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Higher Order Generalization of Positive Linear Operators Defined by a Class of Borel Measures

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Abstract

In the present paper, we introduce a sequence of linear operators, which is a higher order generalization of positive linear operators defined by a class of Borel measures studied in [2]. Then, using the concept of $A-$statistical convergence we obtain some approximation results which are stronger than the aspects of the classical approximation theory.

Key Words: Statistical convergence, A-statistical convergence, positive linear operators, regular matrices, the elements of the Lipschitz class, Korovkin-type approximation theorem.

1. Introduction

Let I be an arbitrary interval of the real line, and let $C(I)$ denote the linear space of all real-valued continuous functions on I. Assume that g is a non-negative increasing function on $[0,\infty)$ with $g(0)=1$. If I is an unbounded interval, then we consider the following function space

$$C_g(I)=\left\{ f\in C(I):\lim_{|y|\to\infty;\ (y\in I)}\frac{|f(y)|}{(g(|y|))^c}=0 \text{ for any } c>0\right\}, \tag{1}$$

which was examined in [2], [3]. Here, we should remark that if $I=[a,+\infty)$ (or $I=(a,+\infty)$), then the item "$|y|\to\infty;\ (y\in I)$" in the definition (1) reduces to "$y\to+\infty$"; however, if $I=(-\infty,a]$ (or $I=(-\infty,a)$), then we have "$y\to-\infty$". On the other hand, if I is an bounded interval, then we will use the space $C(I)$ instead of $C_g(I)$.

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Now, for each fixed $x \in I$, let $\{\mu_{n,x} : n \in \mathbb{N}\}$ be a collection of measures defined on $(I, \mathcal{B})$, where $\mathcal{B}$ is the sigma field of Borel measurable subsets of I. Assume that, for any $\delta > 0$, the condition

$$\sup_{n\in\mathbb{N}} \int\limits_{I \setminus I_\delta} g(|y|)d\mu_{n,x}(y) < \infty \tag{2}$$

holds, where $I_\delta := [x-\delta, x+\delta] \cap I$. In the condition (2), the boundedness is pointwise with respect to x; that is, it is bounded for each fixed $x \in I$. With this terminology, in [2], some approximation properties of the following positive linear operators defined on $C_g(I)$ were investigated:

$$L_n(f;x) = \int\limits_I f(y)d\mu_{n,x}(y) \ , \ n \in \mathbb{N} \text{ and } f \in C_g(I). \tag{3}$$

Define the space $C_g^{[r]}(I)$ by

$$C_g^{[r]}(I) = \left\{ f : f^{(r)} \in C_g(I) \right\}, \ \ (r = 0, 1, 2, ...).$$

If $r = 0$, then observe that $C_g^{[0]}(I) = C_g(I)$. We now consider the $r-$th order generalization of the operators L_n defined by (3) as follows

$$L_n^{[r]}(f;x) = \sum_{k=0}^{r} \int\limits_I f^{(k)}(y) \frac{(x-y)^k}{k!} d\mu_{n,x}(y), \tag{4}$$

where $f \in C_g^{[r]}(I)$, $(r = 0, 1, 2, ...)$, $n \in \mathbb{N}$, and also the function g satisfy the condition (2). We note that this kind of generalization was also considered in [11]. It is easy to see that if $r = 0$, then we have

$$L_n^{[0]}(f;x) = L_n(f;x).$$

The main goal of the present paper is to investigate various approximation properties of the linear operators $L_n^{[r]}$ defined by (4) with the help of the concept of A-statistical convergence. Recently, it has been shown that regular (non-matrix) summability transformations are also quite effective on the approximation of positive linear operators (see [2], [3], [4], [5], [9]). Especially, using the concept of the A-statistical convergence, where

A is a non-negative regular matrix, instead of the ordinary convergence in the approximation theory gives us many advantages, since A statistical convergence is stronger than the usual convergence.

Before proceeding further, we recall the concept of A-statistical convergence.

Let $A := (a_{jn})$, $j, n \in \mathbb{N}$, be a non-negative regular summability, i.e. $\lim Ax = L$ whenever $\lim x = L$, where $Ax := ((Ax)_j)$ is called an A-transform of $x := (x_n)$ and is given by

$$(Ax)_j := \sum_{n=1}^{\infty} a_{jn} x_n,$$

provided that the series convergence for each $j \in \mathbb{N}$ (see [10]). Then a sequence $x := (x_n)$ is called A-statistical convergent to a number L if, for every $\varepsilon > 0$,

$$\lim_j \sum_{n:|x_n - L| \geq \varepsilon} a_{jn} = 0.$$

We denote this limit by $st_A - \lim x = L$ [7] (see also [12], [14]). If we take $A = C_1$, the Cesàro matrix of order one, then C_1-statistical convergence is equivalent to statistical convergence [6], [8]. Also replacing the matrix A by the identity matrix, A-statistical convergence coincides with the ordinary convergence. Kolk [12] proved that A-statistical convergence is stronger than ordinary convergence in the case of which $\lim_j \max_n |a_{jn}| = 0$.

2. A-Statistical Approximation Properties

In this section using A-statistical convergence we investigate some approximation properties of the operators $L_n^{[r]}$ defined by (4).

We note that a function $f \in C(I)$ belongs to $Lip_M(\alpha)$, $0 < \alpha \leq 1$, provided

$$|f(y) - f(x)| \leq M\,|y - x|^{\alpha} \quad (x, y \in I \text{ and } M > 0). \tag{5}$$

Then we obtain the following result.

Theorem 2.1 *Let I be an arbitrary interval of the real line, and let r be a non-negative integer. Assume that*

$$\int_I d\mu_{n,x}(y) = 1 \quad \textit{(for each } x \in I \textit{ and } n \in \mathbb{N}\textit{).} \tag{6}$$

Then for all $f \in C_g^{[r]}(I)$ *such that* $f^{(r)} \in Lip_M(\alpha)$, $0 < \alpha \leq 1$, *and for each* $x \in I$, *we have*

$$\left| L_n^{[r]}(f;x) - f(x) \right| \leq C L_n(|x-y|^{\alpha+r};x)$$

where

$$C = \frac{M\alpha}{\alpha+r} \frac{B(\alpha,r)}{(r-1)!}, \tag{7}$$

and $B(\alpha,r)$ *is the beta function.*

Proof. By (4) and (6), we get

$$f(x) - L_n^{[r]}(f;x) = \int_I \left\{ f(x) - \sum_{k=0}^{r} f^{(k)}(y) \frac{(x-y)^k}{k!} \right\} d\mu_{n,x}(y). \tag{8}$$

Applying the Taylor's formula (see [11]) we may write that

$$f(x) - \sum_{k=0}^{r} f^{(k)}(y) \frac{(x-y)^k}{k!} = \frac{(x-y)^r}{(r-1)!} \int_0^1 (1-t)^{r-1} \left[f^{(r)}(y+t(x-y)) - f^{(r)}(y) \right] dt. \tag{9}$$

Since $f^{(r)} \in Lip_M(\alpha)$, we get from (5) that

$$\left| f^{(r)}(y+t(x-y)) - f^{(r)}(y) \right| \leq M t^{\alpha} |x-y|^{\alpha}. \tag{10}$$

Considering (10) in (9), and using the beta integral, we conclude that

$$\left| f(x) - \sum_{k=0}^{r} f^{(k)}(y) \frac{(x-y)^k}{k!} \right| \leq |x-y|^{\alpha+r} \frac{M\alpha}{\alpha+r} \frac{B(\alpha,r)}{(r-1)!}. \tag{11}$$

So combining (11) with (8), we have

$$\begin{aligned} \left| f(x) - L_n^{[r]}(f;x) \right| &\leq \frac{M\alpha}{\alpha+r} \frac{B(\alpha,r)}{(r-1)!} \int_I |x-y|^{\alpha+r} d\mu_{n,x}(y) \\ &= \frac{M\alpha}{\alpha+r} \frac{B(\alpha,r)}{(r-1)!} L_n(|x-y|^{\alpha+r};x), \end{aligned}$$

which gives the desired result. □

Theorem 2.2 *Let $A = (a_{jn})$ be a non-negative regular summability matrix, and let I be an arbitrary interval of the real line. Let $0 < \alpha \leq 1$ and let r be a non-negative integer. Assume that the condition (6) is satisfied. Assume further that $g : [0, \infty) \to \mathbb{R}$, $g(y) = e^y$ and $h_x : I \to \mathbb{R}$, $h_x(y) = |x - y|^{\alpha+r}$ for each fixed $x \in I$. If the condition*

$$st_A - \lim_n L_n(h_x, x) = 0 \tag{12}$$

holds, then for all $f \in C_g^{[r]}(I)$ such that $f^{(r)} \in Lip_M(\alpha)$, we have

$$st_A - \lim_n \left| L_n^{[r]}(f; x) - f(x) \right| = 0.$$

Proof. Let $x \in I$ be fixed. By the definitions of the functions g and h_x, observe that $h_x \in C_g(I)$. Now, for a given $\varepsilon > 0$, define the following sets:

$$U := \left\{ n \in \mathbb{N} : \left| L_n^{[r]}(f; x) - f(x) \right| \geq \varepsilon \right\}$$

and

$$V := \left\{ n \in \mathbb{N} : L_n(h_x; x) \geq \frac{\varepsilon}{C} \right\},$$

where the constant C is given by (7). So it follows from Theorem 2.1 that $U \subseteq V$. Therefore, we get, for all $j \in \mathbb{N}$, that

$$\sum_{n \in U} a_{jn} \leq \sum_{n \in V} a_{jn}. \tag{13}$$

Note the condition (12) implies $\lim_j \sum_{n \in V} a_{jn} = 0$. So, we conclude from (13) that $\lim_j \sum_{n \in U} a_{jn} = 0$, whence the result. □

If we use the test functions $e_i(y) = y^i$, $(i = 0, 1, 2)$, instead of (12) in Theorem 2.1, then we have the following approximation result via $A-$statistical convergence.

Theorem 2.3 *Under the conditions of Theorem 2.2, if*

$$st_A - \lim_n |L_n(e_i, x) - e_i(x)| = 0, \quad (i = 0, 1, 2), \tag{14}$$

then for all $f \in C_g^{[r]}(I)$ such that $f^{(r)} \in Lip_M(\alpha)$, we have

$$st_A - \lim_n \left| L_n^{[r]}(f; x) - f(x) \right| = 0.$$

Proof. We first note that, by the definition of g, the test functions e_i, $(i = 0, 1, 2)$ belong to $C_g(I)$. So, by Theorem 1 in [2], the condition (14) yields that for all $h \in C_g(I)$

$$st_A - \lim_n |L_n(h, x) - h(x)| = 0. \tag{15}$$

In particular, take $h := h_x$, which is defined in Theorem 2.2. Since $h_x(x) = 0$, it follows from (15) that

$$st_A - \lim_n |L_n(h_x, x)| = 0,$$

which gives (12). Therefore the proof follows from Theorem 2.2. □

Corollary 2.4 *If I is closed and bounded interval of the real line, say $I = [a, b]$, and also*

$$st_A - \lim_n \|L_n(e_i, \cdot) - e_i\|_{C[a,b]} = 0, \quad (i = 0, 1, 2),$$

then for all $f \in C^{[r]}[a, b]$ such that $f^{(r)} \in Lip_M(\alpha)$ we have

$$st_A - \lim_n \left\| L_n^{[r]}(f, \cdot) - f \right\|_{C[a,b]} = 0,$$

where $\|\cdot\|_{C[a,b]}$ denotes the usual sup norm on $[a, b]$.

Special Cases

(a) Choosing $r = 0$ in Theorem 2.3, we get Theorem 1 in [2].

(b) The choice of $r = 0$ in Corollary 2.4 reduces to Corollary 2 in [2].

(c) If we replace the matrix A by the Cesàro matrix of order one and choose $r = 0$ in Corollary 2.4, then we get the statistical approximation theorem introduced by Gadjiev and Orhan (see Theorem 1 in [9]).

(d) If we replace the matrix A by the identity matrix and also choose $r = 0$ in Corollary 2.4, then we get the classical Korovkin-type approximation theorem (see, for instance, [1], [13]).

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