\bar{\sigma}(V \setminus \Lambda)$ is defined as

$$Z_\Lambda(\bar{\sigma}(V \setminus \Lambda)) = \sum_{\sigma(\Lambda) \in \Omega(\Lambda)} \exp(-\beta H_\Lambda(\sigma(\Lambda) | \bar{\sigma}(V \setminus \Lambda))), \quad (3.2)$$

where $\Omega(\Lambda)$ is the set of all configuration in volume Λ and $\beta = \frac{1}{T}$ is the inverse temperature.

We consider the configuration $\sigma(V_n)$ and the partition functions Z_{V_n} in volume V_n and for the brevity we will denote them as σ_n and $Z^{(n)}$ respectively. Let us decompose the partition function $Z^{(n)}$ into following summands:

$$Z^{(n)} = \sum_{i=1}^q Z_i^{(n)},$$

where

$$Z_i^{(n)} = \sum_{\sigma_n \in \Omega(V_n): \sigma(x^0)=i} \exp(-\beta H_{V_n}(\sigma_n | \bar{\sigma}(V \setminus V_n))). \quad (3.3)$$

We set

$$\theta = \exp(\beta J); \theta_1 = \exp(\beta J_1); \theta_2 = \exp(\beta h).$$

We will consider case $q = 3$ and $k = 2$.

Let $S(x^0) = \{x^1, x^2\}$. If $\sigma(x^0) = i$, $\sigma(x^1) = j$ and $\sigma(x^2) = m$, then from (3.1) and (3.2) we have following

$$Z_i^{(n)} = \sum_{j,m=1}^3 \exp(J\delta_{ij} + J\delta_{im} + J_1\delta_{jm} + h\delta_{1i}) Z_j^{(n-1)} Z_m^{(n-1)},$$

so that

$$\begin{aligned} Z_1^{(n)} &= \theta_2 [\theta^2 \theta_1 (Z_1^{(n-1)})^2 + 2\theta Z_1^{(n-1)} Z_2^{(n-1)} \\ &+ 2\theta Z_1^{(n-1)} Z_3^{(n-1)} + 2Z_2^{(n-1)} Z_3^{(n-1)} + \theta_1 (Z_2^{(n-1)})^2 + \theta_1 (Z_3^{(n-1)})^2], \end{aligned}$$

$$\begin{aligned} Z_2^{(n)} &= \theta_1 (Z_1^{(n-1)})^2 + 2\theta Z_1^{(n-1)} Z_2^{(n-1)} \\ &+ 2Z_1^{(n-1)} Z_3^{(n-1)} + 2\theta Z_2^{(n-1)} Z_3^{(n-1)} + \theta^2 \theta_1 (Z_2^{(n-1)})^2 + \theta_1 (Z_3^{(n-1)})^2, \end{aligned}$$

$$Z_3^{(n)} = \theta_1(Z_1^{(n-1)})^2 + 2\theta Z_2^{(n-1)}Z_3^{(n-1)} + 2\theta Z_2^{(n-1)}Z_3^{(n-1)} + 2\theta(Z_1^{(n-1)})^2(Z_3^{(n-1)})^2 + \theta_1(Z_1^{(n-1)})^2 + \theta^2\theta_1(Z_3^{(n-1)})^2.$$

After replacements $u_n = \frac{Z_2^{(n)}}{Z_1^{(n)}}$ and $v_n = \frac{Z_3^{(n)}}{Z_1^{(n)}}$, we have the following system of recurrent equations:

$$\begin{aligned} \theta_2 u_n &= \frac{\theta_1 + 2\theta u_{n-1} + 2v_{n-1} + 2\theta u_{n-1}v_{n-1} + \theta^2\theta_1 u_{n-1}^2 + \theta_1 v_{n-1}^2}{\theta^2\theta_1 + \theta_1 u_{n-1}^2 + \theta_1 v_{n-1}^2 + 2u_{n-1}v_{n-1} + 2\theta u_{n-1} + 2\theta v_{n-1}} \\ \theta_2 v_n &= \frac{\theta_1 + 2\theta u_{n-1} + 2v_{n-1} + 2\theta u_{n-1}v_{n-1} + \theta_1 u_{n-1}^2 + \theta^2\theta_1 v_{n-1}^2}{\theta^2\theta_1 + \theta_1 u_{n-1}^2 + \theta_1 v_{n-1}^2 + 2u_{n-1}v_{n-1} + 2\theta u_{n-1} + 2\theta v_{n-1}}. \end{aligned}$$

If $u = \lim u_n$ and $v = \lim v_n$, then

$$\begin{aligned} \theta_2 u &= \frac{\theta_1 + 2\theta u + 2v + 2\theta uv + \theta^2\theta_1 u^2 + \theta_1 v^2}{\theta^2\theta_1 + 2\theta u + 2\theta v + 2uv + \theta_1 u^2 + \theta_1 v^2} \\ \theta_2 v &= \frac{\theta_1 + 2u + 2\theta v + 2\theta uv + \theta_1 u^2 + \theta^2\theta_1 v^2}{\theta^2\theta_1 + 2\theta u + 2\theta v + 2uv + \theta_1 u^2 + \theta_1 v^2}. \end{aligned} \tag{3.4}$$

The solutions of this system of nonlinear equations describe translation-invariant Gibbs measures [3]. Note that, if there is more than one positive solution for equation (3.4), then there is more than one translation-invariant Gibbs measure corresponding to these solutions. We say that a phase transition occurs for model (2.1), if equation (3.4) has more than one positive solution. The number of solutions of equation (3.4) depends on the parameter $\beta = \frac{1}{kT}$. The phase transition usually occurs for low temperature. It is possible to find an exact value of T^* such that a phase transition occurs for all $T < T^*$, where T^* is called a critical value of temperature.

Finding the exact value of the critical temperature for some models is to exactly solve the models.

4. Translation-Invariant Gibbs Measures

In this sections we describe solutions of the system of nonlinear equations

$$\theta_2 u = \frac{\theta_1 + 2\theta u + 2v + 2\theta uv + \theta^2\theta_1 u^2 + \theta_1 v^2}{\theta^2\theta_1 + 2\theta u + 2\theta v + 2uv + \theta_1 u^2 + \theta_1 v^2} \tag{4.1}$$

$$\theta_2 v = \frac{\theta_1 + 2u + 2\theta v + 2\theta uv + \theta_1 u^2 + \theta^2 \theta_1 v^2}{\theta^2 \theta_1 + 2\theta u + 2\theta v + 2uv + \theta_1 u^2 + \theta_1 v^2}.$$

Subtracting these equations, we have

$$\theta_2(u - v) = \frac{2(\theta - 1)(u - v) + \theta^2 \theta_1(u^2 - v^2)}{\theta^2 \theta_1 + 2\theta u + 2\theta v + 2uv + \theta_1 u^2 + \theta_1 v^2}, \quad (4.2)$$

so that equality $u = v$ give us some solutions of (4.1). For $u = v$ we have

$$\theta_2 u = \frac{(\theta^2 \theta_1 + 2\theta + \theta_1)u^2 + 2(\theta + 1)u + \theta_1}{2(\theta_1 + 1)u^2 + 4\theta u + \theta^2 \theta_1}. \quad (4.3)$$

Let us consider the function

$$f(u) = \frac{(\theta^2 \theta_1 + 2\theta + \theta_1)u^2 + 2(\theta + 1)u + \theta_1}{2(\theta_1 + 1)u^2 + 4\theta u + \theta^2 \theta_1}. \quad (4.4)$$

Elementary analysis of this function give us the following:

- 1) If $\theta > 1$, $f(u)$ is increasing; and if $\theta < 1$, $f(u)$ is decreasing for $u > 0$;
- 2) If $\theta_1 > \theta_1^*$, where

$$\theta_1^* = \frac{\theta^2 + \theta + 1 + \sqrt{9\theta^4 + 26\theta^3 + 35\theta^2 + 50\theta + 33}}{(\theta^2 + 2)(\theta + 1)},$$

then there exists an inflection point $u^* > 0$ such that $f''(u) > 0$ for $0 < u < u^*$ and $f''(u) < 0$ for $u > u^*$.

Lemma. *Let $\theta > 1$, and there is an inflection point. There exist $\eta_1(\theta, \theta_1)$, $\eta_2(\theta, \theta_1)$ with $0 < \eta_1(\theta, \theta_1) < \eta_2(\theta, \theta_1)$ such that equation (4.1) has three solutions if $\eta_1(\theta, \theta_1) < \theta_2 < \eta_2(\theta, \theta_1)$; has two solutions if either $\theta_2 = \eta_1(\theta, \theta_1)$ or $\theta_2 = \eta_2(\theta, \theta_1)$ and has single solution in other cases. In fact,*

$$\eta_i(\theta, \theta_1) = \frac{1}{u_i^*} f(u_i^*)$$

where u_1^*, u_2^* are the solutions of equation

$$u f'(u) = f(u).$$

Proof follows from properties 1 and 2 of function f . Equation $uf'(u) = f(u)$ is equivalent to

$$Au^4 + Bu^3 + Cu^2 + Du + E = 0, \tag{4.5}$$

where $A = 2(\theta_1 + 1)(\theta^2\theta_1 + 2\theta + \theta_1) > 0$, $B = 8(\theta + 1)(\theta_1 + 1) > 0$, $D = 8\theta\theta_1 > 0$, and $E = \theta^2\theta_1 > 0$.

As

$$C = -\theta^4\theta_1^2 - 2\theta_1\theta^3 - \theta_1\theta^2 + 8\theta^2 + 8\theta + 6\theta_1^2 + 6\theta_1,$$

then equation (4.5) has no positive roots if $C > 0$, and have two positive roots u_1^* , u_2^* if $C < 0$. It is easy to show that exist θ_1^{**} such that for all $\theta_1 > \theta_1^{**}$ coefficient C is negative, where $\theta > \sqrt{2}$.

Thus for $\theta > \sqrt{2}$, $\theta_1(\theta) > \max(\theta_1^*, \theta_1^{**})$ and $\eta_1(\theta, \theta_1) < \theta_2 < \eta_2(\theta, \theta_1)$, equation (4.5) has three positive roots.

Now let us come back to equation (4.3). After cancelling by $(u - v)$, we have

$$t = \frac{\theta_1\theta_2s^2 + (2\theta_2\theta - \theta_1(\theta^2 - 1))s + \theta^2\theta_1\theta_2 - 2(\theta - 1)}{2\theta_2(\theta_1 - 1)}. \tag{4.6}$$

where $u + v = s$ and $uv = t$. After dividing the first equation of (4.1) into the second, and simplifying we have

$$t = \frac{\theta_1s^2 + 2s + \theta_1}{\theta^2\theta_1 + \theta_1 - 2\theta}, \tag{4.7}$$

From (4.7) and (4.6) follow

$$\begin{aligned} \frac{\theta_1(\theta + 1)s + 2}{\theta_2} &= \frac{\theta_1[\theta_1(\theta + 1) - 2]s^2 + 2[\theta_1(\theta^2 + \theta - 2) - 2(\theta + 1)]s}{\theta_1(\theta^2 + 1) - 2\theta} \\ &+ \frac{\theta_1[\theta_1(\theta^3 + \theta^2 + 2\theta + 2) - 2(\theta^2\theta + 1)]}{\theta_1(\theta^2 + 1) - 2\theta}. \end{aligned} \tag{4.8}$$

It is easy to show that if $\theta_1 > \frac{\theta+1}{\theta^2+\theta-2}$ then all coefficients in (4.8) are positive.

Note that the system of equations

$$\begin{aligned} u + v &= s \\ uv &= t \end{aligned} \tag{4.9}$$

have solutions when $t < \frac{s^2}{4}$, so that from (4.7) follow

$$(\theta_1(\theta^2 - 3) - 2\theta)s^2 - 8s - 4\theta_1 > 0. \tag{4.10}$$

It is easy to check that for $\theta > \sqrt{3}$ and

$$\theta_1 > \frac{\theta + \sqrt{9\theta^2 - 24}}{\theta^2 - 3}, \tag{4.11}$$

inequality (4.10) is valid.

From $\theta_1 > \frac{\theta+1}{\theta^2+\theta-2}$ and $\theta_1 > \frac{\theta+\sqrt{9\theta^2-24}}{\theta^2-3}$ follow that, for

$$\theta_1 > \frac{\theta + \sqrt{9\theta^2 - 24}}{\theta^2 - 3},$$

all coefficients of (4.8) are positive and system (4.9) have solutions.

Now let us consider equation (4.8). The elementary analysis gives the following:

1) if

$$\theta_2 < \frac{2[\theta_1(\theta^2 + 1) - 2\theta]}{\theta_1[\theta_1(\theta^3 + \theta^2 + 2\theta + 2) - 2(\theta^2 + \theta + 1)]},$$

then equation (4.8) has single root;

2) if

$$\theta_2 > \frac{2[\theta_1(\theta^2 + 1) - 2\theta]}{\theta_1[\theta_1(\theta^3 + \theta^2 + 2\theta + 2) - 2(\theta^2 + \theta + 1)]},$$

there is k_0 such that the line $y - \frac{2}{\theta_2} = k_0 s$ is a tangent for parabola from the right-hand side of (4.8); and then for $\theta_2 < \frac{\theta_1(\theta_1+1)}{k_0}$, equation (4.8) has two positive roots.

Thus equation (4.8) has two positive roots, when

$$\theta > \sqrt{3}, \theta_1 > \frac{\theta + \sqrt{9\theta^2 - 24}}{\theta^3 - 3},$$

and

$$\frac{2[\theta_1(\theta^2 + 1) - 2\theta]}{\theta_1[\theta_1(\theta^3 + \theta^2 + 2\theta + 2) - 2(\theta^2 + \theta + 1)]} < \theta_2 < \frac{\theta_1(\theta + 1)}{k_0}$$

where k_0 is defined above.

Recalling results for equation (4.1), we have following. The system of equation (4.1) has seven solutions if $\theta > \sqrt{3}$, $\theta_1 > A$ and $B < \theta_2 < C$, where

$$A = \max(\theta_1^*, \theta_1^{**}, \frac{\theta + \sqrt{9\theta^2 - 24}}{\theta^2 - 3})$$

$$B = \max(\eta_1, \frac{2[\theta_1(\theta^2 + 1) - 2\theta]}{\theta_1[\theta_1(\theta^3 + \theta^2 + 2\theta + 2) - 2(\theta^2 + \theta + 1)]}) \quad (4.12)$$

$$C = \min(\eta_2, \frac{\theta_1(\theta + 1)}{k_0}).$$

A more detailed analysis similar to [2] shows that only three solutions are stable.

Thus we have proved the following theorem

Theorem. *Assume that conditions (4.12) are satisfied then for the model (2.1), and then there are three translation-invariant Gibbs measure, i.e. there is phase transition takes place.*

It is easy to show that these three measures correspond to boundary conditions

$$\sigma(V \setminus V_n) \equiv i, \quad i = 1, 2, 3.$$

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Nasir GANIKHODJAEV
 Institute of Mathematics
 29 F. Hodjaev str.
 Tashkent 700125, Uzbekistan
 and Faculty of Science
 IIUM, 53100 Kuala Lumpur, MALAYSIA
 e-mail: nasirgani@yandex.ru
 Hasan AKIN
 Department of Mathematics
 Arts and Science Faculty
 Harran University, Sanliurfa, 63200, TURKEY
 e-mail: akinhasan@harran.edu.tr
 Seyit TEMİR
 Department of Mathematics
 Arts and Science Faculty
 Harran University, Sanliurfa, 63200, TURKEY
 e-mail: temirsevit@harran.edu.tr

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