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The Compact Metric Space of the Lattice of Varieties and *-Varieties of C*-Algebras

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Abstract

Variety of Banach algebras is a non-empty class of Banach algebras in which there exist a family of laws such that all of its members satisfy all of the laws. In this paper, we have used merely mathematical items such as Banach algebras and varieties including Banach algebras in order to change the space of all varieties of Banach algebras into a compact metric space. We prove some theorems in the metric space of zero at infinity varieties, define the *-varieties of *-algebra and prove many theorems about *-varieties of C*-algebras.

Key Words: Variety of Banach algebras, H-variety, Q-algebras, IR-algebras.

1. Introduction

Throughout this paper, by a polynomial we mean a polynomial in several non-commuting variables without constant term. For each Banach algebra A and polynomial p, we define

$$\|p\|_A = \sup\{\|p(x_1, ..., x_n)\| : x_i \in A, \|x_i\| \leq 1\}.$$

By a law we mean a formal expression $\|p\| \leq K$, where $K \in R$ and p is a polynomial. We say that A satisfies the above law, if $\|p\|_A \leq K$. $\|p\| \leq K$ is homogeneous, if p is a homogeneous polynomial.

Let v be a variety and let $\{L_\alpha\}_{\alpha\in I}$ be the families of laws which determine v. We define

$$|p|_v = \inf\{K : \exists\alpha \in I; (\|p\| \leq K) \in L_\alpha\}$$

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where p is a polynomial. The family $\{|p|_v\}_p$ is a family of laws which determine v. By means of the mapping $p \to |p|_v$, we can compare the elements of the lattice of all varieties. An H-variety is one generated by a family of homogeneous laws. If A is a Banach algebra, the smallest variety containing A is denoted by $v(A)$, called the variety generated by A. Each variety v is singly generated, i.e., there exists a Banach algebra A such that $v = v(A)$. The class of all varieties with inclusion is a complete lattice. In this paper, we regard two Banach algebras identical if they are isometric and isomorphic.

2. The Lattice of Varieties

In this section, we shall introduce some main results about varieties, for proofs of which the reader can refer to [3], [4] and [6].

The following theorem shows that the variety of all IQ-algebras and all IR-algebras is generated by $\mathbb{C}$(complex numbers) and $B(H)$, respectively.

Theorem 2.1 ([3]). *The variety of all IQ-algebras and all IR-algebras is generated by C(complex numbers) and $B(H)$, respectively, where H is the separable infinite dimensional Hilbert space.*

The following theorem explains why we can let a variety be defined by condition on all polynomials.

Theorem 2.2 ([4], $Theorem$2.3). *A non-empty class of Banach algebras is a variety if and only if it is closed under closed subalgebras, quotient algebras, products(direct sum) and images under isometric isomorphisms.*

Let A be a Banach algebras. Then $S(A), Q(A)$ and $P(A)$ denote, respectively, the class of all Banach algebras isometrically isomorphic to closed subalgebras of A, quotient algebras of A and direct sums of copies of A.

To show some useful relationships in order to obtain the characteristics of the varieties of Banach algebras we have the following theorem.

Theorem 2.3 ([6], $Theorem$2.1). *Let A be a Banach algebra. Then*

$$v(A) = QSP(A).$$

Each variety is determined by a family of laws. But among such families one is particular noteworthy; namely, the family of laws with minimal right-hand sides K. The function giving these right-hand sides is as the following.

For each variety v and polynomial p,

$$|p|_v = \sup\{\|p\|_A : A \in v\}.$$

The following theorem shows that this supremum is always obtained.

Theorem 2.4 ([6], $Theorem$2.4). *For each variety v, there is an $A \in v$ such that for all polynomials p*

$$|p|_v = \|p\|_A.$$

An easy corollary of the above theorem is as follows: Let v_1 and v_2 be two varieties. Then $v_1 \subseteq v_2$ if and only if, for each polynomial p

$$|p|_{v_1} \leq |p|_{v_2}$$

We note that, partially ordered by inclusion principle, the class of all varieties is a complete lattice.

Lemma 2.5 ([6], $Lemma$2.7). *Let $\{A_\alpha\}_{\alpha \in I}$ be a family of Banach algebras and $A = \oplus_{\alpha \in I} A_\alpha$. Then the supremum of $\{v(A_\alpha)\}_{\alpha \in I}$ in the lattice of varieties is $v(A)$.*

We shall denote the variety of all Banach algebras by **1**. Let N_n(for each $n \in N$, $n \neq 1$) be the variety determined by the law

$$\|X_1 ... X_n\| = 0$$

The following theorem explains the maximum and minimum of the class of all varieties.

Theorem 2.6 ([6], $Theorem$3.3). *Let L be the class of all varieties, then N_2 is the minimum of L, and $L \backslash \{N_2, \mathbf{1}\}$, has no maximum and minimum.*

The following theorem shows that there is an uncountable chain between any variety and H- variety.

Theorem 2.7 ([6], $Theorem$4.2). *Let v_1 be a variety, v_2 an H- variety, and $v_1 \subseteq v_2$. If there exists a homogeneous polynomial p_0 such that $|p_0|_{v_1} < |p_0|_{v_2}$, there will be an uncountable chain between v_1 and v_2.*

An immediate corollary is as follows. Between any two H-varieties there is an uncountable chain of H-varieties which can be expressed in the following theorem.

Theorem 2.8 ([6], $Theorem 4.4$). *Let A and B be two Banach algebras obeying sets of algebraic identities $\sum A$ and $\sum B$, respectively, such that $\sum A \setminus \sum B \neq \emptyset$ and $\sum B \setminus \sum A \neq \emptyset$, there is a pair of uncountable chains between $inf\{H(A), H(B)\}$ and $sup\{H(A), H(B)\}$.*

The following theorem shows existence of an uncountable anti-chain of varieties of Banach algebras.

Theorem 2.9 ([6], $Theorem 4.5$). *There exists an uncountable family of varieties $\{v_\alpha\}_{\alpha \in I}$ such that if $\alpha, \beta \in I$ and $\alpha \neq \beta$, v_α is not a subset of v_β and v_β is not a subset of v_α.*

So it can be concluded that the lattice of all varieties is long and wide.

We can show that there is no H-variety between the varieties, of IQ-algebras and IR-algebras. The following theorem can be considered, to show this fact, while it explains that such a restriction to homogeneous polynomials is not possible in the characterization of the IQ-algebras and IR-algebras.

Theorem 2.10 ([6], $Theorem 4.7$). *Between the varieties of all IQ-algebras and IR-algebras there is no H-variety.*

3. The Metric Space of Varieties

Now we shall show that the lattice of all varieties is a metric space. It is known that each variety can be determined just by non-homogeneous polynomials, but it is not possible to determine all of the varieties just by homogeneous polynomials. To start, we need to have definitions for unit disk of the set of all polynomials, the set of all homogeneous polynomials, the set of all non homogeneous polynomials and zero at infinity variety.

Definition 3.1 *If we consider L the lattice of all varieties, and L_H the lattice of all H-varieties. P the set of all polynomials, P_H the set of all homogeneous polynomials. P_{NH} the set of all non homogeneous polynomials, we can define*

$$P_1 = \{p \in P : |p|_1 < 1\}$$

$$P_{H_1} = \{p \in P_H : |p|_1 < 1\}$$

$$P_{NH_1} = \{p \in P_{NH} : |p|_1 < 1\}.$$

Let $v \in L$. We define $\Phi_v : P_1 \to C$ as

$$\Phi_v(p) = |p|_v.$$

It is easy to show that the mapping $v \to \Phi_v$ from L into $l^\infty(P_1, C)$ is one to one. Hence $l^\infty(P_1, C)$ induces the following metric to L:

$$d_L(v, w) = d(\Phi_v, \Phi_w) = \|\Phi_v - \Phi_w\|$$

$$= \ sup_{p \in P_1} |\Phi_v(p) - \Phi_w(p)| = \ sup_{p \in P_1} ||p|_v - |p|_w|.$$

Therefore (L, d_L) is a metric space.

Similarly (L_H, d_H) with

$$d_H(v, w) = \ sup_{p \in P_{H_1}} ||p|_v - |p|_w|,$$

and (L, d_{NH}) with

$$d_{NH}(v, w) = \ sup_{p \in P_{H_1}} ||p|_v - |p|_w|$$

are metric spaces.

Definition 3.2 *Let $0 \leq x \leq 1$ and $v \neq N_2$ be a variety. v_x can be defined as the variety determined by the laws*

$$\|p\| \leq x^{i-1} |p|_v,$$

where p is a homogeneous polynomial of degree $i > 1$.

Definition 3.3 *We say that a variety v is zero at infinity if for each $\varepsilon > 0$ there exists $N > 0$ such that for all $p \in P_{H_1}$, if deg(p) $> N$ then $|p|_v < \varepsilon$.*

Now, we prove the main theorem.

4. Compact Metric Space of Varieties

Theorem 4.1 *The set of all varieties is a complete metric space.*

Proof. Let $v \in L$. We define $\varphi_v : P \to \mathbb{R}^+$ as

$$\varphi_v(p) = |p|_v.$$

The mapping $\psi : v \to \varphi_v$ from L into $L^\infty(P_1)$ is one to one. Hence $L^\infty(P_1)$ induces the following metric to L:

$$d_L(v, w) = d(\varphi_v, \varphi_w) = \|\varphi_v - \varphi_w\|_\infty$$

ψ is well-defined. If $v = w$, then for all polynomials $p \in P$, we have $|v|_v = |p|_w$. The mapping $v \to \varphi_v$ is continuous.

Let $\varphi = \{\varphi_v | v \in L\}$. We show that (φ, d) is a closed in $L^\infty(P_1)$ metric space.

Let $\{v_n\}_{n=1}^\infty$ be a sequence of varieties and $\{\varphi_{v_n}\}_{n=1}^\infty$ be a Cauchy sequence in φ metric subspace.

Let $\varphi_{v_n} \overset{\|\ \|_\infty}{\longrightarrow} f$, where $f \in L^\infty(P_1)$. We show that there exists $\varphi_v \in \varphi$, such that $f = \varphi_v$. Since $\varphi_{v_n} \overset{\|\ \|_\infty}{\longrightarrow} f$. Then we have, for each $\varepsilon > 0$, there exist $N > 0$, for all n, if $n \geq N$, Then $\|\varphi_{v_n} - f\|_\infty < \varepsilon$. Thus $\sup_p|\varphi_{v_n}(p) - f(p)| < \varepsilon$ and $\sup_p||p|_{v_n} - f(p)| < \varepsilon$. Therefore the sequence $\{|p|_{v_n}\}_{n=1}^\infty$ is uniformly convergent. Let $M_n(p) = \sup_{k\geq n}|p|_{v_k}$ and $\{|p|_{v_n}\}_{n=1}^\infty$ be an increasing sequence and $v = \vee_{n=1}^\infty v_n$. For each $\varepsilon > 0$, there exits $N > 0$, such that, $|p|_{v_n} > |p|_v - \varepsilon$ and for all n, if $n \geq N$

$$|p|_v - \varepsilon < |p|_{v_n} \leq M_n(p) \leq |p|_v < |p|_v + \varepsilon.$$

Thus

$$|p|_v - \varepsilon < M_n(p) \leq |p|_v < |p|_v + \varepsilon.$$

Hence $\lim_{n\to\infty} M_n(p) = |p|_v$, but $\lim_{n\to\infty} M_n(p) = f(p)$.

Therefore $f(p) = |p|_v$ and $f(p) = \varphi_v(p)$ for all polynomials $p \in P_1$, giving $f = \varphi_v$.

Let $\{|p|_{v_n}\}_{n=1}^\infty$ be a decreasing sequence and $m_n(p) = \inf_{k\geq n}|p|_{v_k}$. We have, $\lim_{n\to\infty} m_n(p) = f(p)$.

For each $\varepsilon > 0$, there exist $N_1 > 0$, such that, $\inf_n|p|_{v_n} + \varepsilon > |p|_{v_{N_1}}$ and for all n, if $n \geq N_1$,

$$\inf_{k\geq 1}|p|_{v_k} - \varepsilon < \inf_{k\geq 1}|p|_{v_k} < m_n(p) < |p|_{v_{N_1}} < \inf_{k\geq 1}|p|_{v_k} + \varepsilon$$

Therefore, for all polynomials $p \in P_1$, we have $\lim_{n\to\infty} M_n(p) = \inf_{k\geq 1}|p|_{v_k}$. Hence for all polynomial $p \in P_1$, $f(p) = \inf_{k\geq 1}|p|_{v_k}$ and f is a non-negative real-valued function on the set of all polynomials and a family of laws $\{\|p\| \leq f(p)\}_p$ determines a variety. But the variety $\wedge_{k\geq 1}v_k$ is determined by a family of laws $\{\|p\| \leq \inf_{k\geq 1}|p|_{v_k}\}_{p\in P} = \{\|p\| \leq f(p)\}_{p\in P}$

There exist a variety $v', v' = \{A|\|p\|_A \leq f(p)\}_p$ and $|p|_{v'} = f(p),\ f = \varphi_{v'}$. Now, let $\{|p|_{v_n}\}_{n=1}^{\infty}$ be not monotone. Since $\{|p|_{v_n}\}_{n=1}^{\infty}$ is convergent, hence $\{|p|_{v_n}\}_{n=1}^{\infty}$ have a decreasing subsequence or increasing subsequence.

Thus (φ, d) is a closed set in $L^{\infty}(P_1)$ metric space. Since $L^{\infty}(P_1)$ is a complete metric space, (φ, d) is a complete metric subspace and (L, d_L) is a complete metric space. □

Theorem 4.2 *The subspace (L_H°, d_H) of zero at infinity H-varieties is closed in (L, d_L).*

Proof. Let $\{v_n\}_{n=1}^{\infty}$ be a sequence of zero at infinity varieties. Let $v_n \xrightarrow{d} v$. Thus for each $\varepsilon > 0$, there exists $N > 0$, such that for all $n \in \mathbb{N}$, if $n \geq N$, then $\sup_{p\in P_1}||p|_{v_n} - |p|_v| < \varepsilon$.

Since the metric spaces (L, d_L) and (L_{NH}, d_{NH}) are equivalent,

$$\sup_{q\in P_{NH_1}}||q|_{v_n} - |q|_v| < \varepsilon.$$

Let p be a homogeneous polynomial, where $q_i = p\ \ (i = 1, 2, ...)$. Since $\sup_{q\in P_{NH_1}}||q|_{v_n} - |q|_v| < \varepsilon$, for all q, such that $q_i = p$, we have $||q|_{v_n} - |q|_v| < \varepsilon$. Thus $|\inf_{q|q|_{v_n}} - \inf|q|_v| < \varepsilon$ and $||p|_{v_n} - |p|_v| < \varepsilon$, $||p|_{v_N} - |p|_v| < \varepsilon$.

Since v_N is a zero at infinity variety, for each $\varepsilon > 0$ there exists $N_1 > 0$ such that for all polynomials $p \in P_{H_1}$, if $\deg(p) > N_1$, then $|p|_{v_N} < \varepsilon$. Thus $|p|_v < \varepsilon$, and v is a zero at infinity. □

Corollary 4.3 *The set of all zero infinity varieties is a complete metric subspace.*

Definition 4.4 *Let v be zero at infinity variety. We define*

$$[N_2, v] = \{w|N_2 \subseteq w \subseteq v, w \text{ be a variety}\}.$$

It is obvious that, the members of $[N_2, v]$ are zero at infinity varieties.

Theorem 4.5 *Let v be zero at infinity. Then $[N_2, v]$ is a closed set.*

Proof. Let $\{v_n\}_{n=1}^{\infty}$ be a sequence of zero at infinity varieties and for all $n \in \mathbb{N}$, $v_n \in [N_2, v]$, hence $N_2 \subseteq v_n \subseteq v$. If $v' \not\subset v$, then there exists $c \in v'$ such that $c \notin v$. So there exists $p_0 \in P_{H_1}$ such that $\|p_0\|_c > |p_0|_v$, and $|p_0|_{v'} \geq \|p_0\|_c > |p_0|_v \geq |p_0|_{v_n}$.

Since $v_n \xrightarrow{d_H} v'$. Therefore we have, for each $\varepsilon > 0$, there exists $N > 0$ such that for all n, if $n \geq N$, then $\sup_{p \in P_1} ||p|_{v_n} - |p|_{v'}| < \varepsilon$. Thus for all polynomials $p \in P_{H_1}$ $||p|_{v_n} - |p|_{v'}| < \varepsilon$.

Now if $\varepsilon = |p_0|_{v'} - |p_0|_v$, then $-(|p_0|_{v'} - |p_0|_v) < |p_0|_{v_n} - |p_0|_{v'} < |p_0|_{v'} - |p_0|_v$. Thus for all $n \geq N$, $|p_0|_v < |p_0|_{v_n}$. This is a contradiction. □

Corollary 4.6 *Let v be zero at infinity variety. Then $[N_2, v]$ is a complete metric subspace.*

Proof. $[N_2, v] \subseteq L_H^{\circ}$ and $[N_2, v]$ is a closed set. □

Corollary 4.7 *For all $n \in \mathbb{N}$, $n > 2$, the closed intervals $[N_2, v]$ are the complete metric subspaces.*

Theorem 4.8 *The sequence $\{N_n\}_{n=1}^{\infty}$ is not convergent.*

Proof. First we show that $N_n \not\to 1$. For all polynomials $p \in P_1$, we have

$$\sup_{p \in P_1} ||p|_{N_n} - |p|_1| \geq ||p|_{N_n} - |p|_1|$$

if $X_1 X_2 ... X_n$. Then there exists $N > 0$ such that for all n, $n \geq N$

$$d(N_n, 1) \geq ||X_1 X_2 ... X_n|_{N_n} - |X_1 X_2 ... X_n|_1| = |X_1 X_2 ... X_n|_1 = 1.$$

Let $\{N_n\}_{n=1}^{\infty}$ be convergent. Let $N_n \to v$. Then for each $\varepsilon > 0$ there exists $N_1 > 0$, for all n and for all $p \in P_1$, if $n \geq N_1$, $\sup_{p \in P_1} ||p|_{N_n} - |p|_1| < \varepsilon$. For all polynomials $p \in P_1$, $\{|p|_{N_n}\}_{n=1}^{\infty}$ is increasing sequence. Thus $|p|_{N_n} \to |p|_{V_n N_n}$. But $\vee_n N_n = \mathbf{1}$. Therefore $v = \mathbf{1}$. □

Theorem 4.9 *Let $\{v'_n\}_{n=1}^{\infty}, \{v_n\}_{n=1}^{\infty}$ be two sequences of varieties, $v_n \xrightarrow{d} v$ and $v'_n \xrightarrow{d} v'$. Then $v_n \vee v'_n \xrightarrow{d} v \vee v'$, and $v \vee v' = inf_{n\geq 1}(\ sup_{k\geq n} v_k \vee v'_k)$.*

Proof. For all polynomials $p \in P_1$, we have,

$$|p|_{v_n \vee v'_n} = \sup\{|p|_{v_n}, |p|_{v'_n}\}, |p|_{v\vee v'} = \sup\{|p|_v, |p|_{v'}\},$$

since $v_n \xrightarrow{d} v$. Hence for each $\varepsilon > 0$ there exists $N_1 > 0$, such that for all polynomials $p \in P_1$ and for n, if $n \geq N_1$, then $\sup_{p\in P_1} ||p|_{v_n} - |p|_v| < \frac{\varepsilon}{4}$. Also $v'_n \xrightarrow{d} v'$. Thus for each $\varepsilon > 0$ there exists $N_2 > 0$ such that for all polynomials $p \in P_1$ and for all n,if $n \geq N_2$, then $\sup_{p\in P_1} ||p|_{v'_n} - |p|_{v'}| < \frac{\varepsilon}{4}$. Let $N = max\{N_1, N_2\}$. Therefore for all n and for all polynomials $p \in P_1$, if $n \geq N$, then

$$\begin{aligned} \sup_{p\in P_1} ||p|_{v_n\vee v'_n} - |p|_{v\vee v'}| = \sup_{p\in P_1} |\frac{1}{2}(|p|_{v_n} + |p|_{v'_n}) + \frac{1}{2}||p|_{v_n} - |p|_{v'_n}| \\ -\frac{1}{2}(|p|_v + |p|_{v'}) - \frac{1}{2}||p|_v - |p|_{v'}|| \\ < 4 \times \frac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

□

Theorem 4.10 *The set of all varieties is totally bounded.*

Proof. Let $v \in L$. We define $\varphi_v : P \to \mathbb{R}^+$ as

$$\varphi_v(p) = |p|_v.$$

The mapping $v \to \varphi_v$ from L into $L^{\infty}(p_1)$ is one to one. Hence $L^{\infty}(p_1)$ induces the following metric to L:

$$d_L(v, w) = d(\varphi_v, \varphi_w) = \|\varphi_v - \varphi_w\|_{\infty}$$

Now, let $\{v_n\}_{n=1}^{\infty}$ be a sequence of varieties. For all polynomials $p \in P_1$, $\{|p|_{v_n}\}_{n=1}^{\infty}$ is bounded. Then the sequence $\{|p|_{v_n}\}_{n=1}^{\infty}$ have a subsequence $\{|p|_{v_{n_k}}\}_{k=1}^{\infty}$, which is convergent, and also a Cauchy sequence. Thus the sequence $\{v_{n_k}\}_{k=1}^{\infty}$ is Cauchy sequence. □

Corollary 4.11 *The set of all varieties is a compact metric space.*

Corollary 4.12 *The set of all zero at infinity varieties is a compact metric subspace.*

Example 4.13 *A sequence $\{N_n\}_{n=1}^{\infty}$ is zero at infinity of varieties.*
For each $\varepsilon > 0$ there exists $N > 0$ such that for all $p \in P_{H_1}$, if deg(p) $> N$, then $|p|_{N_n} < \varepsilon$.

If $p = X_1X_2...X_m$ be homogeneous polynomials and deg(p) $= m$, if $n > m > N$, then $|p|_{N_n} < \varepsilon$.

Theorem 4.14 *A sequence $\{N_n\}_{n=1}^{+\infty}$ is not a Cauchy sequence.*

Definition 4.15 *There exists $\varepsilon > 0$, for all $N > 0$, there exists, n, m such that if $n, m > N$, then*

$$sup_{p\in P_{H_1}} ||p|_{N_n} - |p|_{N_m}| \not\to 0.$$

If $p = X_1X_2...X_m$ be homogeneous polynomials and $n > m$, then

$$sup_{p\in P_{H_1}} ||p|_{N_n} - |p|_{N_m}| \geq ||X_1X_2...X_m|_{N_n} - |X_1X_2...X_m|_{N_m}|.$$

$$|X_1X_2...X_m|_{N_n} = 1 \not\to 0.$$

5. The Connected Set of Zero at Infinity Varieties

Theorem 5.1 *Let v be zero at infinity variety. Then $\{v_a|0 \leq a \leq 1\}$ is path-connected, subspace of metric space (L_H, d_H).*

Proof. Let $0 \leq a < a' \leq 1$, v_a and v_a' be two zero at infinity varieties. We define the mapping

$$f : [0, 1] \longrightarrow \{v_a|0 \leq a \leq 1\}$$

as follows

$$f(t) = v_{a+t(a'-a)}.$$

We have $f(0) = v_a$ and $f(1) = v'_a$. Now we prove that f is continuous mapping of $[0,1]$ onto $\{v_a | 0 \leq a \leq 1\}$. Let for all $n \in \mathbb{N}$, $t_n, t \in [0,1]$ and $t_n \longrightarrow t$.

Then

$$d(f(t_n), f(t)) = \sup_{p \in P_{H_1}} ||p|_{v_{a+t_n(a'-a)}} - |p|_{v_{a+t(a'-a)}}|$$

$$= \sup|[a + t_n(a'-a)]^{i-1}|p|_v - [a + t(a'-a)]^{i-1}|p|_v|$$

$$= \sup|p|_v|(a + t_n(a'-a))^{i-1} - (a + t(a'-a))^{i-1}|$$

$$= \sup_{p \in P_{H_1}} |p|_v|(a + t_n(a'-a) - a - t(a'-a))[(a + t_n(a'-a))^{i-2} +$$

$$(a + t_n(a'-a)(a + t(a'-a)) + ... + (a + t(a'-a))^{i-2}]|$$

$$= \sup_{p \in P_{H_1}} |p|_v|(t_n - t)(a'-a)||a + t_n(a'-a)^{i-2} + ... + (a + t(a'-a))^{i-2}|$$

$$\leq \sup_{p \in P_{H_1}} |(t_n - t)(a'-a)|(i-2)a'^{i-2} < |t_n - t| \sup_{p \in P_{H_1}} (i-2)a'^{i-2}.$$

Since $(i-2)a'^{i-2}$ is convergent, it is bounded. Thus

$$d(f(t_n), f(t)) \longrightarrow 0.$$

□

Theorem 5.2 *Let L°_H be the set of all zero at infinity varieties. Let $\alpha \in I$ and $v^\alpha_a \in L^\circ_H$, be zero at infinity variety. Let $\{v^\alpha_a | 0 \leq a \leq 1\} = c_\alpha$. Then $\bigcup_{\alpha \in I} c_\alpha$ is connected.*

Proof. For any $\alpha, \beta \in I$ $(\alpha \neq \beta)$, we have $c_\alpha \cap c_\beta \neq \emptyset$. Since if $a = 0$, then $v^\alpha_0 = N_2$ and $v^\beta_0 = N_2$ and $\bigcap_{\alpha \in I} c_\alpha \neq \emptyset$. Therefore the theorem is proved.

Let $\bigcap_{\alpha \in I} c_\alpha = \emptyset$. Let there exists $\alpha_0 \in I$, such that for any $\alpha \in I$, $c'_\alpha = c_\alpha \cup c_{\alpha_n}$. Then for any $\alpha \in I$, c'_α is connected. We have

$$\bigcap_{\alpha\in I} c'_\alpha = c_{\alpha_0} \cup (\bigcap_{\alpha\in I} c_\alpha) = c_{\alpha_0} \neq \emptyset.$$

Thus $\bigcup_{\alpha\in I} c'_\alpha = c_{\alpha_0} \cup (\bigcup_{\alpha\in I} c_\alpha) = \bigcup_{\alpha\in I} c_\alpha$ and $\bigcup_{\alpha\in I} c_\alpha$ is connected. □

Corollary 5.3 *The set of all zero at infinity varieties is a connected set.*

6. Laws of Variety of C*-Algebras

In this section, we shall characterize varieties of C*-algebras by inequalities on norms of polynomials, and also we shall show that each variety of C*-algebras is singly-generated.

Definition 6.1 *An involution on an algebra A is a conjugate-linear map $a \longrightarrow a^*$ on A such that $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in A$. The pair $(A, *)$ is called an involution algebra, or a *-algebra. A Banach *-algebra is a *-algebra A together with a complete sub-multiplicative norm such that $\|a^*\| = \|a\|$ $(a \in A)$. A C*-algebra is a Banach *-algebra for which $\|a^*a\| = \|a\|^2$ $(a \in A)$. A closed *-subalgebra of a C*-algebra is obviously a C*-algebra. We shall therefore call a closed *-algebra of a C*-algebra a C*-subalgebra. If $\{A_i\}_{i\in I}$ is a family of C*-algebra, the direct sum $\oplus_{i\in I} A_i$ is a C*-algebra with the point wise-defined involution.*

Definition 6.2 *Let v be a family of C*-algebras. We say that v is a variety of C*-algebra, if it is closed under taking, (i) direct sums, (ii) C*-algebra, (iii) quotients (by closed ideals) and (iv) *- isomorphic.*

Throughout this paper, the word "polynomials" is taken to mean a complex polynomial in several non-commuting variables without constant term.

Definition 6.3 *By a law we mean a formal expression $\|p\| \leq K$, where p is a polynomial and $K \in R$. If A is a Banach algebra we say that A satisfies the above law if $\|p\|_A \leq K$, which is a homogeneous law for p to be homogeneous polynomial.*

Definition 6.4 *Let v be a family of Banach algebras. We say that v is a variety of Banach algebras, if it is closed under taking, (i) direct sums, (ii) closed subalgebras, (iii) quotients (by closed ideal) and (iv) isometric isomorphic.*

Definition 6.5 *Let v be a non-empty class of Banach algebras. We define $\hat{u}$ to be the class all Banach algebras bi-continuously isomorphic to members of u.*

Theorem 6.6 *Let v^* be a non-empty class of C^*-algebras. Then v^* is a variety of C^*-algebras if and only if there exists a family of laws $\{\|p\| \leq K_P\}_p$ such that*

$$v^* = \{A : A \text{ is a } C^* - \text{ algebras, and } \|p\| \leq K_P \quad \forall p\}$$

Proof. Let v^* be a variety of C*-algebras. Define

$$K_p = \sup\{\|p\|_C \ : \ C \in v^*\}$$

for all polynomials p. For each polynomial p and $\varepsilon > 0$, there is $C(p,\varepsilon) \in v^*$ such that $\|p\|_{C(p,\varepsilon)} > K_p - \varepsilon$. Let $B = \oplus C(p,\varepsilon)$, then $B \in v^*$. $\|p\|_B = \sup\|p\|_{C(p,\varepsilon)}$, so $\|p\|_B = K_p$ for all polynomials p. It is clear that

$$v^* \subseteq \{A : A \text{ is a } C^* - \text{ algebras, and } \|p\| \leq K_P\ \}$$

Now, suppose that A is a C*-algebras such that for all p, $\|p\|_A \leq K_p = \|p\|_B$. Let $X = B_1^{A_1}$ and $\Gamma = B^X$ (direct sum of copies of B). Since $B \in v^*$, $\Gamma \in v^*$. We define $\zeta_a \in \Gamma (a \in A_1)$ by

$$\zeta_a(x) = x(a) \quad (x \in X).$$

Then $\|\zeta_a\|_\infty = 1$. Let U_0 be the *-subalgebra of Γ generated by

$$\{\zeta_a \ \ a \in A_1\} \cup \{\zeta_a^* \ : \ a \in A_1\}.$$

Define a *-homomorphism $\theta : U_0 \to A$ by $\theta(\zeta_a) = a$ and $\theta(\zeta_a^*) = a^*$ we show that θ is well defined. Each element of U_0 is of the form

$$p(\zeta'_{a_1} ... \zeta'_{a_1})$$

where $p = p(X_1, ..., X_n)$ is a polynomial and ζ'_{a_i} is ζ'_{a_i} or $\zeta^*_{a'_i}$. Let for each $1 \leq i \leq n$,

$$a'_i = \begin{cases} a_i & \zeta'_{a_i} = \zeta_{a_i} \\ a_i^* & \zeta'_{a_i} = \zeta^*_{a_i}. \end{cases}$$

We have

$$\|p(a'_1, ..., a'_n)\| \leq \|p\|_A \leq \|p\|_B$$

$$= \sup\{\|p(x'(a_1), ..., x'(a_n))\| \; : x \in X\}.$$

$$= \sup\{\|p(\zeta'_{a_1}, ..., \zeta'_{a_1})\| \; : x \in X\}$$

$$= \|p(\zeta'_{a_1}, ..., \zeta'_{a_1})\|,$$

where

$$x'(a_i) = \begin{cases} x(a_i) & \zeta'_{a_i} = \zeta_{a_i} \\ (x(a_i))^* & \zeta'_{a_i} = \zeta^*_{a_i}. \end{cases}$$

Hence $\theta : U_0 \to A$ is well-defined. Let U be the closure of U in Γ. Then θ extends to a *-homomorphism of U onto A. Hence $U/ker\theta$ is *-isomorphic with A, $A \in v^*$. The converse is straightforward. □

The proof of the above theorem shows that for each variety of C*-algebras v^* there is $B \in v^*$ such that v^* is generated by the laws $\{\|p\| \leq \|p\|_B\}_p$ and v^* the smallest variety of C*-algebras containing B, i.e, $v^* = v^*(B)$. For v to be the variety of Banach algebras generated by the some laws, we have the following theorem.

Theorem 6.7 *Let v^*, w^* be two varieties of C^*-algebras. Then*

(i) v is the smallest variety of Banach algebras such that $v^* = v \cap \mathbf{1}^*$ where $\mathbf{1}^*$ is the variety of all $C^* - algebras$.

(ii) $v^* \subseteq w^*$ if and only if $v \subseteq w$.

Proof. (i) It is clear that $v^* = v \cap \mathbf{1}^*$. Let v' be a variety of Banach algebras such that $v^* = v' \cap \mathbf{1}^*$. Let $v^* = v^*(A)$, so $A \in v'$. Since for all polynomials p, $|p|_v = \|p\|_A \leq |p|_{v'}$, that $v \subseteq v'$.

(ii)Let $w^* = w^*(B)$ and $v^* = v^*(A)$. Since $A \in w^*$, so for all polynomial p,

$$|p|_v = \|p\|_A \leq \|p\|_B = |p|_w$$

Hence $v \subseteq w$. The converse is evident. □

Corollary 6.8 *Let A be a C^*-algebra. Let w be a variety of Banach algebras, such that $w \subseteq v(A)$ and $w \neq v(A)$. Then there exists a C^*-algebras B such that, $B \in v(A) \backslash w$.*

Proof. Straightforward. □

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