Braiding for Categorical and Crossed Lie Algebras and Simplicial Lie Algebras

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Abstract

In this work, we give the notion of *braiding* for categorical Lie algebras and crossed modules of Lie algebras and we give an equivalence between them.

Key Words: Simplicial Lie Algebra, Categorical Lie Algebra Crossed Module.

1. Introduction

Crossed modules of groups were introduced by Whitehead in [19]. The commutative algebra analogue of crossed modules was given by Porter in [18]. Kassel and Loday [15] introduced crossed modules of Lie algebras as computational algebraic objects equivalent to simplicial Lie algebras with Moore complex of length 1. Conduché [6] defined 2-crossed module of groups and he gave a link between 2-crossed modules and simplicial groups. Ellis [11] captured the algebraic structure of a Moore complex of length 2 in his definition of a 2-crossed module of Lie algebras. Akça and Arvasi [1] have defined higher dimensional Peiffer elements for Lie algebras in the image of the Moore complex of a simplicial Lie algebra, and they then gave a functor from simplicial Lie algebras to 2-crossed Lie algebras in terms of hypercrossed complex pairings.

Joyal and Street [13, 14] have defined the notion of *braiding* for a monoidal category and they have showed that braided monoidal categories are equivalent to crossed semi-modules with bracket operations. Brown and Gilbert introduced in [4] the notion of

braided, regular crossed module as an algebraic model for homotopy 3-type equivalent to Conduché's 2-crossed module and simplicial groups with Moore complex of length 2. The reduced case of braided regular crossed module is called a braided crossed module of groups. Of course, braided crossed modules of groups are equivalent to reduced 2-crossed modules and braided categorical groups (cf. [13, 14]) and reduced simplicial groups with Moore complex of length 2 (cf. [6]).

Thus the main points of this paper are:

(i) to define the notion of *braiding* for categorical Lie algebras (without identities) and crossed modules of Lie algebras;

(ii) to construct an equivalence between the category of braided crossed modules of Lie algebras and that of braided categorical Lie algebras by using the equivalence between crossed objects and categorical objects in the category of Lie algebras; and

(iii) by using the method of Akça and Arvasi [1], to give a functor from reduced simplicial Lie algebras to braided crossed modules of Lie algebras in terms of hypercrossed complex pairings.

2.   Braiding for Categorical Lie Algebras and Crossed Modules

All Lie algebras will be over a fixed commutative ring k.

As we mentioned in introduction, Joyal and Street [13, 14] have defined the notion of *braiding* for any monoidal category and showed that braided monoidal categories are equivalent to crossed semi-modules with bracket operations. Now, we give the definition of braided monoidal category from [13].

A *braided monoidal category* is a monoidal category $(V, \otimes, I)$ together with a family of isomorphisms

$$\tau_{A,B} : A \otimes B \to B \otimes A$$

called *braidings*, natural in both variables, such that for all A, B, C in V,

(i) $\tau_{A,I} = \tau_{I,A} = 1_A$,

(ii) $\tau_{A\otimes B,C} = (\tau_{A,C} \otimes 1_B) \circ (1_A \otimes \tau_{B,C})$,

(iii) $\tau_{A,B\otimes C} = (1_B \otimes \tau_{A,C}) \circ (\tau_{A,B} \otimes 1_C)$.

A symmetry is a braiding such that the following diagram commutes:

$$A \otimes B \xrightarrow{\tau_{A,B}} B \otimes A \xrightarrow{\tau_{B,A}} A \otimes B$$
$$A \otimes B \xrightarrow{\quad 1 \quad} A \otimes B \quad ;$$

but not every braiding is a symmetry. If the braiding τ is a symmetry, then $(V, \otimes, I)$ is called a symmetric monoidal category.

One link with homotopy theory and related areas is that the nerve of a braided monoidal category is, up to group completion, a double loop space (cf. Berger, [3]). Kamps and Porter gave a link between braided monoidal categories, Gray categories and 2-crossed modules in [16]. Garzon and Miranda [12] examined braided categorical groups and braided crossed modules of groups and they gave the homotopy properties of them.

In the following $\mathbf{Cat}(\mathbf{LAlg})$ will denote the category of internal categories in the category of Lie algebras. An object of $\mathbf{Cat}(\mathbf{LAlg})$, called a categorical Lie algebra, will be represented by a diagram of Lie algebras and homomorphisms

$$\mathbf{L} : L_1 \underset{I}{\overset{s,t}{\rightrightarrows}} L_0 \,,$$

such that $sI = tI = id_{L_0}$, and the composition of two morphisms $x, y \in L_1$ with $t(x) = s(y)$ will be denoted $x \circ y$. Now we give the notion of braiding for categorical Lie algebras.

Definition 2.1 *A braiding for a categorical Lie algebra*

$$\mathbf{L} : L_1 \underset{I}{\overset{s,t}{\rightrightarrows}} L_0$$

is a map

$$
\begin{aligned}
L_0 \times L_0 &\xrightarrow{\tau} L_1 \\
(a, b) &\longmapsto \tau_{a,b}
\end{aligned}
$$

which satisfies the following conditions.

1. $s\tau_{a,b} = [a, b]$ and $t\tau_{a,b} = [b, a]$, thus $\tau_{a,b} : [a, b] \to [b, a]$ is a morphism in L_1.

2. Given $x, y \in L_1$; $x : a \to a'$, $y : b \to b'$, the following diagram is commutative:

$$[a, b] \xrightarrow{\ [x,y]\ } [a', b']$$

$$\tau_{a,b} \Big\downarrow \qquad\qquad \Big\downarrow \tau_{a',b'}$$

$$[b, a] \xrightarrow[\ [y,x]\ ]{} [b', a']$$

or equivalently for $x, y \in L_1$,

$$[x, y] \circ \tau_{t(x),t(y)} = \tau_{s(x),s(y)} \circ [y, x].$$

3. For $a, b, c \in L_0$

$$\tau_{[a,b],c} = [\tau_{a,c}, I(b)] + [I(a), \tau_{b,c}].$$

4. For $a, b, c \in L_0$

$$\tau_{a,[b,c]} = [I(b), \tau_{a,c}] + [\tau_{a,b}, I(c)].$$

A categorical Lie algebra together with a braiding map is called a braided categorical Lie algebra. Given braided categorical Lie algebras $(\mathbf{L}, \tau), (\mathbf{L}', \tau')$, a morphism between them is a morphism of Lie algebras which is compatible with τ in the sense that the following square is commutative:

$$
\begin{array}{ccc}
L_0 \times L_0 & \xrightarrow{\ \tau\ } & L_1 \\
{\scriptstyle f_0 \times f_0} \downarrow & & \downarrow {\scriptstyle f_1} \\
L_0' \times L_0' & \xrightarrow{\ \tau'\ } & L_1'.
\end{array}
$$

$\mathbf{BCat}(\mathbf{LAlg})$ will denote the category of braided categorical Lie algebras.

The notion of crossed module of Lie algebras was defined by Kassel and Loday [15].

Let M and N be two Lie algebras. By an action of N on M we mean a k-linear map $N \times M \to M$, $(n, m) \mapsto n \cdot m$ satisfying

$$
\begin{aligned}
[n, n'] \cdot m &= n \cdot (n' \cdot m) - n'(n \cdot m) \\
n \cdot [m, m'] &= [n \cdot m, m'] + [m, n \cdot m']
\end{aligned}
$$

for all $m, m' \in M$ and $n, n' \in N$.

Recall from [15] the notion of a crossed module of Lie algebras. A *crossed module of Lie algebras* is a Lie homomorphism $\partial : M \to N$ together with an action of N on M such that

$$CM1) \quad \partial(n \cdot m) = [n, \partial m] \qquad CM2) \quad \partial m \cdot m' = [m, m']$$

for all $m, m' \in M$, $n \in N$.

Now, we give the notion of *braiding* for crossed module of Lie algebras.

Definition 2.2 *A braided crossed module of Lie algebras* (M, N, ∂) *is a crossed module of Lie algebras together with a map* $\{-, -\} : N \times N \to M$ *called braiding map satisfying the following axioms:*

B1. $\partial\{x, y\} + [y, x] = [x, y]$,

B2. $\{x, \partial a\} + a \cdot x = x \cdot a$,

B3. $\{\partial b, y\} + y \cdot b = b \cdot y$,

B4. $\{\partial a, \partial b\} + [b, a] = [a, b]$,

B5. $\{a, [b, c]\} = b \cdot \{a, c\} + \{a, b\} \cdot c$,

B6. $\{[a, b], c\} = \{a, c\} \cdot b + a \cdot \{b, c\}$

for all $a, b, c \in M$ *and* $x, y \in N$.

The morphisms of braided crossed modules of Lie algebras are the morphisms of crossed modules of Lie algebras which are compatible with the braiding map. **BLXM** will denote the category of braided crossed modules of Lie algebras.

The category of braided crossed modules is equivalent to that of braided cat-groups. In the following, we give the Lie algebra case of this result.

Theorem 2.3 *The category of braided categorical Lie algebras is equivalent to the category of braided crossed modules of Lie algebras.*

Proof. Let (M, N, ∂) be a braided crossed module of Lie algebras together with a braiding map $\{-, -\}$. By using the action of N on M, one forms the semi-direct product Lie algebra $M \rtimes N$ with Lie operation given by

$$[(m, n), (m', n')] = ([m, m'] + m \cdot n' + n \cdot m', [n, n'])$$

for all $(m, n), (m', n') \in M \rtimes N$. Let $L_0 = N$ and $L_1 = M \rtimes N$. The source and target maps are given by $s(m, n) = \partial m + n$ and $t(m, n) = n$ respectively. These maps are Lie

algebra homomorphisms. Indeed,

$$
\begin{aligned}
s([(m,n),(m',n')]) &= s(([m,m'] + m \cdot n' + n \cdot m', [n,n'])) \\
&= \partial[m,m'] + \partial(m \cdot n') + \partial(n \cdot m') + [n,n'] \\
&= [\partial m, \partial m'] + [\partial m, n'] + [n, \partial m'] + [n,n'] \\
&= [\partial m + n, \partial m' + n'] \\
&= [s(m,n), s(m',n')],
\end{aligned}
$$

and

$$
t([(m,n),(m',n')]) = [n,n'] = [t(m,n), t(m',n')].
$$

From L_0 to L_1, the identity map can be given by $I(n) = (0,n)$ for $n \in L_0$. The category composition on L_1 can be given by

$$
(m,n) \circ (m',n') = (m + m', n')
$$

if $n = \partial m' + n'$. Thus we have a category object in the category of Lie algebras:

$$
\mathbf{L} : L_1 = M \rtimes N \underset{I}{\overset{s,t}{\rightrightarrows}} N = L_0 \ .
$$

A braiding map for this categorical Lie algebra is given by

$$
\begin{aligned}
\tau : \quad L_0 \times L_0 &\rightarrow \quad L_1 \\
(a,b) \quad &\mapsto \quad \tau_{a,b} = (\{a,b\}, [b,a]),
\end{aligned}
$$

where $\{-,-\}$ is the braiding map for the crossed module ∂. Now, we show that all axioms of braided categorical Lie algebra are satisfied.

1. For $a, b \in L_0$,

$$
\begin{aligned}
s\tau_{a,b} &= s(\{a,b\}, [b,a]) \\
&= \partial\{a,b\} + [b,a] \\
&= [a,b] - [b,a] + [b,a] \quad \text{(by } B1) \\
&= [a,b],
\end{aligned}
$$

and

$$
\begin{aligned}
t\tau_{a,b} &= t(\{a,b\}, [b,a]) \\
&= [b,a].
\end{aligned}
$$

Thus we have $\tau_{a,b} : [a, b] \to [b, a]$.

2. For $x = (a, b) : \partial a + b \to b$ and $y = (a', b') : \partial a' + b' \to b'$, we have

$$[x, y] : [(\partial a + b), (\partial a' + b')] \to [b, b'].$$

For $c = \partial a + b$ and $d = \partial a' + b'$, it must be that

$$\tau_{c,d} \circ [y, x] = [x, y] \circ \tau_{b,b}.$$

We have

$$
\begin{aligned}
[x, y] \circ \tau_{b,b'} &= [x, y] \circ (\{b, b'\}, [b', b]) \\
&= ([a, a'] + a \cdot b' + b \cdot a', [b, b']) \circ (\{b, b'\}, [b', b]) \\
&= ([a, a'] + a \cdot b' + b \cdot a' + \{b, b'\}, [b', b]).
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
\tau_{c,d} \circ [y, x] &= (\{c, d\}, [d, c]) \circ ([a', a] + a' \cdot b + b' \cdot a, [b', b]) \\
&= (\{c, d\} + [a', a] + a' \cdot b + b' \cdot a, [b', b]).
\end{aligned}
$$

Since

$$
\begin{aligned}
\{c, d\} &= \{\partial a + b, \partial a' + b'\} \\
&= \{\partial a + b, \partial a'\} + \{\partial a + b, b'\} \\
&= \{\partial a, \partial a'\} + \{b, \partial a'\} + \{\partial a, b'\} + \{b, b'\} \quad \text{by bilinearity} \\
&= [a, a'] - [a', a] + b \cdot a' - a' \cdot b + a \cdot b' - b' \cdot a + \{b, b'\} \quad \text{by } B4, B2, B3,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\tau_{c,d} \circ [y, x] &= (\{c, d\} + [a', a] + a' \cdot b + b' \cdot a, [b', b]) \\
&= ([a, a'] - [a', a] + b \cdot a' - a' \cdot b + a \cdot b' - b' \cdot a + \{b, b'\} \\
&\quad + [a', a] + a' \cdot b + b' \cdot a, [b', b]) \\
&= ([a, a'] + a \cdot b' + b \cdot a' + \{b, b'\}, [b', b]).
\end{aligned}
$$

Thus we have

$$\tau_{c,d} \circ [y, x] = [x, y] \circ \tau_{b,b}.$$

Therefore, the diagram

$$[s(x), s(y)] \xrightarrow{[x,y]} [t(x), t(y)]$$

$$\tau_{s(x),s(y)} \downarrow \qquad \qquad \downarrow \tau_{t(x),t(y)}$$

$$[s(y), s(x)] \xrightarrow[{[y,x]}]{} [t(y), t(x)]$$

is commutative.

3. For $a, b, c \in L_0$, we have

$$
\begin{aligned}
[\tau_{a,c}, I(b)] + [I(a), \tau_{b,c}] &= [(\{a,c\}, [c,a]), (0,b)] + [(0,a), (\{b,c\}, [c,b])] \\
&= ([\{a,c\}, 0] + \{a,c\} \cdot b + [c,a] \cdot 0, [[c,a], b]) \\
&\quad + ([0, \{b,c\}] + 0 \cdot [c,b] + a \cdot \{b,c\}, [a, [c,b]]) \\
&= (\{a,c\} \cdot b + a \cdot \{b,c\}, [[c,a], b] + [a, [c,b]]) \\
&= (\{[a,b], c\}, [c, [a,b]]) \quad \text{(by } B6 \text{ and Jacobi identity)} \\
&= \tau_{[a,b],c}.
\end{aligned}
$$

4. For $a, b, c \in L_0$, we have

$$
\begin{aligned}
[I(b), \tau_{a,c}] + [\tau_{a,b}, I(c)] &= [(0,b), (\{a,c\}, [c,a])] + [(\{a,b\}, [b,a]), (0,c)] \\
&= ([0, \{a,c\}] + 0 \cdot [c,a] + b \cdot \{a,c\}, [b, [c,a]]) \\
&\quad + ([\{a,b\}, 0] + [b,a] \cdot 0 + \{a,b\} \cdot c, [[b,a], c]) \\
&= (b \cdot \{a,c\} + \{a,b\} \cdot c, [b, [c,a]] + [[b,a], c]) \\
&= (\{a, [b,c]\}, [[b,c], a]) \quad \text{(by } B5 \text{ and Jacobi identity)} \\
&= \tau_{a,[b,c]}.
\end{aligned}
$$

Thus, all axioms of braided categorical Lie algebra are verified. We can define a functor from braided crossed modules to braided categorical Lie algebras:

$$\Delta : \mathbf{BLXM} \to \mathbf{BCat}(\mathbf{LAlg}).$$

Conversely let

$$\mathbf{L} : L_1 \xrightarrow[I]{s,t} L_0$$

be a braided categorical Lie algebra together with a braiding map τ. Then $t : \ker s \to L_0$ is a crossed module with the actions given by $a \cdot x = [a, I(x)]$ and $x \cdot a = [I(x), a]$ for $x \in L_0$

and $a \in \ker s$. Indeed $t(a \cdot x) = t([a, I(x)]) = [ta, x]$ and $a \cdot t(a') = [a, I(ta')] = [a, a']$. The braiding map on this crossed module is given by

$$
\begin{aligned}
\{-, -\} : \quad C_0 \times C_0 &\longrightarrow \quad \ker s \\
(a, b) &\longmapsto \quad I([a, b]) - \tau_{a,b}.
\end{aligned}
$$

Since $s(\{a, b\}) = s(I([a, b]) - \tau_{a,b}) = [a, b] - [a, b] = 0$, we have $\{a, b\} \in \ker s$. The verification of the axioms of braided crossed module is easy. For example the following equalities;
for $a, b \in C_0$

$$
\begin{aligned}
t\{a, b\} &= t(I([a, b]) - \tau_{a,b}) \\
&= tI([a, b]) - t\tau_{a,b} \\
&= [a, b] - [b, a],
\end{aligned}
$$

for $x, y \in \ker s$

$$
\begin{aligned}
\{a, ty\} &= [I(a), y] - \tau_{a,ty} \\
&= a \cdot y - y \cdot a
\end{aligned}
$$

and

$$
\begin{aligned}
\{tx, b\} &= [x, I(b)] - \tau_{tx,b} \\
&= x \cdot b - b \cdot x
\end{aligned}
$$

are axioms $B1, B2$, and $B3$ of braided crossed module respectively. We leave the other axioms to reader as an exercise. Thus we can define a functor from braided categorical Lie algebras to braided crossed modules;

$$
\Theta : \mathbf{BCat}(\mathbf{LAlg}) \to \mathbf{BLXM}.
$$

$\square$

3.   Simplicial and Braided Crossed Lie Algebras

In this section, we define a functor from reduced simplicial Lie algebras to braided crossed modules of Lie algebras in terms of hypercrossed complex pairings.

Simplicial Lie Algebras

A simplicial Lie algebra (cf. [1], [7] and [11]) $\mathbf{L}$ consists of a family of Lie algebras L_n together with face and degeneracy maps $d_i^n : L_n \to L_{n-1}$, $0 \leq i \leq n$ $(n \neq 0)$ and $s_i^n : L_n \to L_{n+1}$, $0 \leq i \leq n$ satisfying the usual simplicial identities;

$$
\begin{array}{llll}
1. & d_i^{n-1} d_j^n & = & d_{j-1}^{n-1} d_i^n, \quad (0 \leq i < j \leq n), \\
2. & s_i^{n+1} s_j^n & = & s_{j+1}^{n+1} s_i^n, \quad (0 \leq i \leq j \leq n), \\
3. & d_i^{n+1} s_j^n & = & s_{j-1}^{n-1} d_i^n, \quad (0 \leq i < j \leq n), \\
4. & d_i^{n+1} s_j^n & = & id, \quad\quad\ (i = j \text{ or } i = j+1), \\
5. & d_i^{n+1} s_j^n & = & s_j^{n-1} d_{i-1}^n \quad (0 \leq j < i - 1 \leq n).
\end{array}
$$

In fact it can be completely described as a functor $\mathbf{L} : \Delta^{op} \to \mathbf{LieAlg}$ where Δ is the category of finite ordinals. We obtain for each $k \geq 0$ a subcategory $\Delta_{\leq k}$ determined by the objects $[j]$ of Δ with $j \leq k$. A $k-$truncated simplicial Lie algebra is a functor from $\Delta_{\leq k}^{op}$ to $\mathbf{LieAlg}$. We denote the category of $k-$truncated simplicial Lie algebra by $\mathbf{Tr}_k\mathbf{SimpLAlg}$. A *reduced simplicial Lie algebra* is a simplicial Lie algebra whose last component is trivial.

The Moore Complex

Given a simplicial Lie algebra $\mathbf{L}$, the Moore complex $(\mathbf{NL}, \partial)$ of $\mathbf{L}$, is the chain complex defined by

$$
NL_n = \bigcap_{i=0}^{n-1} \mathrm{Ker} d_i^n,
$$

with $\partial_n : NL_n \to NL_{n-1}$ induced from d_n^n by restriction. We say that the Moore complex $\mathbf{NL}$ of a simplicial Lie algebra is of length k if $NE_n = 0$ for all $n \geq k+1$, so that a Moore complex of length k is also of length l for $l \geq k$.

Hypercrossed Complex Pairings

The following terminology and notation can be found in [5], [17] for group case, [2] for commutative algebra case and [1], [11] for Lie algebra case.

For the ordered set $[n] = \{0 < 1 < ... < n\}$, let $\alpha_i^n : [n+1] \to [n]$ be the increasing surjective map given by;

$$
\alpha_i^n(j) = \begin{cases} j & \text{if } j \leq i, \\ j-1 & \text{if } j > i. \end{cases}
$$

Let $S(n, n-r)$ be the set of all monotone increasing surjective maps from $[n]$ to $[n-r]$. This can be generated from the various α_i^n by composition. The composition of these generating maps is subject to the following rule: $\alpha_j \alpha_i = \alpha_{i-1} \alpha_j$, $j < i$. This implies that every element $\alpha \in S(n, n-r)$ has a unique expression as $\alpha = \alpha_{i_1} \circ \alpha_{i_2} \circ ... \circ \alpha_{i_r}$ with $0 \leq i_1 < i_2 < ... < i_r \leq n-1$, where the indices i_k are the elements of $[n]$ such that $\{i_1, ..., i_r\} = \{i : \alpha(i) = \alpha(i+1)\}$. We thus can identify $S(n, n-r)$ with the set $\{(i_r, ..., i_1) : 0 \leq i_1 < i_2 < ... < i_r \leq n-1\}$. In particular, the single element of $S(n, n)$, defined by the identity map on $[n]$, corresponds to the empty 0-tuple ( ) denoted by $\emptyset_n$. Similarly, the only element of $S(n, 0)$ is $(n-1, n-2, ..., 0)$. For all $n \geq 0$, let

$$S(n) = \bigcup_{0 \leq r \leq n} S(n, n-r).$$

We say that $\alpha = (i_r, ...i_1) < \beta = (j_s, ..., j_1)$ in $S(n)$ if $i_1 = j_1, ..., i_k = j_k$ but $i_{k+1} > j_{k+1}, (k \geq 0)$ or if $i_1 = j_1, ..., i_r = j_r$ and $r < s$. This makes $S(n)$ an ordered set.

We recall briefly from Akça and Arvasi (cf. [1]) the construction of a family of k-linear morphisms. We define a set $P(n)$ consisting of pairs of elements (α, β) from $S(n)$ with $\alpha \cap \beta = \emptyset$ and $\beta < \alpha$ where $\alpha = (i_r, ...i_1), \beta = (j_s, ...j_1) \in S(n)$. The k-linear morphisms that we will need,

$$\{C_{\alpha,\beta} : NL_{n-\#\alpha} \times NL_{n-\#\beta} \to NL_n : (\alpha, \beta) \in P(n), n \geq 0\}$$

are given as composites:

$$
\begin{aligned}
C_{\alpha,\beta}(x_\alpha \otimes y_\beta) &= p[-, -](s_\alpha \times s_\beta)(x_\alpha, y_\beta) \\
&= p([s_\alpha(x_\alpha), s_\beta(y_\beta)]) \\
&= (1 - s_{n-1}d_{n-1})...(1 - s_0 d_0)([s_\alpha(x_\alpha), s_\beta(y_\beta)]),
\end{aligned}
$$

where

$$s_\alpha = s_{i_r}...s_{i_1} : NL_{n-\#\alpha} \to L_n, s_\beta = s_{j_s}...s_{j_1} : NL_{n-\#\beta} \to L_n,$$

$p : L_n \to NL_n$ is defined by composite projections $p = p_{n-1}...p_0$ with $p_j = 1 - s_j d_j$ for $j = 0, 1, ..., n-1$ and $[-, -] : L_n \times L_n \to L_n$ denotes the Lie bracket.

From [1], we will now consider that the ideal I_n in L_n such that generated by all elements of the form

$$C_{\alpha,\beta}(x_\alpha, y_\beta),$$

where $x_\alpha \in NL_{n-\#\alpha}$ and $y_\beta \in NL_{n-\#\beta}$ and for all $(\alpha, \beta) \in P(n)$.

Consider $C_{\alpha,\beta}(x_\alpha, y_\beta)$ and $C_{\beta,\alpha}(y_\beta, x_\alpha)$, here one uses $[s_\alpha(x_\alpha), s_\beta(y_\beta)]$, the other

$$[s_\alpha(x_\alpha), s_\beta(y_\beta)] = -[s_\beta(y_\beta), s_\alpha(x_\alpha)],$$

so the changing α and β gives the only minus sign.

Proposition 3.1 ([1]) *Let* **L** *be a simplicial Lie algebra and* $n > 0$, *and* D_n *the ideal in* L_n *generated by degenerate elements. We suppose* $L_n = D_n$, *and let* I_n *be the ideal generated by elements of the form*

$$C_{\alpha,\beta}(x_\alpha, y_\beta) \qquad \text{with } (\alpha, \beta) \in P(n),$$

where $x_\alpha \in NL_{n-\#\alpha}, y_\beta \in NL_{n-\#\beta}$ *with* $1 \le r, s \le n$. *Then,* $NL_n = I_n$ *and as a corollary*

$$\partial_n(NL_n) = \partial_n(I_n).$$

According to above proposition for $n = 2, 3$, Akça and Arvasi have showed the image of I_n by ∂_n what it looks like.

Supposing $D_2 = L_2$, take $\beta = (1), \alpha = (0)$ and $x, y \in NL_1 = \ker d_0$. The ideal I_2 is generated by elements of the form

$$C_{(1)(0)}(x \otimes y) = [s_1 x, s_0 y - s_1 y]$$

and these give the generator elements of the ideal I_2. Then the image of I_2 by ∂_2 is

$$d_2[C_{(1)(0)}(x, y)] = [x, s_0 d_1 y - y]$$

where $x \in \ker d_0$ and $y - s_0 d_1 y \in \ker d_1$. Therefore the image of I_2 by ∂_2 is $[\ker d_0, \ker d_1]$. For $n = 3$, the linear morphisms are

$$
\begin{array}{ccc}
C_{(1,0),(2)}, & C_{(2,0),(1)}, & C_{(2,1),(0)} \\
C_{(2),(0)}, & C_{(2),(1)}, & C_{(1),(0)}.
\end{array}
$$

Then the ideal I_3 is generated by the elements: for $x \in NL_1$ and $y \in NL_2$

$$
\begin{array}{rcl}
C_{(1,0),(2)}(x, y) & = & [s_1 s_0 x - s_2 s_0 x, s_2 y], \\
C_{(2,0),(1)}(x, y) & = & [s_2 s_0 x - s_2 s_1 x, s_1 y - s_2 y], \\
C_{(2,0),(1)}(x, y) & = & [s_2 s_1 x, s_0 y - s_1 y + s_2 y];
\end{array}
$$

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and for $x, y \in NL_2$

$$
\begin{aligned}
C_{(1),(0)}(x, y) &= [s_1 x, s_0 y - s_1 y] + [s_2 x, s_2 y], \\
C_{(2),(0)}(x, y) &= [s_2 x, s_0 y], \\
C_{(1),(0)}(x, y) &= [s_2 x, s_1 y - s_2 y].
\end{aligned}
$$

For the image of these elements see to [1].

Then, we can give the following proposition. In the proof of this proposition, we will use the image of the $C_{\alpha,\beta}$ pairings in the Moore complex of a simplicial Lie algebra.

Proposition 3.2 *There is a functor from reduced simplicial Lie algebras to braided crossed modules of Lie algebras.*

Proof. Let **L** be a reduced simplicial Lie algebra with Moore complex **NL**. We construct a braided crossed module of Lie algebras (M, N, ∂). Let $N = NL_1$ and $M = NL_2/\partial_3(NL_3 \cap D_3)$. The two actions of $n \in N$ on $m \in M$ are:

1. the actions $m \cdot \partial_1 n$ and $\partial_1 n \cdot m$ correspond to $[m, s_0 n]$ and $[s_0 n, m]$ respectively.

2. the actions $n \cdot m$ and $m \cdot n$ correspond to $[s_1 n, m]$ and $[m, s_1 n]$ respectively.

It is plainly that the morphism

$$
\partial_2 = d_2 : M \to N
$$

is a crossed module of Lie algebras. The braiding map on this crossed module can be defined by

$$
\begin{aligned}
\{-, -\} : \quad N \times N &\longrightarrow M = NL_2/\partial_3(NL_3 \cap D_3) \\
(x, y) &\longmapsto \{x, y\} = \overline{[s_1 x, s_1 y - s_0 y] - [s_1 y - s_0 y, s_1 x]},
\end{aligned}
$$

where the right hand side denotes a coset in $NL_2/\partial_3(NL_3 \cap D_3)$ represented by an element in NL_2.

Now, we show that all axioms of braided crossed module are verified. In the following calculations, we display the elements omitting the overlines, for the sake of simplification.

B1. For $x, y \in N$, we have

$$\begin{aligned}
\partial_2\{x, y\} &= d_2([s_1x, s_1y - s_0y] - [s_1y - s_0y, s_1x]) \\
&= [x, y - s_0d_1y] - [y - s_0d_1y, x] \\
&= [x, y] - [y, x] + [s_0d_1y, x] - [x, s_0d_1y] \\
&= [x, y] - [y, x] + \partial_1y \cdot x - x \cdot \partial_1y \quad \text{(by the action)} \\
&= [x, y] - [y, x] \quad \text{(by reduced condition).}
\end{aligned}$$

B2. From $\partial_3(C_{(2,1),(0)}(y, a)) = [s_1y, s_0d_2a - s_1d_2a] + [s_1y, a] \in \partial_3(NL_3 \cap D_3)$, we have

$$[s_1y, s_1d_2a - s_0d_2a] \equiv [s_1y, a] \, \text{mod}\partial_3(NL_3 \cap D_3),$$

and from $\partial_3(C_{(0),(2,1)}(a, y)) = [s_0d_2a - s_1d_2a, s_1y] + [a, s_1y] \in \partial_3(NL_3 \cap D_3)$, we have

$$[s_1d_2a - s_0d_2a, s_1y] \equiv [a, s_1y] \, \text{mod}\partial_3(NL_3 \cap D_3).$$

We thus have from $\partial_3(C_{(2,1),(0)}(y, a) - C_{(0),(2,1)}(a, y))$,

$$\begin{aligned}
\{y, \partial_2a\} &= [s_1y, s_1d_2a - s_0d_2a] - [s_1d_2a - s_0d_2a, s_1y] \\
&\equiv [s_1y, a] - [a, s_1y] \, \text{mod}\partial_3(NL_3 \cap D_3) \\
&= y \cdot a - a \cdot y \quad \text{(by the action)}
\end{aligned}$$

for $y \in N$ and $a \in M$.

B3. From $\partial_3(C_{(1,0),(2)}(x, b)) = [s_0y - s_1y, s_1d_2x] - [s_0y - s_1y, x] \in \partial_3(NL_3 \cap D_3)$, we have

$$[s_0y - s_1y, s_1d_2x] \equiv [s_0y - s_1y, x] \, \text{mod}\partial_3(NL_3 \cap D_3)$$

and from $\partial_3(C_{(2),(1,0)}(b, x)) = [s_1d_2x, s_0y - s_1y] - [x, s_0y - s_1y] \in \partial_3(NL_3 \cap D_3)$, we have

$$[s_1d_2x, s_0y - s_1y] \equiv [x, s_0y - s_1y] \, \text{mod}\partial_3(NL_3 \cap D_3).$$

We thus obtain from $\partial_3(C_{(1,0),(2)}(x, b) - C_{(2),(1,0)}(b, x))$,

$$\begin{aligned}
\{\partial_2x, b\} &= [s_1d_2b, s_1x - s_0x] - [s_1x - s_0x, s_1d_2b] \\
&\equiv [s_1x, b] - [b, s_1x] + [s_1s_0d_1x, b] - [b, s_1s_0d_1x] \, \text{mod}\partial_3(NL_3 \cap D_3) \\
&= x \cdot b - b \cdot x + \partial_1x \cdot b - b \cdot \partial_1x \quad \text{(by the action)} \\
&= x \cdot b - b \cdot x \quad \text{(by reduced condition)}
\end{aligned}$$

for $x \in M$ and $b \in N$.

B4. From $\partial_3(C_{(1),(0)}(a,b)) = [s_1 d_2 a, s_0 d_2 b] - [s_1 d_2 a, s_1 d_2 b] + [a,b] \in \partial_3(NL_3 \cap D_3)$, we have

$$[s_1 d_2 a, s_1 d_2 b - s_0 d_2 b] \equiv [a,b] \, \mathrm{mod} \partial_3(NL_3 \cap D_3),$$

and from $\partial_3(C_{(0),(1)}(b,a)) = [s_1 d_2 b, s_1 d_2 a] - [s_0 d_2 b, s_1 d_2 a] + [b,a] \in \partial_3(NL_3 \cap D_3)$, we have

$$[s_1 d_2 b - s_0 d_2 b, s_1 d_2 a] \equiv [b,a] \, \mathrm{mod} \partial_3(NL_3 \cap D_3).$$

Thus from $\partial_3(C_{(1),(0)}(a,b) - C_{(0),(1)}(b,a))$ we have

$$\begin{aligned} \{\partial_2 a, \partial_2 b\} &= [s_1 d_2 a, s_1 d_2 b - s_0 d_2 b] - [s_1 d_2 b - s_0 d_2 b, s_1 d_2 a] \\ &= [a,b] - [b,a] \, \mathrm{mod} \partial_3(NL_3 \cap D_3). \end{aligned}$$

Other axioms can be shown similarly.

If the Moore complex of the simplicial Lie algebra $\mathbf{L}$ is length 2, we can write $\partial_3(NL_3 \cap D_3) = 0$ due to [1], in this case, we would have defined a functor from reduced simplicial Lie algebras with Moore complex of length 2 to braided crossed modules of Lie algebras. Consequently we can define a functor

$$\Gamma : \mathbf{ReSimpLAlg} \rightarrow \mathbf{BLXM}.$$

$\square$

Remark The situations in this paper can be summarized in the following diagram of unbroken arrows:

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